MALAGA SPRING SCHOOL

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TORIC TOPOLOGY

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LECTURE 6
STABLY COMPLEX STRUCTURES

OVERVIEW

1. QUADRATIC INTERSECTIONS

2. SCZCARBA'S THEOREM

3. EXAMPLES

1. QUADRATIC INTERSECTIONS

Work with simple polytope $P \subset \mathbb{R}^n$, with

- facets $\{F_1, \ldots, F_m\}$, defined by
- half-spaces H_i, normals a_i, for $1 \leq i \leq m$.

Affine embedding $i_P \colon \mathbb{R}^n \to \mathbb{R}^V$ maps x to signed distances $(d(x, \partial H_1), \ldots, d(x, \partial H_m))$; so

$$i_P(P) \;=\; i_P(\mathbb{R}^n) \cap \mathbb{R}^m_{\geqslant}$$

displays $P \subset \mathbb{R}^V_{\geqslant}$ as submanifold *with corners*.

Recall that $\mathcal{Z}_P = T^m \times P/\sim$, where $(t,p) \sim (u,p)$ whenever $t^{-1}u \in T^{\sigma(p)}$. There is therefore a pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{\;i_Z\;} & \mathbb{C}^m \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ P & \xrightarrow{\;i_P\;} & \mathbb{R}^m_{\geqslant} \end{array} \quad , \qquad (1)$$

where $q(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$, the embedding i_Z is T^m-equivariant, and the maps q are T^m-orbit maps.

Rewrite $\mathbb{C}^m$ as $\mathbb{R}^{2m}$, and z_j as $q_j + ir_j$, for $1 \leq j \leq m$. So T_j act by rotation.

Then Diagram (1) shows that i_Z embeds $\mathcal{Z}_P$ in $\mathbb{R}^{2m}$ as the space of solutions of $m - n$ quadratic equations [BPR07]

$$\sum_{k=1}^{m} c_{j,k} \left(q_k^2 + r_k^2 - b_k \right) = 0 \qquad (2)$$

for $1 \leq j \leq m - n$, where the $c_{j,k}$ and b_k depend on $H_1, \ldots, H_m$.

Each equation defines a smooth hypersurface, whose intersections are transverse; so $\mathcal{Z}_P$ is an $(m + n)$–dimensional framed submanifold of $\mathbb{R}^{2m}$. The framing is equivariant.

2. SCZCARBA'S THEOREM

Let $N = N^d$ be a smooth manifold. A *stably complex structure* is a *real* isomorphism

$$c\colon \tau(N) \oplus \mathbb{R}^{2k-d} \xrightarrow{\cong} \zeta\,,$$

where ζ is a $U(k)$-bundle for some large k.

Since $\mathbb{R}^{2k-d}$ is oriented (and framed!) by the standard basis, c fixes an orientation of N.

If T^l acts on N by $\alpha\colon T^l \times N \to N$, then c is *T^l–invariant* if the isomorphism

$$c \cdot (d\alpha(t,\,\cdot) \oplus 1) \cdot c^{-1}\colon \zeta \longrightarrow \zeta$$

is complex, for every $t \in T^l$.

We will see that every quasitoric $M(P, \Lambda)$ admits a *canonical* c, so long as P is *oriented*!

A *combinatorial pair* (P, Λ) is an oriented simple polytope P with ordered facets, and an integral $n \times m$ matrix Λ satisfying Condition $(\star)$.

We recall the exact sequence

$$K(\Lambda) \xrightarrow{<} T^m \xrightarrow{\Lambda} T^n,$$

and the principal K-bundle

$$q \colon \mathcal{Z}_P \to M = M(P, \Lambda).$$

Tangents along the fibres of q define a real $(m-n)$–plane bundle ξ. It is K-equivariantly trivial [SCZC64, Corollary (6.2)].

The embedding $i_Z\colon \mathcal{Z}_P \to \mathbb{C}^m$ of (1) gives an equivariant decomposition

$$\tau(Z_P) \oplus \nu(i_Z) \;\cong\; \tau(\mathbb{C}^m)|_{Z_P} \;=\; \mathbb{C}^m,$$

where $\nu(i_Z) \cong \mathbb{R}^{m-n}$ because $\mathcal{Z}_P$ is framed.

Factor out the kernel $K = K(\Lambda) < T^m$ and apply Sczcarba's Theorem (1.1) [SCZC64] to get an isomorphism

$$\tau(M) \oplus (\xi/K) \oplus (\nu(i_Z)/K) \;\cong\; \mathcal{Z}_P \times_K \mathbb{C}^m \quad (3)$$

of real $2m$-plane bundles over $M = M(P, \Lambda)$.

The right-hand side of (3) is $\bigoplus_{i=1}^m \rho_i$, where ρ_i is the complex line bundle defined by the action of K on the ith coordinate.

The equivariant framings give isomorphisms

$$\xi/K \cong \mathbb{R}^{m-n} \quad \text{and} \quad \nu(i_Z)/K \cong \mathbb{R}^{m-n}.$$

Actually, ρ_i is isomorphic to the i th facial 2–plane bundle $\nu(X_i \subset M)$, and is orientated by the i th column of Λ. The orientation is a complex structure; reversal is conjugation!

So (3) reduces to an isomorphism

$$c_{(P,\Lambda)} \colon \tau(M) \oplus \mathbb{R}^{2(m-n)} \;\cong\; \rho_1 \oplus \ldots \oplus \rho_m \,. \quad (4)$$

Different framings of $\mathbb{R}^{2(m-n)}$ may lead to different isomorphisms. Since M is connected and $O(2(m-n))$ has two components, isomorphisms agree whenever the framings induce the same orientations on $\mathbb{R}^{2(m-n)}$.

Choose the orientation compatible with

- $\tau(M)$, given by orientation of P
- $\rho_1 \oplus \ldots \oplus \rho_m$, given by columns of Λ.

The induced structure is invariant under the action of T^n, because i_Z is T^m–equivariant.

An *omniorientation* of a quasitoric M consists of 2^{m+1} items of data:

- an orientation of M
- an orientation of $\nu(X_i \subset M)$, for $1 \leq i \leq m$.

Theorem Every $M(P,\varLambda)$ is omnioriented; the omniorientation induces the canonical stably complex structure $c_{(P,\varLambda)}$ of (4), which is T^n-invariant.

Reversing the orientation of P *negates* $c_{(P,\varLambda)}$, whereas negating the column λ_i *conjugates* ρ_i, for $1 \leq i \leq m$. None of these operations affects the validity of Condition $(*)$ for $\varLambda$.

3. EXAMPLES

3.2. Nonsingular projective toric varieties

In this case, Λ is the matrix of the *fan* $\Sigma = \Sigma_P$ associated to a rational simple polytope P. Its columns are the inward pointing normals to the facets.

Then the summand $(\xi/K) \oplus (\nu(i_Z)/K)$ of (3) admits a canonical complex structure, and is trivial. Furthermore, $\tau(M)$ is itself complex.

So (4) reduces to

$$c_\Sigma \colon \tau(M) \oplus \mathbb{C}^{(m-n)} \;\cong\; \rho_1 \oplus \ldots \oplus \rho_m \,, \qquad (5)$$

which is the *stabilisation* of the canonical complex varietal structure.

3.2. The projective plane

Consider the complex varietal dicharacteristic

$$\Lambda = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

of Lecture 4; it gives Σ_{Δ^2}.

The framed hypersurface $\mathcal{Z}_P \subset \mathbb{C}^3$ is $S^5 \simeq \mathbb{C}^3 \setminus \{0\}$, and every ρ_i is isomorphic to the conjugate Hopf bundle $\overline{\eta}$. So the stably complex structure of (7) becomes

$$c_\Sigma \colon \tau(\mathbb{C}P^2) \oplus \mathbb{C} \;\cong\; \overline{\eta} \oplus \overline{\eta} \oplus \overline{\eta}\,; \qquad (6)$$

this is very familiar!

We began Lecture 4 with the omniorientation

$$\Lambda \;=\; \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

This gives stable complex structure

$$\tau(\mathbb{C}P^2) \oplus \mathbb{R}^2 \;\cong\; \overline{\eta} \oplus \overline{\eta} \oplus \eta\,, \qquad (7)$$

which *cannot* be destabilised.

3.2. The Hirzebruch surface

Consider the complex varietal dicharacteristic

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

of Lecture 4. It gives Σ_Q, where Q is the quadrilateral $(1,0)$, $(2,1)$, $(0,1)$ and $(0,0)$.

The framed intersection $\mathcal{Z}_Q \subset \mathbb{C}^4$ is $(S^3)^2 \simeq (\mathbb{C}^2 \setminus \{0\})^2$, and there are isomorphisms

$$\rho_1 \cong \overline{\eta}_1, \;\; \rho_2 \cong \overline{\eta}_1, \;\; \rho_3 \cong \eta_3, \;\; \rho_4 \cong \overline{\eta}_1\eta_3\,,$$

where the 1st Chern classes $c_1(\eta_1) = v_1$ and $c_1(\eta_3) = v_3$ generate $H^2(Hi^2_\gamma;\mathbb{Z})$.

So the varietal stably complex structure is

$$c_\Sigma\colon \tau(Hi^2_\gamma) \oplus \mathbb{C}^2 \;\cong\; \overline{\eta}_1 \oplus \overline{\eta}_1 \oplus \eta_3 \oplus \overline{\eta}_1\eta_3\,. \quad (8)$$

We began Lecture 4 with the omniorientation

$$\Lambda \quad = \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}.$$

This gives the stable complex structure

$$\tau(Hi^2_\gamma) \oplus \mathbb{R}^4 \;\cong\; \eta_1 \oplus \overline{\eta}_1 \oplus \eta_3 \oplus \eta_1\overline{\eta}_3\,, \qquad (9)$$

which *cannot* be destabilised.

The total Chern class of $\tau(Hi^2_\gamma)$ is

$$(1+v_1)(1-v_1)(1+v_3)(1+v_1-v_3)$$

in $H^*(Hi^2_\gamma;\mathbb{Z})$. But $v_1^2 = v_3(v_1-v_3) = 0$ from Lecture 5, so we obtain

$$(1-v_1^2)(1+v_1+v_3(v_1-v_3)) \;=\; 1+v_1.$$

So $c_1^2(\tau) = c_2(\tau) = 0$, and all Chern numbers are zero. Hence Hi^2_γ is a *boundary* with this stably complex structure !!