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TORIC TOPOLOGY

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LECTURE 5
COHOMOLOGY STRUCTURES

OVERVIEW

1. ADDITIVE STRUCTURE

2. MULTIPLICATIVE STRUCTURE

3. COMPLEX ORIENTATIONS

1. ADDITIVE STRUCTURE

We assume throughout that (M, a) denotes a well-behaved T^n-action on a smooth manifold M^{2n}, with quotient polytope P. As always, P has facets F_i, for $1 \leq i \leq m$.

Each of the *facial submanifolds*

$$X_i \;=\; X_{F_i} \;:=\; q^{-1}(F_i) \;\subset\; M$$

is quasitoric, with respect to the action of $T^n/T_{y_i} \cong T^{n-1}$, where $y_i \in q^{-1}(\mathit{Int}(F_i))$.

The F_i intersect *transversally* in P, so the X_i intersect transversally in M. In particular,

$$X_{i_1} \cap \cdots \cap X_{i_k} \;=\; q^{-1}(F_{i_1} \cap \cdots \cap F_{i_k}) \qquad (1)$$

is transverse, for any object $\sigma = \{F_{i_1}, \ldots, F_{i_k}\}$ of $\mathrm{CAT}(K_P)$. We may write it as $X_\sigma \subset M$; it is also quasitoric.

The additive structure of $H^*(M;\mathbb{Z})$ turns out to depend only on the combinatorial type of P; it can be read off beautifully from CAT(P)!

We exhibit a cell-structure for M in terms of a certain Morse function $M \to \mathbb{R}$, described via an embedding $P \subset \mathbb{R}^n$.

Choose a vector $w \in \mathbb{R}^n$ that is *not* perpendicular to *any* edge of P, and let

$$g(u) \; = \; \langle u, w \rangle \; \text{ for every } u \in \mathbb{R}^n.$$

Usually, we interpret g as the restriction $g|_P \colon P \to \mathbb{R}$.

Each edge is oriented along increasing values of g, so the 1-skeleton $P^{(1)}$ is a directed graph. Each vertex has an *in-degree* $d(x)$, whose inward edges define a face F_x of dimension $d(x)$.

Let $\partial' F_x$ denote the union of those faces which *do not* contain x, and write

$$G_x \;=\; F_x \setminus \partial' F_x\,.$$

The charts U_i of 4(1) show that $G_x \subset U_x$ is diffeomorphic to the orthant pair $\mathbb{R}^{d(x)}_{\geqslant} \subset \mathbb{R}^n_{\geqslant}$.

If we define the subspaces

$$e_x := q^{-1}(G_x) \quad \text{and} \quad M_x = q^{-1}(F_x)$$

in M, then e_x is diffeomorphic to $\mathbb{C}^{d(x)}$, and its closure is M_x (which is also quasitoric).

The e_x therefore define a decomposition of M^{2n} into even dimensional cells. Let there be h_k of them in dimension $2k$.

Theorem 5.1. *$H_{2k-1}(M;\mathbb{Z})$ is zero, and $H_{2k}(M;\mathbb{Z})$ is free of rank h_k, for $1 \leq k \leq n$.*

So what are the integers h_k? Well; assume that P has f_k faces of dimension $n-1-k$ for $-1 \leq k \leq n-1$. Then the identity

$$\sum_{k=-1}^{n-1} f_k u^{n-1-k} \equiv \sum_{k=0}^{n} h_k (u+1)^{n-k}. \qquad (2)$$

holds in $\mathbb{Z}[u]$.

Why? Because the *left* hand side of (2) is the sum of monomials u^d, one for each d–dimensional face of P; so each face F_x of in-degree $d(x)$ contributes $(u+1)^{n-d(x)}$ to the *right* hand side.

The polytopes Δ^2 and I^2 show that $\mathbb{C}P^2$ and Hi^2_γ have integral homology groups

$$\mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z} \qquad \text{and} \qquad \mathbb{Z} \quad 0 \quad \mathbb{Z} \oplus \mathbb{Z} \quad 0 \quad \mathbb{Z}$$

respectively.

2. MULTIPLICATIVE STRUCTURE

The additive analysis also shows that $H_{2i}(M;\mathbb{Z})$ is generated by the fundamental classes $[X_\sigma]$ of the submanifolds (1).

In particular, the $[X_i]$ generate $H_{2n-2}(M;\mathbb{Z})$ (with redundancies), and their Poincaré duals v_i generate $H^2(M;\mathbb{Z})$, where $1 \leq i \leq m$. Moreover, the intersection

$$X_\sigma \;=\; X_{i_1} \cap \cdots \cap X_{i_k}$$

is transverse, so $v_\sigma = v_{i_1} \cdots v_{i_k}$ in $H^{2|\sigma|}(M;\mathbb{Z})$.

We conclude that

Lemma. *As a graded ring, $H^*(M;\mathbb{Z})$ is concentrated in even dimensions and generated by the 2–dimensional classes v_i.*

The explicit multiplicative structure depends on λ (or Λ!)

We begin with the Borel fibrations

$$T^n \longrightarrow ET^n \times M \longrightarrow ET^n \times_{T^n} M \xrightarrow{l} BT^n , \quad (3)$$

and study the Serre spectral sequence

$$E_2^{*,*}(M) \ = \ H^*(ET^n \times_{T^n} M; \mathbb{Z}) \otimes \wedge_{\mathbb{Z}}(a_1, \ldots, a_n) ,$$

which converges to $H^*(M; \mathbb{Z})$.

It helps to consider

$$DJ(K_P) \ := \ ET^V \times_{T^V} \mathcal{Z}_{\mathcal{P}} .$$

Now $K(\Lambda)$ acts freely on $\mathcal{Z}_{\mathcal{P}}$ (by Lemma, Lecture 4) and $K(\Lambda) \to T^V \to T^n$ is split; so

$$\begin{aligned} DJ(K_P) &\simeq ET^n \times_{T^n} (\mathcal{Z}_P / K(\Lambda)) \\ &\cong ET^n \times_{T^n} M . \end{aligned}$$

Hence $H^*(ET^n \times_{T^n} M; \mathbb{Z})$ is isomorphic to the face ring $\mathbb{Z}[K_P]$!

Unravelling the definitions shows that

$$l^*\colon H^*(BT^n) \longrightarrow \mathbb{Z}[K_P]$$

acts in dimension 2 by Λ^{tr}, as homomorphism

$$\mathbb{Z}\langle u_1, \ldots, u_n\rangle \longrightarrow \mathbb{Z}\langle v_1, \ldots, v_m\rangle\,.$$

In terms of complex line bundles, this means

$$l^*\eta_j \;=\; \phi_1^{\lambda_{i,1}} \otimes \cdots \otimes \phi_m^{\lambda_{i,m}} \qquad (4)$$

over $DJ(K_P)$, for $1 \leq j \leq n$.

Now we pull the Serre spectral sequence for $T^n \to ET^n \to BT^n$ back along l^*; it has

$$\begin{aligned} E_2^{*,*}(ET^n) \;&=\; H^*(BT^n;\mathbb{Z}) \otimes \wedge_{\mathbb{Z}}(a_1, \ldots, a_n) \\ &=\; S_{\mathbb{Z}}(u_1, \ldots, u_n) \otimes \wedge_{\mathbb{Z}}(a_1, \ldots, a_n)\,, \end{aligned}$$

with $d_2(a_i) = u_i$. So E_2 realises the Koszul resolution [MAC67, Sections VII.2 & 6] for $\mathbb{Z}$.

But l^* acts by Λ^{tr}, so $E_2^{*,*}(M)$ realises the Koszul resolution for $\mathbb{Z}[K_P]/J$, where J is the ideal generated by

$$\lambda_{i,1}v_1 + \ldots + \lambda_{i,m}v_m \quad \text{for} \quad 1 \leq i \leq m.$$

Theorem 5.2. *As graded rings, $H^*(M;\mathbb{Z})$ is isomorphic to*

$$S_Z(v_1, \ldots, v_m)/(I + J)\,,$$

where I and J are generated by the non-faces of P and the rows of Λ respectively.

So $H^*(\mathbb{C}P^2;\mathbb{Z}) \cong S_{\mathbb{Z}}(v_1)/(v_1^3)$, as

$$I = (v_1v_2v_3) \text{ and } J = (v_1 + v_3, v_2 + v_3)\,;$$

and $H^*(Hi^2_\gamma;\mathbb{Z}) \cong S_{\mathbb{Z}}(v_1, v_3)/(v_1^2, v_3(v_1 - v_3))$, as

$$I = (v_1v_2, v_3v_4) \text{ and } J = (v_1+v_2, -v_1+v_3+v_4)\,.$$

3. COMPLEX ORIENTATIONS

Now suppose that $E^*(-)$ is *complex oriented*, so that $E^*(\mathbb{C}P^\infty) \cong E_*[[v^E]]$; where v^E is the 1st E-theory Chern class of η.

Then a similar analysis works for $E^*(M)$

Theorem 5.2. *As graded E_*-algebras, $E^*(M)$ is isomorphic to*

$$E_*[[v_1^E, \dots, v_m^E]]/(I^E + J^E)\,,$$

where I^E and J^E are generated by the non-faces of P and the Chern classes

$$v^E\Big(\phi_1^{\lambda_{i,1}} \otimes \cdots \otimes \phi_m^{\lambda_{i,m}}\Big)$$

respectively, for $1 \leq j \leq n$.

The latter relations follow from (4), and involve the *formal group law* for E.

For example, let E be complex K-theory; then $K_* \cong \mathbb{Z}[z, z^{-1}]$, and zv^K is represented by the virtual bundle $\eta - \mathbb{C}$ in $K^0(\mathbb{C}P^\infty)$.

So for $\mathbb{C}P^2$, (4) gives

$$v^K(\eta_1 \otimes \eta_3) = v_1^K + v_3^K + z v_1^K v_3^K \,,$$

and similarly for $\eta_2 \otimes \eta_3$. Hence

$$\begin{aligned} K^*(\mathbb{C}P^2) &\cong K_*[[v_1^K, v_2^K, v_3^K]]/(v_1^K v_2^K v_3^K, J^K) \\ &\cong K_*[[v_1^E]]/((v_1^K)^3)\,. \end{aligned}$$

For Hi_γ^2, similar calculations show that

$$K^*(Hi_\gamma^2) \cong K_*[[v_1^K, v_3^K]]/((v_1^K)^2, v_3^K(v_1^K - v_3^K))\,.$$