MALAGA SPRING SCHOOL
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TORIC TOPOLOGY

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LECTURE 1: EXAMPLES

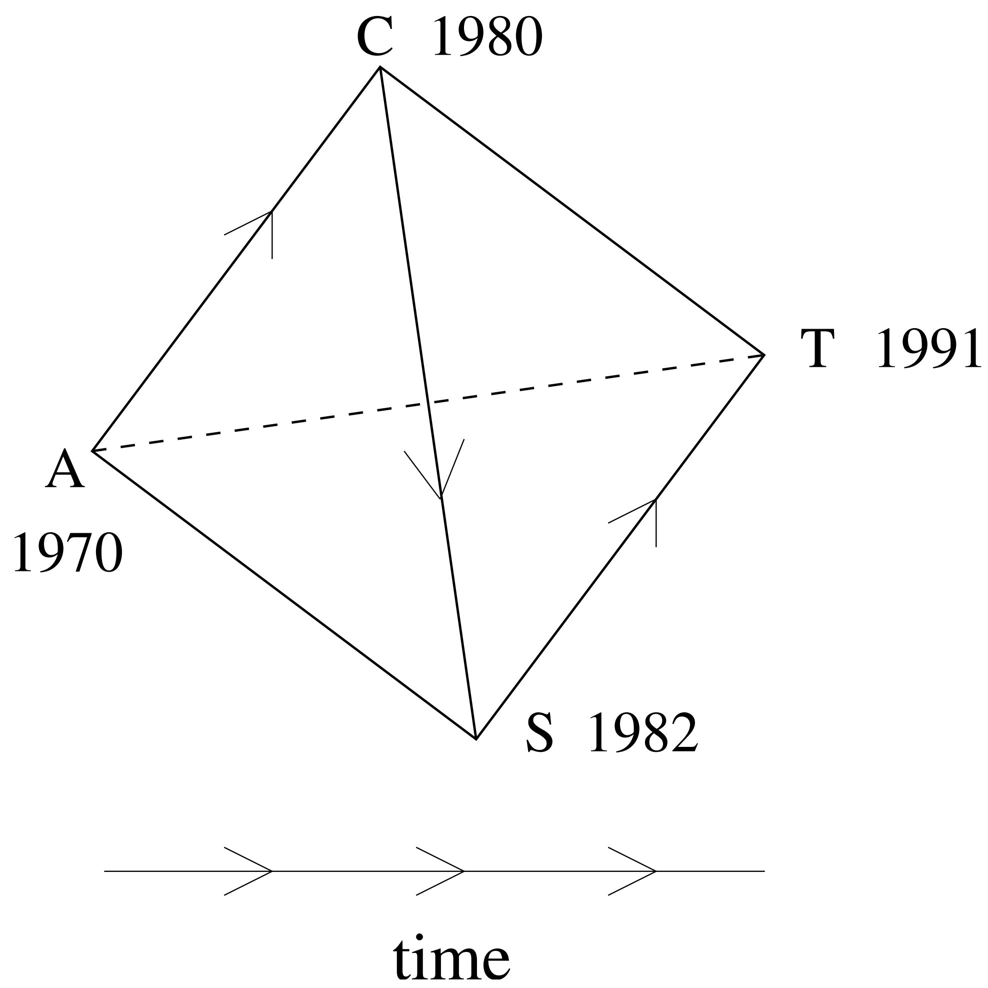
Toric varieties provide a quite different yet elementary way to see many examples and phenomena in algebraic geometry.

William Fulton [FU93]

OVERVIEW

1. THE TORIC TETRAHEDRON
2. THE NON-NEGATIVE ORTHANT
3. THE PROJECTIVE PLANE
4. A HIRZEBRUCH SURFACE
5. A SINGULAR EXAMPLE

1. THE TORIC TETRAHEDRON



1. THE NON-NEGATIVE ORTHANT

$$\begin{array}{c} \mathbb{C}^n \\ \Big\downarrow T^n \\ \mathbb{R}^n_{\geqslant} \end{array}$$

This is the most important example of all!

T^n acts on $\mathbb{C}^n$ *coordinatewise*, via

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n)$$

The orbit space records only the *moduli*

$$(x_1, \ldots, x_n), \text{ where } x_j \in \mathbb{R} \text{ and } x_j \geq 0.$$

The orbit $T^n z$ of any $z \in \mathbb{C}^3$ is controlled precisely by the *number of zeros* in x ...
... and there is a *section*!

2. THE PROJECTIVE PLANE

$$
\begin{array}{c}
\mathbb{C}^3 \\
\big\downarrow T^3 \\
\mathbb{R}^3_{\geqslant}
\end{array}
\qquad\qquad
\begin{array}{c}
S^5 \\
\big\downarrow T_\delta \\
\mathbb{C}P^2 \\
\big\downarrow T^2 \\
\Delta^2
\end{array}
$$

- S^5 is the unit sphere in $\mathbb{C}^3$
- $\Delta^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ in $\mathbb{R}^3_{\geqslant}$;

these describe the action of T^3 on S^5.

- $T_\delta < T^3$ is the diagonal circle
- $\mathbb{C}P^2 = \{[z_1, z_2, z_3] : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$
- T^2 is T^3/T_δ.

Note that T_δ acts *freely* on S^5.

- There is a *real part*

$$\begin{array}{ccc} & & S^2 \\ \mathbb{R}^3 & & \downarrow (C_2)_\delta \\ \downarrow \mathbb{C}_2^3 & & \mathbb{R}P^2 \\ \mathbb{R}^3_{\geqslant} & & \downarrow C_2^2 \\ & & \Delta^2 \end{array}$$

which is fixed by the involution $z \mapsto \overline{z}$; so $C_2 < T$ is the subgroup $\{\pm 1\}$.

- ... and a *section* $s\colon \Delta^2 \xrightarrow{\subseteq} S^2$:

$$\begin{array}{ccc} \Delta^2 & \xrightarrow[=]{} & \Delta^2 \\ s \downarrow & & \downarrow s \\ S^2 & \xrightarrow{\subseteq} & S^5 \\ (C_2)_\delta \downarrow & & \downarrow T \\ \mathbb{R}P^2 & \xrightarrow{\subseteq} & \mathbb{C}P^2 \end{array}$$

The tangent bundle of $\mathbb{C}P^2$ satisfies

$$\tau(\mathbb{C}P^2) \oplus \mathbb{C} \;\cong\; \eta \oplus \eta \oplus \eta \, , \qquad (1)$$

where η is the conjugate *Hopf bundle*, whose total space consists of pairs

(L, z), where $L < \mathbb{C}^3$ is a line, and $z \in L$.

(1) describes the canonical *stably complex structure* on $\mathbb{C}P^2$; *there are others!*

The integral cohomology of $\mathbb{C}P^2$ satisfies

$$H^*(\mathbb{C}P^2; \mathbb{Z}) \;\cong\; \mathbb{Z}[x]/(x^3) \, ,$$

where $x = c_1(\eta)$ is the first Chern class of η.

3. A HIRZEBRUCH SURFACE

$$\begin{array}{ccc} & & S^3 \times S^3 \\ \mathbb{C}^4 & & \big\downarrow T^2_\gamma \\ \big\downarrow T^4 & & Hi^2_\gamma \\ \mathbb{R}^4_\geqslant & & \big\downarrow T^2 \\ & & \Delta^1 \times \Delta^1 \end{array}$$

where $S^3 \times S^3$ lies in $\mathbb{C}^2$, and

$$T^2_\gamma \;=\; \{(t_1, t_1; t_3, t_1^{-1} t_3) : t_1, t_3 \in T\} \;<\; T^4$$

acts *freely*. So $T^2 = T^4/T^2_\gamma$, and $\Delta^1 \times \Delta^1$ is

$$I^2 \;=\; \{(x_1, x_2; x_3, x_4) : x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1\},$$

which is a curvilinear square in $\mathbb{R}^4_\geqslant$.

$$Hi^2_\gamma = \{(L, L') : L < \mathbb{C}^2,\, L' < L \oplus \mathbb{C}\} = \mathbb{C}P(\eta \oplus \mathbb{C}),$$

where $L = \langle z_1, z_2 \rangle$ and $L' = \langle z_4 z_1, z_4 z_2, z_3 \rangle$.

- There is a *real part*

$$
\begin{array}{ccc}
 & & S^1 \times S^1 \\
 & & \downarrow (C_2^2)_\gamma \\
\mathbb{R}^4 & & \\
\downarrow \mathbb{C}_2^4 & & \mathbb{R}Hi_\gamma^2 \\
\mathbb{R}^4_\geqslant & & \downarrow C_2^2 \\
 & & I^2
\end{array}
$$

- ... and a *section* $s\colon I^2 \xrightarrow{\subseteq} S^1 \times S^1$:

$$
\begin{array}{ccc}
I^2 & \underset{=}{\longrightarrow} & I^2 \\
s \downarrow & & \downarrow s \\
S^1 \times S^1 & \xrightarrow{\subseteq} & S^3 \times S^3 \\
(C_2^2)_\gamma \downarrow & & \downarrow T_\gamma^2 \\
\mathbb{R}Hi_\gamma^2 & \xrightarrow{\subseteq} & Hi_\gamma^2
\end{array}
$$

The tangent bundle of Hi^2_γ satisfies

$$\tau(Hi^2_\gamma) \oplus \mathbb{C}^2 \;\cong\; \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_4 \;, \qquad (2)$$

where the ρ_j are certain complex line bundles.

(2) describes one of *two* canonical stably complex structure on Hi^2_γ; and *there are others!*

The integral cohomology of Hi^2_γ satisfies

$$H^*(Hi^2_\gamma;\mathbb{Z}) \;\cong\; \mathbb{Z}[x_1,x_2]/(x_1^2, x_2^2 - x_1x_2) \;,$$

where $x_1 = c_1(\rho_1)$ and $x_2 = c_1(\rho_2)$.

4. A SINGULAR EXAMPLE

Consider the *weighted circle*

$$T\langle 1,1,3\rangle \;=\; \{(t,t,t^3) : t \in T\} \;<\; T^3.$$

$$\begin{array}{ccc} & & S^5 \\ \mathbb{C}^3 & & \big\downarrow T\langle 1,1,3\rangle \\ \big\downarrow T^3 & & \mathbb{P}(1,1,3) \\ \mathbb{R}^3_{\geqslant} & & \big\downarrow T^2 \\ & & \Delta^2 \end{array}$$

- $\mathbb{P}(1,1,3)$ is a *weighted projective space*

- T^2 is $T^3/T\langle 1,1,3\rangle$

Note that $T\langle 1,1,3\rangle$ does acts *not* act freely.

- There is a commutative diagram

$$\begin{array}{ccc} S^5 & \xrightarrow{\phi'} & S^5 \\ {\scriptstyle T_\delta}\downarrow & & \downarrow{\scriptstyle T\langle 1,1,3\rangle} \\ \mathbb{C}P^2 & \xrightarrow{\phi} & \mathbb{P}(1,1,3) \end{array}$$

where $\phi'(z_1, z_2, z_3) = (z_1, z_2, z_3^3)/|-|$.

- $\mathbb{P}(1,1,3) \cong \mathbb{C}P^2/C_3$
- $\mathbb{P}(1,1,3)$ has a real part and a section ...
- ... but no tangent bundle!

For more details, see [KA73] and [BFR09].