“A brief history of homogenization and micromechanics, with applications”

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Overview

• What is homogenization?
• A little history
• Some results as examples
• Problems using homogenization (low-frequency limit of dynamic problems):
  • Waves in bone
  • Waves in pre-stressed composites
• Summary and the “future”
What is homogenization?
Assign material properties, e.g.
Young’s modulus for linear elasticity: \( \sigma = E e \),
Viscosity for Newtonian fluids: \( \sigma = \tilde{\mu} \dot{e} \).

With conservation eqns, these gives pdes for deformation of body.

Continuum assumption works IF \( \lambda \gg a \), where,
\( \lambda \) is characteristic lengthscale of applied loading,
\( a \) is molecular or atomistic lengthscale.
Inhomogeneous media

For *simple* (homogeneous) media (e.g. water, steel, rubber, etc) these material properties are “constants”.

But what about *inhomogeneous* media when \( \sigma = E(x)e \) or \( \sigma = \mu(x)\dot{e} \)?

Using these in equations of motion give complicated pdes to solve...

E.g.s: Industrial Composites and biological media
Is it necessary to solve these complicated pdes?

Do we really need this level of detail?
Consider the following extension of the continuum assumption:
If \( \lambda \gg a \) where now
\( \lambda \) is characteristic lengthscale of applied loading,
\( a \) is the microscale of the inhomogeneous material,

then we can HOMOGENIZE the material.

I.e.

\[
\sigma = E(x)e \quad \rightarrow \quad \bar{\sigma} = E_\ast \bar{e}
\]

and thus \( E_\ast \) is seen as an effective Young’s modulus.

Extension to more general linear elasticity contexts:

\[
\sigma_{ij} = C_{ijkl}(x)e_{kl} \quad \rightarrow \quad \bar{\sigma}_{ij} = C^\ast_{ijkl}\bar{e}_{kl},
\]

or, to avoid clutter, omit the indices:

\[
\sigma = C'(x)e \quad \rightarrow \quad \bar{\sigma} = C_\ast \bar{e}
\]
A little history and some results
Pre-1960s

Consider now a two-phase composite, of the host/inclusion type. Two materials perfectly bonded together.

Phase 0: Host phase $C = C_0$ so that $\sigma = C_0 e$

Phase 1: Inclusion phase, $C = C_1$ so that $\sigma = C_1 e$.

Early 1960s, only results:
simple bounds
dilute results when interactions between inclusions is negligible

We will now derive an important result in homogenization theory.
**Effective properties in terms of strain concentration**

Stress in the composite can be written as:

\[
\sigma(x) = \chi_0(x)\sigma(x) + \chi_1(x)\sigma(x),
\]

\[
= \chi_0(x)C_0e(x) + \chi_1(x)C_1e(x)
\]

Integrate over the whole body \( V = V_0 \cup V_1 \).

\[
\bar{\sigma} = \frac{1}{|V|} \int_V \sigma \ dV = \frac{C_0}{|V|} \int_{V_0} e \ dV_0 + \frac{C_1}{|V|} \int_{V_1} e \ dV_1
\]

and since \( V_0 = V \setminus V_1 \),

\[
\bar{\sigma} = \frac{C_0}{|V|} \left[ \int_V e \ dV - \int_{V_1} e \ dV_1 \right] + \frac{C_1}{|V|} \int_{V_1} e \ dV_1
\]

\[
= C_0\bar{e} + (C_1 - C_0)\frac{|V_1|}{|V|} \frac{1}{|V_1|} \int_{V_1} e \ dV_1
\]

\[
= C_0\bar{e} + (C_1 - C_0)\phi\bar{e}_1
\]

[where \( \phi = \frac{|V_1|}{|V|} \)]

\[
= [C_0 + \phi(C_1 - C_0)A] \bar{e}.
\]

[where \( \bar{e}_1 = Ae \)].
**Concentration tensor**

Consider a single-inclusion problem:

Given far-field displacement $u = \bar{e}x$, (so far-field strain is $\bar{e}$), we find $e_1(x)$ in the form (linear problem) $\bar{e}_1 = \mathcal{A}\bar{e}$.

So, if we use $A = \mathcal{A}$, then we get:

$$C^d_\ast = C_0 + \phi(C_1 - C_0)A.$$

where the superscript $d$ indicates dilute (well-separated inclusions).
Eshelby’s seminar paper of 1957

Single, isolated ellipsoidal inclusion in an isotropic host phase. Impose far field strain $\bar{\epsilon}$, $(u = \bar{\epsilon} \mathbf{x})$.

Given this, Eshelby showed that:

- the strain $e_1$ inside the inclusion is uniform,
  i.e. each component of $e_1$ is constant.

Hence $\mathcal{A}$ is uniform for isolated ellipsoids.

Eshelby also hypothesized that:
- this was also the case for anisotropic host phases (proved mid-60s)
- this result holds only for ellipsoids (only proved in 2008)
Simple bounds

Note again:

\[ \bar{\sigma} = [C_0 + \phi(C_1 - C_0)A]\bar{\epsilon}, \quad \bar{\epsilon}_1 = A\bar{\epsilon} \]

If we take \( A = I \) (Identity tensor), i.e. uniform strain, then

\[ C^V_* = C_0 + \phi(C_1 - C_0) = (1 - \phi)C_0 + \phi C_1 \]

Similarly, can show that the assumption of uniform stress gives

\[ C^R_* = [(1 - \phi)C_0^{-1} + \phi C_1^{-1}]^{-1} \]

In 1951 Hill showed that these were strict bounds on effective properties (using variational principles):

\[ C^R_* \leq C_* \leq C^V_* \]
An example - spherical particles inside a host phase

Take \( C_r = \kappa_r I_1 + \mu_r I_2, \ r = 0, 1 \)
and \( C_\ast = \kappa_\ast I_1 + \mu_\ast I_2 \), and so we have

\[
\left( \frac{(1 - \phi)}{\kappa_0} + \frac{\phi}{\kappa_1} \right)^{-1} \leq \kappa_\ast \leq (1 - \phi) \kappa_0 + \phi \kappa_1,
\]

\[
\left( \frac{(1 - \phi)}{\mu_0} + \frac{\phi}{\mu_1} \right)^{-1} \leq \mu_\ast \leq (1 - \phi) \mu_0 + \phi \mu_1.
\]

Tungsten-Carbide/Cobalt alloy:

Properties \((10^{10} \, N/m^2)\)

\( \mu_1 = 28.8, \)
\( \kappa_1 = 42, \)
\( \mu_0 = 8, \)
\( \kappa_0 = 17.2, \)
A dilute dispersion of spheres

For an isolated sphere, it is (relatively) straightforward to show that

\[ A = \left( \frac{3\kappa_0 + 4\mu_0}{3\kappa_1 + 4\mu_0} \right) I_1 + \left( \frac{5\mu_0(3\kappa_0 + 4\mu_0)}{3\kappa_0(3\mu_0 + 2\mu_1) + 4\mu_0(2\mu_0 + 3\mu_1)} \right) I_2 \]

\[ \kappa^d_* = \kappa_0 + \phi(\kappa_1 - \kappa_0) \left( \frac{3\kappa_0 + 4\mu_0}{3\kappa_1 + 4\mu_0} \right) , \]

\[ \mu^d_* = \mu_0 + \phi(\mu_1 - \mu_0) \left( \frac{5\mu_0(3\kappa_0 + 4\mu_0)}{3\kappa_0(3\mu_0 + 2\mu_1) + 4\mu_0(2\mu_0 + 3\mu_1)} \right) \]

So pre-1962:
1960s - an important decade

This was a very important time for research in this area:

- Hill (Cambridge),
- Hashin, Shtrikman, Dow, Rosen, Tsai and Halpin (NASA),
- Budiansky (Harvard)

Important breakthroughs due to increased use of fibre reinforced composites and need for knowledge of effective properties of such materials:
**Hashin-Shtrikman bounds**

Hashin and Shtrikman developed bounds based on the following form:

\[ \sigma(x) = C_m e(x) + \tau(x) \]

- \( C_m \) is a reference material
- \( \tau(x) \) is known as the stress polarization tensor.

They developed a variational principle in terms of \( \tau(x) \).

Using information about the macroscopic anisotropy of the composite and taking \( \tau(x) \) to be piecewise constant, they found the now much-used Hashin-Shtrikman bounds.

Much work was done after this by Walpole (1966, 1969) and Willis (1970s, 1980s) in order to clarify and extend these early results.
Hashin-Shtrikman bounds

\[ \kappa_+ \]

\[ \kappa_- \]

\[ \kappa^V \]

\[ \kappa^R \]

\[ \kappa^d \]
Approximate schemes

1965 - Hill, Budiansky, developed self-consistent schemes to account for interaction between inhomogeneities (albeit approximately).

\( A \) is approximated in the following manner:
1) Embed the inclusion in the effective medium.
2) Solve the single-inclusion problem with \( u = \bar{e}x \) as before.
3) This gives implicit equations for \( \kappa^* \) and \( \mu^* \).

\[
\kappa^{SC} = \kappa_0 + \phi(\kappa_1 - \kappa_0) \left( \frac{3\kappa^* + 4\mu^*}{3\kappa_1 + 4\mu^*} \right),
\]

\[
\mu^{SC} = \mu_0 + \ldots...
\]
Self-consistent scheme

\[
\kappa^+ \quad \kappa^*_{SC} \quad \kappa^*_{V} \quad \kappa^*_{R} \quad \kappa^d_{*} \quad \kappa^-_{*}
\]
1970s - Classical Asymptotic homogenization

Developed for composites whose microstructure is periodic. (Sanchez-Palencia, Bensoussan et al, Bakhvalov and Panasenko,......)

(Periodic) microscale of bar is $O(a)$. Cross-section of bar is $O(q)$. Choose $\eta = q/a \ll 1$ so that dispersive effects are negligible.

Linear (time harmonic) wave propagation governed by:

$$\frac{d}{dx} \left( E(x) \frac{du}{dx} \right) + \rho(x) \omega^2 u = 0$$
Consider low frequency propagation, i.e. $\epsilon = ak \ll 1$.

Using method of asymptotic homogenization, introduce two lengthscales

$$\xi = x \quad \text{and} \quad z = \epsilon x$$

and use the expansion

$$u(x) = u_0(z) + \epsilon u_1(\xi, z) + O(\epsilon^2)$$

It is simple to show that:

$$\frac{d}{dz} \left( E_* \frac{du_0}{dz} \right) + \rho_* u_0 = 0$$

where

$$E_* = \frac{E_1 E_0}{(1 - \phi) E_1 + \phi E_0}$$

is the effective Young’s modulus of the bar.
**1980s - present day**

- Huge growth in the subject area.
- Better **bounds** using microstructure information and numerous advances in approximate static and dynamic schemes.
- Development of **design** of materials, metamaterials, etc.
- Applications in poroelasticity, piezoelectric materials, reaction-diffusion problems, etc.
- Ideas have touched almost every area of applied maths.
- **Multi-scale** problems now a huge area of research.
- Discuss **modern** ideas (the future) later.
Applications

• Effective elastic moduli of Cortical bone
• Multiple scattering in pre-stressed (random) composites
Effective Elastic Moduli of Cortical Bone
Introduction
Clearly there are several levels of microstructure. You will see any number of these scales depending upon your level of observation.
Can we replace
\[ \sigma = C(z, \xi)e \]
by
\[ \bar{\sigma} = C_*(z)\bar{e} \]?

Previous work: Crolet et al. (1993, 96), Hellmich (2002-06).
MAH and generalized Hooke’s law

Use the method of asymptotic homogenization (MAH) (bar problem above).

Pores arranged on a hexagonal lattice.
We can derive the effective elastic properties of such a material.

Difficulty comes in solving the so-called cell problem: here we use complex analysis.

Generalized Hooke’s law: \( \sigma_{ij} = C^*_{ijkl}e_{kl} \)

\[ \sigma_{11} = c^*_{11}e_{11} + c^*_{12}e_{22} + c^*_{13}e_{33}, \]
\[ \sigma_{33} = c^*_{13}e_{11} + c^*_{13}e_{22} + c^*_{33}e_{33}, \]
\[ \sigma_{12} = c^*_{66}e_{12} \]

Hexagonal lattice: transverse isotropy - 5 elastic properties.
Consider results for the two-phase, hexagonal lattice case (Grimal (2010)).

\[
\begin{align*}
C_{1212}^{\text{matrix}} &= 9.25, && C_{1122}^{\text{matrix}} &= 11, \\
C_{1133}^{\text{matrix}} &= 11.9, && C_{3333}^{\text{matrix}} &= 38.1,
\end{align*}
\]

Hard (isotropic) pore:

\[
E_{\text{pore}} = 0.13 \\
\nu_{\text{pore}} = 0.49
\]

Soft pore:

\[
E_{\text{pore}} \to 0 \\
\nu_{\text{pore}} \to 0
\]
Hashin-Rosen bounds on $\mu_{12}^* = c_{66}^*$
Anisotropy parameter

\[
\frac{c_{33}^*}{c_{11}^*}
\]

- Asymptotic (hard)
- Asymptotic (soft)
- FEM (hard)
- FEM (soft)

porosity
Multiple scattering (and homogenization) in a (random) pre-stressed medium
**Background**

Effective (incremental) dynamic behaviour of pre-stressed, inhomogeneous rubbery composites (small-on-large).

Difficult to characterize: large def., constitutive behaviour,...
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Effective (incremental) dynamic behaviour of pre-stressed, inhomogeneous rubbery composites (small-on-large).

Difficult to characterize: large def., constitutive behaviour,...

**Literature:** Ogden, 1984, Destrade and Saccomandi, 2007 CISM

Almost exclusively: homogeneous initial deformations.

What about inhomogeneous deformations? Recently:

Parnell, W.J. 2007, IMA JAM

Bigoni D., Gei, M. and Movchan, A.B. 2008, JMPS

Bertoldi, K. and Boyce, M. 2008, Phys Rev E

Influence of pre-stress on the pass-band structure in *periodic* solids.
Applications

- Oil and geophysical industry
- NDT for pre-stressed materials
- Use of rubber composites in automotive, aerospace and defence industries (often in pre-stressed states)
- Biological tissues (lung, tendon, etc) are all nonlinear (visco)elastic, pre-stressed in vivo and are natural “composites”
Pre-stress and inhomogeneous deformation
Problem description

(a)

(b)
**Incompressibility and Constitutive behaviour**

Assume host medium is incompressible. Impose a hydrostatic pressure $\Sigma_{rr} = -p_\infty$ at infinity (far-field) and a fixed stretch $L$ axially:

$$R = R(r) \quad \Theta = \theta \quad Z = \frac{z}{L}$$

Constitutive behaviour:
Take a neo-Hookean medium:

$$W_{SE} = \frac{\mu}{2} (I_1 - 3),$$

Stresses $\Sigma_{rr}, \Sigma_{\theta\theta}$ come from derivatives of this potential.
**Radial stress, \( L = 1 \)**

Only 1 equilibrium equation:

\[
\frac{d\Sigma_{rr}}{dr} + \frac{1}{r} (\Sigma_{rr} - \Sigma_{\theta\theta}) = 0
\]

which can be integrated to give \( \Sigma_{rr} \):

![Graph showing the radial stress distribution](image-url)
Deformed radius

Together with $\Sigma_{rr} \rightarrow -p_\infty$ as $r \rightarrow \infty$ this gives

$$\frac{p_\infty}{\mu} = \frac{1}{2L} \left( \frac{A^2}{La^2} - 1 + \log \left( \frac{A^2}{La^2} \right) \right),$$

so that $a/A$ can be determined in terms of $p_\infty/\mu$ and $L$. 

\[ \begin{array}{c|c}
L & a/A \\
\hline
0.7 & \text{---} \\
1 & \text{--} \\
1.5 & \text{---}
\end{array} \]
**Incremental deformation**

Superpose small amplitude waves on the finite deformation:

\[
\hat{u} = u + \eta u'
\]

where \( u \) is the finite displacement, \( \eta \ll 1 \) and

\[
u' = (0, 0, w(r, \theta)) \exp(i\omega t).
\]
**Incremental deformation**

Superpose small amplitude waves on the finite deformation:

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where \( u \) is the finite displacement, \( \eta \ll 1 \) and

\[ u' = (0, 0, w(r, \theta)) \exp(i\omega t). \]

Transpires that the modified wave equation is:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ \left( r + \frac{M}{r} \right) \frac{\partial w}{\partial r} \right] + \frac{1}{r^2 + M} \frac{\partial^2 w}{\partial \theta^2} + k^2 w = 0.
\]

where \( M = A^2/L - a^2, k^2 = LK^2 \).
**Incremental deformation**

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\hat{u} = u + \eta u'
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\]

where \( M = A^2/L - a^2 \), \( k^2 = LK^2 \).

Can think of this as:

\[
(\nabla^2 + \mathcal{L}_M + k^2)w = 0
\]

where \( \mathcal{L}_M = 0 \) for no pre-stress.
Line source in the pre-stressed medium

Line source of strength \( c \), at \((r_0, \theta_0)\) \((c = C/L \) at \((R_0, \Theta_0))\):

\[
\frac{\mu}{L} (\nabla^2 + \mathcal{L}_M + \rho \omega^2) w = \frac{C}{Lr_0} \delta(r - r_0) \delta(\theta - \theta_0).
\]
Line source in the pre-stressed medium

Line source of strength $c$, at $(r_0, \theta_0)$ ($c = C/L$ at $(R_0, \Theta_0)$):

$$\frac{\mu}{L}(\nabla^2 + L \mathcal{M} + \rho \omega^2)w = \frac{C}{Lr_0} \delta(r - r_0)\delta(\theta - \theta_0).$$

Introducing the mapping $R^2 = L(r^2 + M) \Theta = \theta$, we get

$$\nabla^2 W + K^2 W = \frac{C}{\mu R_0} \frac{1}{\delta(R - R_0)\delta(\Theta - \Theta_0)}$$
Line source in the pre-stressed medium

Line source of strength $c$, at $(r_0, \theta_0)$ ($c = C/L$ at $(R_0, \Theta_0)$):

$$\frac{\mu}{L} (\nabla^2 + L M + \rho \omega^2)w = \frac{C}{L r_0} \delta(r - r_0) \delta(\theta - \theta_0).$$

Introducing the mapping $R^2 = L(r^2 + M), \Theta = \theta$, we get

$$\nabla^2 W + K^2 W = \frac{C}{\mu R_0} \frac{1}{R_0} \delta(R - R_0) \delta(\Theta - \Theta_0)$$

So that

$$W(R, \Theta) = W_i + \sum_{n=-\infty}^{\infty} i^n a_n H_n(K R) e^{in(\Theta - \Theta_0)}$$
**Line source in the pre-stressed medium**

Line source of strength \( c \), at \((r_0, \theta_0)\) \((c = C/L \text{ at } (R_0, \Theta_0))\):

\[
\frac{\mu}{L}(\nabla^2 + L_M + \rho \omega^2)w = \frac{C}{Lr_0} \delta(r - r_0)\delta(\theta - \theta_0).
\]

Introducing the mapping \( R^2 = L(r^2 + M), \Theta = \theta \), we get

\[
\nabla^2 W + K^2 W = \frac{C}{\mu R_0} \frac{1}{R_0} \delta(R - R_0)\delta(\Theta - \Theta_0)
\]

So that

\[
W(R, \Theta) = W_i + \sum_{n=-\infty}^{\infty} i^n a_n H_n(KR)e^{in(\Theta - \Theta_0)}
\]

and therefore

\[
w(r, \theta) = w_i + \sum_{n=-\infty}^{\infty} i^n a_n H_n \left( k\sqrt{r^2 + M} \right) e^{in(\theta - \theta_0)}
\]
Multiple scattering
Multiple scattering

$N$ well separated voids within a pre-stressed medium.
What is the effect of the pre-stress on the effective wavenumber?
Multiple scattering

\( N \) well separated voids within a pre-stressed medium. What is the effect of the pre-stress on the effective wavenumber?

In the deformed configuration the solution is therefore

\[
w = w_i + \sum_{j=1}^{N} a_j^j Z_n H_n \left( k \sqrt{r_j^2 + M} \right) e^{i n \theta_j}
\]

where \((r_j, \theta_j)\) is the local coordinate system of the \( j \)th void.
Solution in the vicinity of the $s$th void is

$$w(r_s, \theta_s) = w_i + \sum_{n=-\infty}^{\infty} a_n^s Z_n H_n(k \sqrt{r_s^2 + M}) e^{in\theta_s}$$

$$+ \sum_{j=1, j \neq s}^{N} \sum_{n=-\infty}^{\infty} a_n^j Z_n H_n(k \sqrt{r_j^2 + M}) e^{in\theta_j}$$
Solution in the vicinity of the $s$th void is

$$w(r_s, \theta_s) = w_i + \sum_{n=-\infty}^{\infty} a_n^s Z_n H_n(k \sqrt{r_s^2 + M}) e^{i n \theta_s}$$

$$+ \sum_{j=1, j \neq s}^{N} \sum_{n=-\infty}^{\infty} a_n^j Z_n H_n(k \sqrt{r_j^2 + M}) e^{i n \theta_j}$$

Voids are well-separated, so

$$w(r_s, \theta_s) \approx w_i + \sum_{n=-\infty}^{\infty} a_n^s Z_n H_n(k \sqrt{r_s^2 + M}) e^{i n \theta_s}$$

$$+ \sum_{j=1, j \neq s}^{N} \sum_{n=-\infty}^{\infty} a_n^j Z_n H_n(k r_j) e^{i n \theta_j}.$$
Graf’s addition theorem implies

\[ w(r_s, \theta_s) \approx w_i + \sum_{n=\infty}^{\infty} a_n^s Z_n H_n(k \sqrt{r_s^2 + M}) e^{i n \theta_s} \]

\[ + \sum_{j=1, j \neq s}^{N} \sum_{n=\infty}^{\infty} a_n^j Z_n \sum_{m=\infty}^{\infty} H_{n-m}(k R_{js}) e^{i (n-m) \theta_{js}} J_m(k r_s) e^{i m \theta_s} \]

where \( R_{js} \) is the distance between the \( j \)th and \( s \)th void and \( \theta_{js} \) is the angle between the \( j \)th and \( s \)th void.
Graf's addition theorem implies

\[ w(r_s, \theta_s) \approx w_i + \sum_{n=\infty}^{\infty} a_n^s Z_n H_n(k \sqrt{r_s^2 + M}) e^{in\theta_s} \]

\[ + \sum_{j=1, j \neq s}^{N} \sum_{n=\infty}^{\infty} a_j Z_n \sum_{m=\infty}^{\infty} H_{n-m}(kR_{js}) e^{i(n-m)\theta_{js}} J_m(kr_s) e^{im\theta_s} \]

where \( R_{js} \) is the distance between the \( j \)th and \( s \)th void and \( \theta_{js} \) is the angle between the \( j \)th and \( s \)th void.

Zero traction on \( r_s = a \) gives

\[ a_m^s + \sum_{j=1, j \neq s}^{N} \sum_{n=\infty}^{\infty} a_j Z_n H_{n-m}(kR_{js}) e^{i(n-m)\theta_{js}} = F_i \]

where

\[ Z_n = P \frac{J'_n(ka)}{H'_n(Pka)} \]

\[ P = \frac{A}{a \sqrt{L}}. \]
Effective wavenumber

Take ensemble averages, use QCA and hole-correction pair-correlation function, and assuming

\[ \langle a_n \rangle = C_n e^{ik_* x_s} \]

in the low frequency limit we obtain for the effective wavenumber \( k_* \)

\[ \frac{k_*^2}{K^2} = \frac{\rho_* \mu_0}{\rho_0 \mu_*} = L(1 - LP^2 \phi) \left( \frac{1 + LP^3 \phi}{1 - LP^3 \phi} \right) \]
Effective wavenumber

Take ensemble averages, use QCA and hole-correction pair-correlation function, and assuming

$$\langle a_n^s \rangle = C_n e^{ik_*x_s}$$

in the low frequency limit we obtain for the effective wavenumber $k_*$

$$\frac{k_*^2}{K^2} = \frac{\rho_* \mu_0}{\rho_0 \mu_*} = L(1 - LP^2\phi) \left( \frac{1 + LP^3\phi}{1 - LP^3\phi} \right)$$

Given that $\rho_* = (1 - \phi)\rho_0$, this gives

$$\frac{\mu_*}{\mu_0} = \frac{(1 - \phi)(1 - LP^3\phi)}{L(1 - LP^2\phi)(1 + LP^3\phi)},$$

which for no pre-stress ($P = L = 1, \phi = \phi_0$) is

$$\frac{\mu_*}{\mu_0} = \frac{1 - \phi_0}{1 + \phi_0}.$$
Effective Shear Modulus

\[
\frac{\mu_*}{\mu_0}
\]

\[
\phi_0 = 0.01
\]

\[
\phi_0 = 0.1
\]

\[
\phi_0 = 0.2
\]
Effective Wavespeed

\[ \frac{c_*/c_0}{\phi_0} \]

- \( \phi_0 = 0.01 \)
- \( \phi_0 = 0.1 \)
- \( \phi_0 = 0.2 \)
Summary

• Homogenization - a classical subject but with much still to offer
• History - Eshelby is a classical paper!

Applications:
• Waves in bone
• Pre-stressed random composites
• Flexural waves in thin plates (sea-ice?)
• Design of “tunable” wave-filters

Current “hot topics”
• metamaterials - unnatural material properties.
• cloaking materials
• nanomaterials
• fractal media - e.g. the gecko’s feet
• soft tissue modelling
Related Publications

- Parnell, W.J. and Abrahams, I.D. 2011
  *An introduction to homogenization in Continuum Mechanics*,
  Cambridge University Press

- Parnell, W.J., Vu, M.-B., Naili, S. and Grimal, Q. 2010

- Parnell, W.J. and Abrahams, I.D.

- Parnell, W.J. and Grimal, Q. 2009

- Parnell, W.J. 2007
  *Effective wave propagation in a pre-stressed nonlinear elastic composite bar*, IMA J. Appl. Math. 72, 223-244.