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TORIC TOPOLOGY

Nigel Ray

nigel.ray@manchester.ac.uk

LECTURE 2
ABELIAN GROUP ACTIONS

The problem then is to develop the properties remaining invariant under the transformations of the principal group

Felix Klein (1872)

OVERVIEW

1. ACTIONS ON SETS

2. SECTIONS AND DERIVED SETS

3. IMPORTING THE TOPOLOGY

4. EXAMPLES

1. ACTIONS ON SETS

G is a finitely generated abelian group, and $\mathcal{S}(G)$ its partially ordered set (or *poset*) of subgroups, ordered by inclusion $\leq$.

We consider *lattices* $\mathbb{Z}^n$, and mod 2 vector spaces $\mathbb{Z}^n/2$. So $\mathcal{S}(\mathbb{Z}^n)$ and $\mathcal{S}(\mathbb{Z}^n/2)$ are large! In particular,

$$H \in \mathcal{S}(\mathbb{Z}^n) \implies H \cong \mathbb{Z}^k$$

and

$$H \in \mathcal{S}(\mathbb{Z}^n/2) \implies H \cong \mathbb{Z}^k/2$$

for some $k \leq n$. So the *rank* of H is defined in both cases.

Often we embed $\mathbb{Z} < \mathbb{R}$, and $\mathbb{Z}/2 < \mathbb{C}$ as the cyclic group $C_2 = \{\pm 1\}$.

A *G-set* (X, a) is a set X with an action

$$a\colon G \times X \longrightarrow X$$

that obeys standard rules; write $a(g, x)$ as $g \cdot x$.

For each $x \in X$, its *isotropy subgroup* is

$$G_x \;:=\; \{g \in G : g \cdot x = x\} \;\leq\; G\,,$$

and its *orbit* is

$$Gx \;:=\; \{g \cdot x : g \in G\} \;\subseteq\; X\,.$$

So there is a bijection $G/G_x \longleftrightarrow Gx$, induced by $g \mapsto g \cdot x$.

For example, $\mathbb{Z}^n$ is a C_2^n-set with respect to *coordinatewise multiplication*

$$(s_1, \ldots, s_n) \cdot (k_1, \ldots, k_n) = (s_1 k_1, \ldots, s_n k_n) \quad (1)$$

where $s_j = \pm 1$.

So $0 \in \mathbb{Z}^n$ is a *fixed point*; moreover

$$(C_2^n)_{(0,\ldots,0,k_{j+1},\ldots,k_n)} \;\cong\; C_2^j \;\text{ for }\; 0 \leq j \leq n\,,$$

and $C_2^{n-j} \longleftrightarrow C_2^n(0, \ldots, 0, k_{j+1}, \ldots, k_n)$.

2. SECTIONS

The orbits of a define a partition of X, whose blocks are the elements of the *orbit set*

$$X/G \; := \; \{Gx : x \in X\}\,.$$

The quotient function $q \colon X \to X/G$ is specified by $q(x) = Gx$.

A *section* $s \colon X/G \to X$ chooses a representative $s(Gx)$ for each orbit, and is therefore a right inverse for q; so $q \cdot s = 1_{X/G}$.

From (1), $\mathbb{Z}^n/C_2^n$ may be identified with

$$\mathbb{Z}^n_{\geq} \; := \; \{(l_1, \ldots, l_n) : l_j \geq 0 \text{ for } 1 \leq j \leq n\}\,.$$

Also $q(k_1, \ldots, k_n) = (|k_1|, \ldots, |k_n|)$, and the natural inclusion $i \colon \mathbb{Z}^n_{\geq} < \mathbb{Z}^n$ is a section.

So $(C_2^n)_{(l_1,\ldots,l_n)} = C(l) < C_2^n$, where

$$C(l) \; := \; \{(s_1, \ldots, s_n) : s_j = 1 \text{ if } l_j \neq 0\}\,.$$

If (X, a) has a section, we can rebuild it from its *characteristic function*

$$\lambda\colon X/G \to \mathcal{S}(G)\,,$$

defined by $\lambda(Gx) := G_x$

To do this, we first consider any $\lambda\colon Q \to \mathcal{S}(G)$, for some set Q. Its *derived set* is

$$D(\lambda) \;=\; (G \times Q)/\!\sim\,, \tag{2}$$

where $(g, q) \sim (h, q)$ whenever $g^{-1}h \in \lambda(q)$.

Then $D(\lambda)$ is a G-set with respect to $g \cdot [h, q] = [gh, q]$, and the isotropy subgroups are $G_{[h,q]} = \lambda(q)$ for any $h \in G$ and $q \in Q$.

The orbits are the subsets $\{[h, q] : h \in G\}$, and projection $D(\lambda) \to Q$ is the quotient map; a canonical section is $s(q) = [1, q]$ for any $q \in Q$.

So example (1) yields $\lambda((l_1,\ldots,l_n)) = C(l)$, and therefore has derived set

$$D(\lambda) \;=\; (C_2^n \times \mathbb{Z}_{\geq}^n)/\sim\,, \tag{3}$$

where $(r,l) \sim (s,l)$ whenever $r_j = s_j$ for all j such that $l_j \neq 0$.

Write any $(k_1,\ldots,k_n) \in \mathbb{Z}^n$ as

$$(s_1,\ldots,s_n)\cdot(l_1,\ldots,l_n)\,,$$

where $s_j = \pm 1$, and $l_j = |k_j|$; for instance, $(0,-1,2,-3) = (\pm 1,-1,1,-1)\cdot(0,1,2,3)$.

This decomposition gives a bijection

$$\mathbb{Z}^n \;\longleftrightarrow\; (C_2^n \times \mathbb{Z}_{\geq}^n)/\sim\,, \tag{4}$$

which recovers the C_2^n-set $\mathbb{Z}^n$ from

- its orbit space $\mathbb{Z}_{\geq}^n$, and
- the characteristic function

$$\lambda\colon \mathbb{Z}_{\geq}^n \longrightarrow \mathcal{S}(C_2^n)\,.$$

The constructions $(X/G, \lambda)$ and $D(\lambda)$ are mutually inverse, and set up a correspondence between characteristic functions and G-sets with section. In particular, any section s for (X, a) leads to a G-equivariant bijection

$$f_s \colon D(\lambda) \longrightarrow X\,, \tag{5}$$

via $f_s[g, Gx] = g \cdot s(Gx)$.

Then $D(\lambda)$ is the *derived form* of (X, a).

By construction, $D(\lambda)$ is universal for G-sets (X, a_X) that admit a function $X \to Q$ which is constant on orbits, and a section $s_X \colon Q \to X$ with $G_{s_X(q)} = \lambda(q)$ for every $q \in Q$.

3. IMPORTING THE TOPOLOGY

Suppose that a is the continuous action of a compact torus $T = T^n$ on a space X. Every isotropy subgroup $T_x < T$ is closed, so the *characteristic map* $\lambda\colon X/T \to \mathcal{C}(T)$ takes values in the poset of closed subgroups.

The topology ensures that closures satisfy

$$\overline{\lambda^{-1}(U)} = \bigcup_{V \geq U} \lambda^{-1}(V) \qquad (6)$$

Alternatively, suppose given some map $\lambda\colon Q \to \mathcal{C}(T)$. Then the derived set $D(\lambda)$ of (2) acquires a topology for which projection onto Q is a quotient map, and the T-action and canonical section are continuous.

Characteristic maps correspond to T-spaces with section; $D(\lambda)$ is the *derived space*.

- Homotopy theory also plays a rôle.

If X/T is contractible, then the action often has a section s, and the correspondence gives a T-equivariant homeomorphism

$$f_s\colon (T \times X/T)/\sim \longrightarrow X\,. \tag{7}$$

Sometimes the action a is *free*, meaning that

$$t \cdot x = x \implies t = 1\,.$$

In this case, $T_x = \{1\}$ for every $x \in X$, and the derived form is simply

$$D(\lambda) \;=\; T \times X/T\,.$$

Up to homotopy equivalence, X may be replaced by $ET \times X$, on which T acts freely by $t \cdot (e, x) = (t \cdot e, t \cdot x)$. Then

$$q\colon ET \times X \;\longrightarrow\; ET \times_T X$$

is a T-bundle over the *Borel construction* (or *homotopy quotient*); this is homotopy equivalent to X/T *if a is free*.

4. EXAMPLES

4.1. The non-negative orthant

is obtained from (4) by extending $\mathbb{Z}$ to $\mathbb{C}$, and C_2 to T^1. So T^n acts on $\mathbb{C}^n$ coordinatewise, and $(T^n)_{(z_1,\dots,z_n)} = T(z) < T^n$, where

$$T(z) \;:=\; \{(t_1,\dots,t_n) : t_j = 1 \text{ if } z_j \neq 0\}\,.$$

The orbit space is the *non-negative orthant*

$$\mathbb{R}^n_{\geq} \;:=\; \{(r_1,\dots,r_n) : r_j \geq 0 \text{ for } 1 \leq j \leq n\}\,.$$

Also $q(z_1,\dots,z_n) = (|z_1|,\dots,|z_n|)$, and the natural inclusion $\mathbb{R}^n_{\geq} < \mathbb{C}^n$ is a section.

So we recover $\mathbb{C}^n$ from the homeomorphism

$$\mathbb{C}^n \;\longleftrightarrow\; (T^n \times \mathbb{R}^n_{\geq})/\!\sim \;=\; D(\lambda)\,, \qquad (8)$$

which is T^n-equivariant; it is given by the natural section, and the characteristic map $\lambda(T^n z) = T(z)$ for any z in $\mathbb{C}^n$.

The associated decomposition of z is

$$(z_1,\dots,z_n) = (e^{i\theta_1},\dots,e^{1\theta_n}) \cdot (|z_1|,\dots,|z_n|)\,.$$

4.2. The projective plane

$T^3/T_\delta \cong T^2$ acts on $\mathbb{C}P^2$ by

$$(u_1, u_2) \cdot [z_1, z_2, z_3] \;=\; [u_1 z_1, u_2 z_2, z_3]\,.$$

The isotropy subcircles are given by

$$T^2_{[0,z_2,z_3]} = T^{(1,0)}, \quad T^2_{[z_1,0,z_3]} = T^{(0,1)},$$
$$\text{and} \quad T^2_{[z_1,z_2,0]} = T^{(1,1)} = T_\delta \qquad (9)$$

in terms of tangents to T^2 at 1; furthermore, $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$ are fixed.

The orbit space is the *curvilinear simplex* Δ^2 in $\mathbb{R}^3_\geqslant$; also $q([z_1, z_2, z_3]) = (|z_1|, |z_2|, |z_3|)$, and the section is $s(r_1, r_2, r_3) = [r_1, r_2, r_3]$.

The T^2-equivariant homeomorphism

$$\mathbb{C}P^2 \;\longleftrightarrow\; (T^2 \times \Delta^2)/\!\sim \;=\; D(\lambda)$$

requires s, where λ is defined by (9). The associated decomposition of $[z]$ is

$$[z_1, z_2, z_3] = (e^{i\theta_1}, e^{i\theta_2}) \cdot [|z_1|, |z_2|, |z_3|]\,.$$

4.3. A Hirzebruch surface

$T^2 \cong T^4/T^2_\gamma$ acts on Hi^2_γ by

$$(u_1, u_2) \cdot [z_1, z_2; z_3, z_4] \; = \; [z_1, u_1 z_2; z_3, u_2 z_4] \, .$$

The isotropy subcircles are given by

$$T^2_{[0,z_2;z_3,z_4]} = T^{(1,-1)}, \quad T^2_{[z_1,0;z_3,z_4]} = T^{(1,0)},$$

$$T^2_{[z_1,z_2;0,z_4]} = T^{(0,1)}, \;\; T^2_{[z_1,z_2;z_3,0]} = T^{(0,1)} \quad (10)$$

in terms of tangents to T^2 at 1; $[1,0;1,0]$, $[1,0;0,1]$, $[0,1;1,0]$, and $[0,1;0,1]$ are fixed.

The orbit space is the *curvilinear square* I^2 in $\mathbb{R}^4_\geqslant$; also $q([z_1, z_2; z_3, z_4]) = (|z_1|, |z_2|; |z_3|, |z_4|)$, with section $s(r_1, r_2; r_3, r_4) = [r_1, r_2; r_3, r_4]$.

The T^2 equivariant homeomorphism

$$Hi^2_\gamma \;\longleftrightarrow\; (T^2 \times I^2)/\!\sim \; = \; D(\lambda) \, , \qquad (10)$$

requires s, where λ is defined by (10). The associated decomposition of $[z]$ is

$$[z_1, z_2; z_3, z_4] = (e^{i\theta_1}, e^{i\theta_2}) \cdot [|z_1|, |z_2|; |z_3|, |z_4|] \, .$$