MALAGA SPRING SCHOOL

MAY 2010

TORIC TOPOLOGY

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LECTURE 4

QUASITORIC MANIFOLDS

OVERVIEW

1. DERIVED FORM

2. ALTERNATIVE VIEWPOINTS

3. DICHARACTERISTICS

4. TORIC VARIETIES

1. DERIVED FORM

Our two examples $\mathbb{C}P^2$ and $B_2 := Hi^2_\gamma$ of *quasitoric manifolds* belong to families

$$\{\mathbb{C}P^n : n \geq 1\} \quad \text{and} \quad \{B_n : n \geq 1\}\,.$$

The B_n are *Bott-Samelson manifolds* [BS58], which arise in *Bott towers* [GK94], [CR05].

For any *simple* n–polytope P, we recall the category $\mathrm{CAT}(K_P)$, whose objects $\sigma = \{F_{i_1}, \ldots, F_{i_k}\}$ have *rank* k, for $0 \leq k \leq n$.

Then we take a rank-preserving functor

$$\Lambda\colon \mathrm{CAT}(K_P) \longrightarrow \mathrm{FST}(T^n)\,;$$

where $\mathrm{FST}(T^n)$ is the category of *framed subtori* $T^k < T^n$. Each circle $\Lambda(F_j)$ is framed by a primitive $\Lambda_j \in \mathbb{Z}^n$, and each k–torus $\Lambda(\sigma)$ is framed by the $n \times k$ matrix

$$\Lambda_\sigma \ := \ [\Lambda_{i_1}, \ldots, \Lambda_{i_k}]\,;$$

we insist that $\Lambda_\sigma(\mathbb{Z}^k) < \mathbb{Z}^n$ be a *summand*. $(\star)$

1.1. Examples (see 2.4.2, 2.4.3)

If $P = \Delta^2$, then a suitable Λ is given by

0	1	2
Δ^2	F_1 F_2 F_3	F_1F_2 F_1F_3 F_2F_3

$\longrightarrow$

0	1	2
$\{1\}$	$T^{(1,0)}$ $T^{(0,1)}$ $T^{(1,1)}$	T^2

;

Condition $(\star)$ applies to the matrix

$$\Lambda \quad = \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

If $P = I^2$, then a suitable Λ is given by

0	1	2
I^2	F_1 F_2 F_3 F_4	F_1F_3 F_2F_4 F_3F_2 F_4F_1

$\longrightarrow$

0	1	2
$\{1\}$	$T^{(1,-1)}$ $T^{(1,0)}$ $T^{(0,1)}$ $T^{(0,1)}$	T^2

;

Condition $(\star)$ applies (with care!) to the matrix

$$\Lambda \quad = \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

Every $p \in P$ determines an object

$$\sigma(p) \ := \{F_{i_1}, \ldots, F_{i_k}\}$$

of $\mathrm{CAT}(K_P)$, by choosing $F(p) = F_{i_1} \cap \cdots \cap F_{i_k}$ to be the smallest face in which it lies. So Λ gives $\lambda\colon P \to \mathcal{C}(T^n)$ by $\lambda(p) = \Lambda(\sigma(p)) < T^n$.

We define the *quasitoric manifold*

$$M(P, \Lambda) \ := \ (T^n \times P)/ \sim$$

as the derived space $D(\lambda)$; thus $(t, p) \sim (u, p)$ exactly when $t^{-1}u \in \Lambda(\sigma(p))$.

The section and projection are

$$s(p) = [1, p] \quad \text{and} \quad q([t, p]) = p\,;$$

the isotropy subgroups are $T_{[t,p]} = \Lambda(\sigma(p))$.

So Examples 1.1 give $\mathbb{C}P^2$ and Hi^2_γ ... and may be generalised combinatorially!

2. ALTERNATIVE VIEWPOINTS

It follows from the properties of Λ that the fixed points of T^n on $M(P,\Lambda)$ are the points $[t, x_i]$, where P has vertices x_i, for $1 \leq i \leq r$.

Now embed P in $\mathbb{R}^n$ as a convex manifold with corners. Since P is *simple*, it admits an open cover $\{U_i : 1 \leq i \leq r\}$ for which

- each U_i contains the vertex x_i
- there are diffeomorphisms $\phi_i \colon \mathbb{R}^n_{\geqslant} \longrightarrow U_i$.

So we may pull $M(P,\Lambda)$ back to get

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\Phi_i} & M(P,\Lambda) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{R}^n_{\geqslant} & \xrightarrow{\phi_i} & U_i \subset P \end{array} \tag{1}$$

for $1 \leq i \leq r$, giving homeomorphisms Φ_i.

The Φ_i are *weakly equivariant*, insofar as Λ defines $f_i \in \mathrm{Aut}(T^n)$ for $1 \leq i \leq r$, such that

$$\Phi_i(t \cdot z) \;=\; f_i(t) \cdot \Phi_i(z) \qquad (2)$$

for every $t \in T^n$ and $z \in \mathbb{C}^n$.

So $M(P, \Lambda)$ actually admits

- local isomorphisms with $(\mathbb{C}^n, T^n)$, as in (2)
- a projection q onto P, with fibres the orbits.

These conditions are usually used to *define* quasitoric manifolds, and motivate the phrase

"well-behaved torus action".

Thirdly, consider the moment angle complex

$$\mathcal{Z}_P \;=\; T^V \times P/\sim,$$

where $V = \{F_1, \ldots, F_m\}$ and $(t, p) \sim (u, p)$ if

- $\sigma(p) = \{F_{i_1}, \ldots, F_{i_k}\}$, and
- $t_i = u_i$ unless $i = i_j$ for some $1 \leq j \leq m$.

By Condition ($\star$), the short exact sequence

$$K(\Lambda) \longrightarrow T^V \xrightarrow{\Lambda} T^n$$

is split, and $K(\Lambda) \cong T^{m-n}$.

Lemma. *The kernel $K(\Lambda)$ acts freely on $\mathcal{Z}_P$, with orbit space $M(P,\Lambda)$.*

Proof. Assume $k \in K(\Lambda)$ fixes $[t,p] \in \mathcal{Z}_P$. So $(kt,p) \sim (t,p)$, and k must lie in $T^{\sigma(p)} < T^V$. But Λ is monic on $T^{\sigma(p)}$, by ($\star$); so $k = 1$.

The orbit space is $M(P,\Lambda)$ because $\Lambda(T^{\sigma(p)}) = T_{[1,p]}$, by construction. □

So we recover the quotient maps

$$\begin{array}{c} \mathcal{Z}_P \\ \Big\downarrow {\scriptstyle K(V)} \\ M(P,\Lambda) \\ \Big\downarrow {\scriptstyle T^n} \\ P \end{array}$$

of Lecture 1.

Examples 4.1.1 revisited:

For $\mathbb{C}P^2$, use $P = \Delta^2$ and $K_1 \longrightarrow T^3 \xrightarrow{\Lambda} T^2$ with $\Lambda = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $K_1 = \{(u, u, u^{-1})\}$.

For Hi^2_γ, use $P = I^2$ and $K_2 \longrightarrow T^4 \xrightarrow{\Lambda} T^2$ with $\Lambda = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}$ and $K_2 = \{(u, u^{-1}, v, uv^{-1})\}$.

Respectively, they give

$$\begin{array}{ccc} \mathcal{Z}_P = S^5 & & \mathcal{Z}_P = S^3 \times S^3 \\ \downarrow K_1 & & \downarrow K_2 \\ M(P,\Lambda) = \mathbb{C}P^2 & \text{and} & M(P,\Lambda) = Hi^2_\gamma \, . \\ \downarrow T^2 & & \downarrow T^2 \\ \Delta^2 & & I^2 \end{array}$$

3. DICHARACTERISTICS

We have constructed $M(P, \Lambda)$ in derived form.

But a well-behaved torus action (M, a) over P does *not* determine a unique Λ, just the isotropy subgroups, and hence $\lambda\colon P \to \mathcal{C}(T^n)$.

So Λ arises by framing the isotropy circles $T_{y_i} < T^n$ (where $q(y_i) \in \mathit{Int}(F_i)$), and hence to orienting $X_i := q^{-1}(F_i) \subset M$, for $1 \leq i \leq m$. This is tantamount to *directing* each T_{y_i}.

Nevertheless, we may express any (M, a) in derived form, because

Lemma. *(M, a) admits a section $s\colon P \to M$.*

Proof. *Int*(P) is convex, and hence contractible. So there is a homeomorphism

$$h\colon q^{-1}(\mathit{Int}(P)) \longrightarrow T^n \times \mathit{Int}(P)\,,$$

and hence a restricted section $s(p) = h(1, p)$. Now extend over P, with care! $\square$

4. TORIC VARIETIES

Certain non-singular toric varieties provide good examples of quasitoric manifolds.

Let $P' \subset \mathbb{R}^n$ be a simple polytope, with

- facets $F'_1, \ldots, F'_m$, and
- inward pointing normals w'_1, ..., w'_m.

Perturbing the facets gives $F_1, \ldots, F_m$, which bound a simple polytope P and have *primitive integral* normals w_1, ..., w_m.

Then $\text{CAT}(P) \cong \text{CAT}(P')$, so P and P' are *combinatorially equivalent*; and $[w_1 \ \ldots \ w_m]$ is an integral $n \times m$ matrix $N(P)$.

For *unsupportive* P (such as the polars of certain cyclic polytopes), $N(P)$ cannot satisfy Condition ($\star$) [DJ91]. Otherwise, $N(P)$ defines a quasitoric manifold $M(P, N(P))$.

Such an $M(P, N(P))$ is a *non-singular projective toric variety*. These include $\mathbb{C}P^2$ and Hi^2_γ, so long as careful choices are made.

For $\mathbb{C}P^2$, choose Δ^2 with vertices $(0,1)$, $(1,0)$ and $(0,0)$, and

$$N(\Delta^2) \;=\; \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix};$$

then $K_1 = \{(u,u,u)\}$.

For Hi^2_γ, choose $P = I^2$ to be the quadrilateral Q with vertices $(1,0)$, $(2,1)$, $(0,1)$ and $(0,0)$, and

$$N(Q) \;=\; \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix};$$

then $K_2 = \{(u,u,v,uv)\}$.

These $N(P)$ differ from the Λ of Examples 4.1.1 only by the signs of certain columns. The quasitoric manifolds are the same.