QUASITORIC MANIFOLDS AND COBORDISM THEORY

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Our aim is to investigate quasitoric manifolds, and their quaternionic analogues, in the setting of cobordism theory.

A quasitoric manifold is said to be reducible if it can be viewed as the total space of an equivariant bundle with quasitoric fibre and quasitoric base space. Buchstaber, Panov and Ray have conjectured that any quasitoric SU-manifold is a boundary in the complex cobordism ring. We prove this conjecture for complex projective space $\mathbb{C}P^n$, and for reducible quasitoric manifolds with fibre $\mathbb{C}P^1$.

We introduce the notion of a quaternionic tower, as the quaternionic analogue of a certain family of quasitoric manifolds known as Dobrinskaya towers. We compute their F-cohomology ring for a quaternionic oriented ring spectrum F, and the properties of various subfamilies are investigated. Approaches to placing the towers within a quaternionic analogue of the theory of toric topology are considered.

We undertake a study of one particular subfamily of quaternionic tower in the quaternionic cobordism ring MSp_*. This allows us to construct a new set of manifolds, and by studying their possible stably quaternionic structures, we prove that they are simply-connected geometric representatives for an important class of torsion elements in MSp_*.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

Toric topology encompasses the study of topological spaces that admit well-behaved torus actions. The subject was initiated by the work of Davis and Januskiewicz [16], who introduced the notion of a quasitoric manifold as the topological analogue of a construction in algebraic geometry. In the fifteen years since their pioneering paper, the subject has rapidly flourished, and links between more established areas of topology have begun to be forged.

One of the first applications of toric topology was to the theory of complex cobordism, the study of which stretches far back into twentieth century mathematics; for instance, the coefficient ring MU_* of complex cobordism was first determined by Milnor [37] and Novikov [40] in the early sixties. Quasitoric manifolds M are amenable to the methods of complex cobordism because once M and certain of its submanifolds are oriented in a particular manner, M is furnished with a canonical stably complex structure. This fact allowed Buchstaber and Ray [9] to fabricate an alternative basis for MU_* using quasitoric manifolds. Their work led to a solution to the topological analogue of a long standing problem in algebraic geometry, first posed by Hirzebruch in 1958 (see e.g. [5, Section 5.3]).

Our thesis continues the study of quasitoric manifolds in complex cobordism theory. Buchstaber, Panov and Ray [7] have conjectured that every quasitoric manifold with an SU-structure is a boundary in the complex cobordism ring, and it is this problem which we investigate for complex projective space $\mathbb{C}P^n$, and for the class of reducible quasitoric manifolds introduced by Dobrinskaya [17]. A quasitoric manifold M^{2n} is said to be *reducible* if it can be viewed as the total space of an equivariant bundle, with quasitoric fibres and quasitoric base space.

In the second half of our thesis, we consider the effect of replacing the complex numbers by the quaternions in some of the constructions of toric topology. In particular, we study a quaternionic analogue of a family of quasitoric manifolds known as Dobrinskaya towers. The seemingly innocuous fact that the quaternions do not commute throws up many obstacles, but the quaternionic viewpoint does allow us to construct a collection of *quaternionic towers* with interesting applications in quaternionic cobordism theory.

As in the case of complex cobordism, the coefficient rings of the other classical cobordism theories $MG^*(-)$, which arise from the Thom spectra MG, for $G = O, SO, SU, Spin$, have long been understood (see e.g. [54]). In contrast, the coefficient ring MSp_* of quaternionic cobordism, remains something of an enigma.

In the past thirty years there have been several attempts to compute this ring. Gargantuan calculations with spectral sequences that converge to MSp_* were undertaken by Kochman [30] and Vershinin [58], while more recent efforts utilised the transfer map [4], and the theory of cobordism with singularities [59]. Though these approaches offered many new insights into the structure of MSp_*, a full description of the ring is still lacking.

However, it is clear that a collection of elements $\varphi_m \in MSp_{8m-3}$ defined by Ray in [42], play a crucial role in MSp_*. The φ_m are multiplicatively indecomposable elements of order 2, and they have been the starting point for all of the investigations described above.

Finding geometrical representatives for elements in MSp_* is a key problem. Once we have obtained such representatives, we would hope that we can gain further information about the cobordism ring, which would have remained hidden, or been harder to obtain by using purely algebraic methods.

Ray went on to realise the torsion elements φ_m geometrically in [47], with the aid of a special subfamily of quaternionic tower. He conjectured that it would be possible to create another collection of geometrical representatives by constructing $(4n+1)$-dimensional manifolds Y^{4n+1} that are closely related to the original family.

Armed with our knowledge of quaternionic towers, we undertake a thorough investigation into the quaternionic cobordism of Y^{4n+1}, and work towards a confirmation of Ray's claim that they are geometric representatives for the torsion elements φ_m, when $n = 2m - 1$.

There are several common themes and ideas, which tie our thesis together. The spaces we study are either quasitoric manifolds, or manifolds inspired by constructions in toric topology; whether quasitoric or not, manifolds constructed from sequences of iterated bundles recur throughout our work. Once we have obtained our manifolds, in all cases, our interest is in studying their algebraic topology using the tools of cobordism theory, both complex and quaternionic.

The contents of each chapter can be summarised as follows.

Chapter 2 introduces the language, notation and background material that we rely upon throughout our thesis. We introduce the various cohomology theories we will require, and the idea of a stable tangential structure is explored in detail. Since iterated bundle constructions and the Borel construction often feature in our work, we establish some of their general properties at this early stage.

The essential ideas of toric topology are detailed in Chapter 3, from the viewpoint recently given by Buchstaber, Panov and Ray [6]. The ordinary cohomology ring of a quasitoric manifold is computed, and we explore their stably complex structures. This is the foundation for the following two chapters.

Chapter 4 is concerned with the study of Dobrinskaya towers. They are introduced as families of quasitoric manifolds, and we utilise the results of the previous chapter to describe their cohomology rings and stably complex structures. We then construct the towers out of a sequence of iterated bundles, and show that the two constructions agree up to diffeomorphism. The special subfamilies of Bott towers and bounded flag manifolds are introduced.

In Chapter 5 we offer a proof of Buchstaber, Panov and Ray's conjecture for complex projective space $\mathbb{C}P^n$ and for the class of reducible quasitoric manifolds N^{2n} with fibre the complex projective line $\mathbb{C}P^1$. This latter class includes certain Dobrinskaya towers and in particular, all Bott towers. We begin with a rigorous definition of a reducible quasitoric manifold, and a review related results of Dobrinskaya. After establishing some cohomological properties of N^{2n} in terms of those of the base space of the reducible quasitoric manifold, we encode the possible choices of omniorientation on N^{2n} by attaching a collection of signs to its dicharacteristic matrix. We deduce a result on the Chern classes of the stable tangent bundle of N^{2n}, which leads to confirmation of the conjecture for N^{2n}. In the final section we study SU-structures on complex projective space, and prove the conjecture for $\mathbb{C}P^n$. We would like here to express our gratitude to Dmitry Leykin, with whom the concept of attaching signs to the dicharacteristic matrix was developed.

The process of transferring the ideas of toric topology to the realm of the quaternions gets underway in Chapter 6. We investigate the properties of quaternionic line bundles, which are then used to construct a quaternionic analogue of the Dobrinskaya tower as an iterated bundle construction. The F-cohomology ring for a quaternionic oriented ring spectrum F is computed. As in Chapter 4, we spend a little time investigating the subfamilies of quaternionic Bott tower and bounded quaternionic flag manifold. Finally, we attempt to realise our towers as the quaternionic analogue of a quasitoric manifold, and we survey other authors' approaches to such a construction.

Our thesis concludes with Chapter 7, in which a subfamily of quaternionic tower is studied in MSp_*. The torsion elements $\varphi_m \in MSp_{8m-3}$ are introduced and we explain how the quaternionic bounded flag manifold can be used to create a geometrical representative for φ_m. The related manifolds Y^{4n+1} are constructed and we study their geometry and F-cohomology rings. In the remainder of the chapter we follow the programme of Ray, Switzer and Taylor [48] to study the different stably quaternionic structures on Y^{4n+1}, and this allows us to prove Ray's conjecture that Y^{8m-3} is a simply connected geometrical representative for φ_m.

As the first three chapters contain background material that is the foundation for our thesis, there is little new here. We believe that some of the findings of Chapter 4 on Dobrinskaya towers have not appeared elsewhere, but they can be deduced easily from a combination of results in [12] and [10].

Chapter 5 seems to be the first attempt at a proof of Buchstaber, Panov and Ray's conjecture, since it was posited in [7]. Unless otherwise stated, we believe that the results of this chapter concerning reducible quasitoric manifolds are original. In particular the idea of incorporating signs into the dicharacteristic matrix, which leads to a proof of the conjecture for the particular family of manifolds, does not seem to have appeared elsewhere before. Our proof of the conjecture in the case of complex projective space is deduced from observations about the topology of $\mathbb{C}P^n$, which we believe are well-known, though we have not found explicit references for all of these facts in the literature. Nevertheless, we believe that our interpretation of these results in the context of toric topology, leading to our proof of the conjecture for $\mathbb{C}P^n$, is original.

We believe that our construction of a quaternionic analogue of a Dobrinskaya tower in Chapter 6, and calculations of its F-cohomology ring, are original. Results related to special subfamilies of the towers are also new, though we acknowledge that Ray [47] has previously worked with the subfamily of bounded quaternionic flag manifolds. The final section concerns our tentative steps towards a quaternionic analogue to the theory of toric topology. Though we believe that exactly such an approach as ours has not been tried previously, several authors have made very similar attempts, and so this section mixes old and new as we discuss and adapt their ideas.

Other than our concluding Chapter 7, we are not aware of any other studies of the manifolds Y^{4n+1}. The first section recounts some well-known background material from several sources. The rest of the chapter employs the aforementioned machinery of Ray, Switzer and Taylor, and so it definitely the case that our methods, and many of the proofs we offer, are often not in themselves original. However, we give the first serious treatment of the geometry and F-cohomology of Y^{4n+1}, and we make many new calculations related to stably quaternionic structures on these manifolds, which culminates in what we believe to be the first proof of Ray's conjecture.

Chapter 2

Notation and prerequisites

In this chapter we establish some of the fundamental concepts and notation that feature throughout our thesis. Some of our terminology is nonstandard, so we begin with a discussion of nomenclature. The next section describes the various cohomology theories that appear in our thesis. Following this, notation related to bundles is established, and we define certain structures on the tangent bundle of a manifold. A brief section follows, where we detail the Borel construction, illustrating the concept with a well-known example. Finally we explore the properties of iterated bundle constructions, which will feature prominently in later chapters.

2.1 Nomenclature

Throughout our thesis we work with a class of manifolds, which were described by Davis and Januskiewicz as *toric* [16]. We prefer to follow the more recent convention of using the phrase *quasitoric manifold*, since the original terminology conflicts with the class of smooth toric varieties in algebraic geometry.

In Chapter 7 we apply certain generalisations of quasitoric manifolds to $MSp^*(-)$, the cobordism theory that arises from the Thom spectrum MSp (see Example 2.2.9 below). We refer to $MSp^*(-)$ as the *quaternionic cobordism ring functor*, rather than *symplectic* cobordism, as it is classically known. We justify this as follows. We have that quaternionic cobordism theory arises from the Thom spectrum of the infinite symplectic group Sp, which mirrors the convention that complex cobordism theory arises from the infinite unitary group U. Though this is undeniably neat, there is a more compelling reason to overturn tradition: in modern parlance the phrase *symplectic manifold* refers to a manifold equipped with a symplectic form, and has little relation to the objects studied in $MSp^*(-)$. Therefore we feel it is sensible to refer to the study of *stably quaternionic* manifolds as quaternionic cobordism theory, thus avoiding any confusion with concepts arising from symplectic geometry.

2.2 Oriented cohomology theories

In this section we establish the basic properties of the various cohomology theories that we will encounter in subsequent chapters. A comprehensive reference for the material is the book of Adams [1], while characteristic classes in our cohomology theories are described by Connor and Floyd [14] and Switzer [56].

A *ring spectrum* E is a spectrum equipped with a smash product map $E \wedge E \to E$. We work exclusively with ring spectra that are commutative and associative up to homotopy, and are equipped with a map $S^0 \to E$ that acts as a unit up to homotopy.

Associated to a ring spectrum E is a cohomology theory $E^*(-)$ and a homology theory $E_*(-)$ (see e.g. [56, 8.33]); we denote the coefficient ring $E^*(pt) \cong E_*(pt)$ by E_*. The reduced and unreduced cohomology rings of a space X will be written as $E^*(X)$ and $E^*(X_+)$ respectively.

Definition 2.2.1. The cohomology theory $E^*(-)$ is *complex oriented* if there exists an element $x^E \in E^2(\mathbb{C}P^\infty)$, such that $E^*(\mathbb{C}P^1)$ is a free E_*-module generated by $i^*x^E \in E^2(\mathbb{C}P^1)$, where i is the inclusion map $\mathbb{C}P^1 \to \mathbb{C}P^\infty$. Furthermore, in such circumstances we say that the ring spectrum E is *complex oriented*.

We will take ζ_n to be the universal $U(n)$-bundle over $BU(n)$, the classifying space of the nth unitary group $U(n)$. The direct limit of the natural inclusions $BU(n) \to BU(n+1)$ is given by the union $BU = \bigcup_{n\geq 1} BU(n)$, which forms the classifying space of the infinite unitary group U. Then we let ζ denote the virtual bundle over

BU given by the limit of the virtual bundles $\zeta_n - \mathbb{C}^n$ over $BU(n)$. By construction, the restriction of ζ to any $BU(n)$ is $\zeta_n - \mathbb{C}^n$.

In the case when $n = 1$, the universal line bundle ζ_1 is simply the canonical complex line bundle over $BU(1) \cong \mathbb{C}P^\infty$, whose fibre over each complex line in $\mathbb{C}P^\infty$ is the line itself. We will also write ζ_1 for the restriction of this bundle to any n-dimensional complex projective space $\mathbb{C}P^n$.

Example 2.2.2. When E is the Eilenberg-Mac Lane spectrum H associated with a ring R, we obtain ordinary cohomology $H^*(-;R)$ and homology $H_*(-,R)$. A complex orientation can be given by taking x^H to be the usual generator in $H^2(\mathbb{C}P^\infty)$.

We will often abbreviate the integral cohomology ring $H^*(X;\mathbb{Z})$ to $H^*(X)$.

Example 2.2.3. Complex K-theory is a complex oriented cohomology theory with coefficient ring K_* isomorphic to $\mathbb{Z}[z, z^{-1}]$, where $z \in K_2$ is represented by the virtual bundle $\zeta_1 - \mathbb{C}$ over the 2-sphere $S^2 \cong \mathbb{C}P^1$ (see e.g. [56, 13.92]). The complex orientation x^K in $K^2(\mathbb{C}P^\infty)$ is represented by the product of z^{-1} with $\zeta_1 - \mathbb{C} \in K^0(\mathbb{C}P^\infty)$.

Example 2.2.4. The Thom spectrum MU arising from the infinite unitary group U gives rise to the cohomology theory of complex cobordism $MU^*(-)$, and the corresponding homology theory, complex bordism $MU_*(-)$. We take x^U to be the complex cobordism class representing the well-known homotopy equivalence $\mathbb{C}P^\infty \to MU(1)$, thus determining a complex orientation for $MU^*(-)$.

Complex cobordism is the *universal* complex oriented cohomology theory in the following sense. For any complex oriented ring spectrum E, there is a one-to-one correspondence between complex orientations $x^E \in E^2(\mathbb{C}P^\infty)$ and maps of ring spectra $g^E : MU \to E$, such that $g^E_*(x^U) = x^E$.

We study complex cobordism in Chapter 5.

Any complex oriented ring spectrum E gives rise to Chern classes in $E^*(-)$ (see e.g. [56, (16.27)]). Given a $U(n)$-bundle θ over a space X, we will write $c_i^E(\theta) \in E^{2i}(X)$ for the *ith Chern class* of θ, for $1 \leq i \leq n$. The *total Chern class* $c^E(\theta)$ of θ is defined to be the sum $1 + c_1^E(\theta) + c_2^E(\theta) + \cdots + c_n^E(\theta)$ in $E^*(X)$. In the universal case of $\mathbb{C}P^\infty$ we will often denote $c_i^E(\zeta_1)$ simply by c_i^E in $E^{2i}(\mathbb{C}P^\infty)$. When there is no danger of confusion, we may also use the symbol c_i^E to denote the Chern class $c_i^E(\zeta)$ in $E^{2i}(BU)$. Furthermore, if the context is clear, we may drop the superscript E to simplify our notation.

In analogy with Definition 2.2.1 we have the following notion.

Example 2.2.5. The cohomology theory $F^*(-)$ is *quaternionic oriented* if there exists an element $y^F \in F^4(\mathbb{H}P^\infty)$, such that $F^*(\mathbb{H}P^1)$ is a free F_*-module generated by $i^*y^F \in F^4(\mathbb{H}P^1)$, where i is the inclusion map $\mathbb{H}P^1 \to \mathbb{H}P^\infty$. Furthermore, in such circumstances we say that the ring spectrum F is *quaternionic oriented.*

Denote by ξ_n the universal $Sp(n)$-bundle over $BSp(n)$, the classifying space of the nth symplectic group $Sp(n)$. The direct limit of the natural inclusions $BSp(n) \to BSp(n+1)$ is given by the union $BSp = \bigcup_{n\geq 1} BSp(n)$, and forms the classifying space of the infinite symplectic group Sp. Then let ξ denote the virtual bundle over BSp given by the limit of the virtual bundles $\xi_n - \mathbb{H}^n$ over $BSp(n)$. By construction, the restriction of ξ to any $BSp(n)$ is $\xi_n - \mathbb{H}^n$.

In the case when $n = 1$, the universal line bundle ξ_1 is simply the canonical quaternionic line bundle over $BSp(1) \cong \mathbb{H}P^\infty$, whose fibre over each quaternionic line in $\mathbb{H}P^\infty$ is the line itself. We will also write ξ_1 for the restriction of this bundle to any n-dimensional quaternionic projective space $\mathbb{H}P^n$.

Example 2.2.6. Ordinary cohomology $H^*(-;R)$ and homology $H_*(-,R)$ as described in Example 2.2.2 are quaternionic oriented by taking the usual generator in $H^4(\mathbb{H}P^\infty)$ as the quaternionic orientation y^H.

Example 2.2.7. Real K-theory $KO^*(-)$ is a quaternionic oriented cohomology theory that will be central to our calculations in Chapter 7, so we establish here some fundamental properties.

The coefficient ring KO_* is given by

$$\mathbb{Z}[\alpha, \beta, \gamma]/(2\alpha, \alpha^3, \alpha\beta, \beta^2 - 4\gamma), \tag{2.2.8}$$

where α, β and γ are the elements represented respectively by the canonical virtual line bundle over S^1, the virtual quaternionic line bundle $\xi_1 - \mathbb{H}$ over S^4, and the real virtual bundle given by the tensor product $(\xi_1 - \mathbb{H}) \otimes_{\mathbb{H}} (\xi_1 - \mathbb{H})$ over $S^4 \wedge S^4 \cong S^8$.

The quaternionic orientation $y^{KO} \in KO^4(\mathbb{H}P^\infty)$ is represented by the product of γ^{-1} with the real virtual bundle $(\xi_1 - \mathbb{H}) \otimes_{\mathbb{H}} (\xi_1 - \mathbb{H}) \in KO^0(S^4 \wedge \mathbb{H}P^\infty) \cong KO^{-4}(\mathbb{H}P^\infty)$.

We relate quaternionic K-theory and real K-theory by the well-known isomorphism $KSp^0(-) \cong KO^{-4}(-)$, given by mapping a virtual quaternionic bundle $\theta \in KSp^0(X)$ to $(\xi_1 - \mathbb{H}) \otimes_{\mathbb{H}} \theta$ in $KO^0(S^4 \wedge X) \cong KO^{-4}(X)$ (see e.g. [14]).

Example 2.2.9. The Thom spectrum MSp that arises from the infinite symplectic group Sp, yields the cohomology theory of quaternionic cobordism $MSp^*(-)$ and the dual theory of quaternionic bordism $MSp_*(-)$. In analogy with the complex case, we take the element y^{Sp} to be the quaternionic cobordism class that represents the homotopy equivalence $\mathbb{H}P^\infty \to MSp(1)$.

Among quaternionic oriented cohomology theories, quaternionic cobordism is universal in the analogous fashion to complex cobordism. In other words, for any quaternionic oriented ring spectrum F, there is a one-to-one correspondence between complex orientations $y^F \in F^4(\mathbb{H}P^\infty)$ and maps of ring spectra $g^F : MSp \to F$, such that $g^F_*(y^{Sp}) = y^F$.

We study quaternionic cobordism theory in Chapter 7.

Any quaternionic oriented ring spectrum F gives rise to quaternionic Pontryagin classes in $F^*(-)$ (see e.g. [56, 16.34]). Given an $Sp(n)$-bundle bundle θ over a space X, we denote by $p^F_i(\theta)$ in $F^{4i}(X)$ the *ith quaternionic Pontryagin class* of θ for $1 \le i \le n$. The *total quaternionic Pontryagin class* $p^F(\theta)$ of θ is given by $1 + p^F_1(\theta) + p^F_2(\theta) + \cdots + p^F_n(\theta)$ in $F^*(X)$. In the universal case of $\mathbb{H}P^\infty$ we will often denote $p^F_i(\xi_1)$ simply by p^F_i in $F^{4i}(\mathbb{H}P^\infty)$. If there is no danger of confusion, we may also use the symbol p^F_i to denote the quaternionic Pontryagin class $p^F_i(\xi)$ in $F^{4i}(BSp)$. Furthermore, if the context is clear, we may drop the superscript F to simplify our notation.

Remark 2.2.10. In our thesis we work exclusively with quaternionic Pontryagin classes, which should not be confused with *real* Pontryagin classes. The latter are defined in terms of the Chern classes of the complexification of a real bundle. Stong explores the relationship between real and quaternionic Pontryagin classes in [53].

2.3 Tangential structures

In later chapters we study quasitoric manifolds and related constructions in the setting of cobordism theory, where the existence of particular structures on the stable tangent bundle of a manifold is of fundamental importance. In this section we establish the necessary concepts and notation needed to describe tangential structures on manifolds.

We begin with the definition of a stably complex structure, and the related notions of an SU-structure and stably quaternionic structure follow. A particular tangential structure on a sphere bundle is then introduced, and in conclusion, we set up some machinery to study the effect of changing structure on the stable tangent bundle.

Further details on the material in this section can be found in the books of Stong [54] and Switzer [56].

We will write $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$ for a trivial n-dimensional real, complex and quaternionic vector bundle, respectively, over any space X; in the case $n = 1$, we will usually omit the superscript so that $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the appropriate trivial line bundles.

Given a vector bundle θ that is equipped with a suitable Riemannian metric (see e.g. [38, Chapter 2]), we will denote by $S(\theta)$ its sphere bundle, whose fibres are the unit spheres in the fibres of θ. Similarly $D(\theta)$ is the disc bundle of unit discs in the fibres of θ. The Thom space of θ is the quotient space $D(\theta)/S(\theta)$ and is written as $T(\theta)$.

For a manifold M, we write $\tau(M)$ for its tangent bundle, though we will occasionally simplify this to τ if M is understood.

2.3.1 Stably complex structures

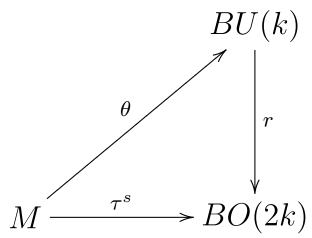
The following notion will be of particular importance in complex cobordism theory.

Definition 2.3.1. A *stably complex structure* on an n-dimensional manifold M is a real isomorphism of vector bundles

$$\tau(M) \oplus \mathbb{R}^{2k-n} \cong \theta, \tag{2.3.2}$$

where θ is a k-dimensional complex vector bundle. We say such manifolds M are *stably complex*.

We denote by $\tau^s(M)$ the *stable tangent bundle* $\tau(M) \oplus \mathbb{R}^{2k-n}$ of M, again simplifying to τ^s when there is no danger of confusion. Often we use the same symbol to denote both a bundle and its classifying map. If the stable tangent bundle of M is classified by $\tau^s\colon M \to BO(2k)$, we can view a stably complex structure on M as a lift of τ^s to $BU(k)$ via the classifying map of θ, illustrated by the commutative diagram



in which r is realification, giving $r\theta = \tau^s$. Then two isomorphisms of the form (2.3.2) are considered equivalent if they are homotopic as lifts of the map τ^s, and accordingly, in such cases we call the structures *homotopic*.

2.3.2 Special unitary & stably quaternionic structures

We can impose stricter conditions on the stable tangent bundle of a manifold to define additional tangential structures.

Suppose that we have a stably complex manifold M. If the first Chern class $c_1(\tau^s) \in H^2(M)$ of the stable tangent bundle of M is zero, then the stably complex structure is *special unitary*; we say M is a *special unitary manifold*. Often we will abbreviate these notions to *SU-structure* and *SU-manifold* respectively. We study SU-structures on quasitoric manifolds in Chapter 5.

Furthermore, we can extend the idea of a stably complex structure to the quaternionic setting, which will be crucial for the constructions of Chapter 6 and our work with quaternionic cobordism in Chapter 7. In analogy with Definition 2.3.1 we have the following notion.

Definition 2.3.3. A *stably quaternionic structure* on a manifold M is a real isomorphism of vector bundles

$$\tau(M) \oplus \mathbb{R}^{2k-n} \cong \theta, \tag{2.3.4}$$

where θ is a k-dimensional quaternionic vector bundle. Such M are said to be *stably quaternionic.*

As in section 2.3.1, we will term two isomorphisms of the form (2.3.4) *homotopic* if they are homotopic as lifts of $\tau^s \colon M \to BO(4k)$ to $BSp(k)$.

2.3.3 Tangential structures on sphere bundles

Throughout this section we work with a bundle θ, which has fibre $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We will assume that θ is equipped with a suitable Riemannian metric. Let $E(\theta)$ denote the total space of θ, and label by π the projection to the base space, which is taken to be some manifold $B(\theta)$.

We write $\tau_F(\theta)$ for the bundle of vectors in $E(\theta)$ that are tangent to the fibres of θ. The bundle of vectors orthogonal to fibres in θ will be denoted by $\tau_\perp(\theta)$; it is obviously isomorphic to $\pi^*(\tau B(\theta))$. Furthermore there is an isomorphism of $O(in)$-bundles,

$$\tau E(\theta) \cong \tau_F(\theta) \oplus \tau_\perp(\theta), \tag{2.3.5}$$

where $i = 1, 2$ or 4 when $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, respectively.

We now apply a result of Szczarba [57, Theorem 1.2] in the particular case of the sphere bundle $S(\theta)$ to obtain the following.

Corollary 2.3.6. *There is an isomorphism*

$$\tau_F(S(\theta)) \oplus \mathbb{R} \cong \pi^*(\theta).$$

This isomorphism and the splitting (2.3.5) combine to give an important proposition that we will rely upon throughout the rest of our thesis.

Proposition 2.3.7. *There exists a real isomorphism of vector bundles*

$$\tau S(\theta) \oplus \mathbb{R} \cong \pi^*(\theta \oplus \tau B(\theta)). \tag{2.3.8}$$

Similarly, we can apply Szczarba's methods to determine an real isomorphism

$$\tau D(\theta) \cong \pi^*(\theta \oplus \tau B(\theta)), \tag{2.3.9}$$

on the tangent bundle of the disc bundle of θ.

In the case that θ is a complex vector bundle such that $\tau B(\theta)$ is stably complex, the isomorphism (2.3.8) leads to a stably complex structure on $S(\theta)$. In a similar fashion, for quaternionic θ we can obtain a stably quaternionic structure on $S(\theta)$ via (2.3.8). These structures bound since they extend via (2.3.9) to $D(\theta)$.

2.3.4 Changes of tangential structure

In Chapter 7 we will consider the effect of changing the tangential structure on a manifold, using the techniques described in the memoir of Ray, Switzer and Taylor [48]. In this section we give a brief exposition of the theory that underlies their methods, taking some ideas from an earlier paper by Ray [46] as our starting point.

Let G, H be stable subgroups of the infinite orthogonal group O such that $G < H$, and choose integers i_G, i_H, such that $H(i_H n)$ and $G(n)$ act on $\mathbb{R}^{i_G n}$. A typical example is when $H = U$ and $G = Sp$, then $i_U = 2$ and $i_{Sp} = 4$.

Consider the fibration

$$H(i_H n)/G(n) \xrightarrow{\iota} BG(n) \xrightarrow{f} BH(i_H n). \tag{2.3.10}$$

The map f classifies the universal $G(n)$-bundle and ι is the inclusion of the fibre. The nullhomotopic composition $f\iota$ classifies a trivial $H(i_H n)$-bundle over $H(i_H n)/G(n)$ and induces a map of Thom spaces

$$S^{i_H n} \wedge H(i_H n)/G(n)_+ \longrightarrow MG(n).$$

Let $n \to \infty$ and take the adjoint to obtain $j\colon H/G_+ \to \Omega^\infty MG$, which in turn induces a map

$$J\colon [X_+, H/G] \longrightarrow MG^*(X_+). \tag{2.3.11}$$

which we term the *J-homomorphism*.

In our thesis, we will only consider situations where H/G is equipped with an infinite loop structure, so that the set of homotopy classes of maps $[X_+, H/G]$ forms a group. In the case of $H/G = O/U$ or U/Sp, the standard equivalences of Bott periodicity give $[X_+, H/G] \cong KO^{-2}(X_+)$ or $KO^{-3}(X_+)$ respectively (see Lemma 2.3.13 below), and a suitable infinite loop structure on O/Sp is described in [43, Section 2]. Henceforth we will denote $[X_+, H/G]$ by $H/G^*(X_+)$.

Now suppose we have an n-dimensional manifold M^n, whose stable tangent bundle carries a *G-structure* g, that is, a class of lift g of the classifying map $\tau^s\colon M^n \to BO$ to BG. This is the generalisation of concepts we defined earlier: in the case when $G = U$ or Sp we have a stably complex or stably quaternionic structure respectively on M^n, as given by Definitions 2.3.1 and 2.3.4. We write $[M^n, g] \in MG_n$ for the equivalence class under the G-bordism relation of a manifold M^n with G-structure g.

With the notion of a G-structure established, we can state the main aim of this section. Suppose we have a manifold M^n, whose tangent bundle carries a fixed H-structure and a G-structure g. We will work towards defining a function $\Psi_g\colon H/G^0(M^n_+) \longrightarrow MG_n$, whose image describes all the possible changes to the G-structure on M^n. Then we will detail a systematic procedure for computing the image of Ψ_g, and use this as the basis for our investigations in Chapter 7.

There is a useful geometric description of $H/G^0(M^n_+)$, which allows us to study the effect of a change of G-structure on a manifold M^n, which already has a given G-structure g.

Lemma 2.3.12. *[48, Lemma 2.1] We can identify $H/G^0(M^n_+)$ with the homotopy classes of G-structures on M^n, considered as a manifold with fixed H-structure.*

Proof. An element $\delta \in H/G^0(M^n_+)$ is represented by a map $M^n \to H/G$. This can be thought of as a new G-structure on the trivial H-bundle over M^n, as it gives a lift to BG by the inclusion of the fibre $H/G \to BG$. Adding this trivial H-bundle to the stable tangent bundle τ^s of M^n does not affect the H stability class of τ^s, and so we have identified $H/G^0(M^n_+)$ and the set of stable G-structures on τ^s. □

The new G-structure on M^n induced by δ is written as $\delta + g$.

Since $SO/U \simeq \Omega^2(BO)$ and $U/Sp \simeq \Omega^3(BO)$ (see e.g. [29, Theorem 5.22]) we can relate certain homotopy classes of tangential structures to real K-theory, the cohomology theory of Example 2.2.7.

Lemma 2.3.13. *The set of homotopy classes $SO/U^0(M^n_+)$ of stably complex structures on M^n as a fixed SO-manifold is in 1-1 correspondence with $KO^{-2}(M^n)$.*

When a stably complex structure on M^n is fixed, the set of homotopy classes $U/Sp^0(M^n_+)$ of stably quaternionic structures on M^n is in 1-1 correspondence with $KO^{-3}(M^n)$.

Now that we have a convenient description of the homotopy classes of G-structures on a manifold M^n, we are almost ready to fulfil our goal of establishing the function $\Psi_g\colon H/G^0(M^n_+) \longrightarrow MG_n$. As far as possible we wish to retain the geometric viewpoint of [46, Section 2]. Therein the group $MG_n(H/G_+)$ is described as the bordism group of n-dimensional manifolds M^n which carry independent G-structures g_1, g_2 that are equivalent (i.e. they are homotopic, in the sense of Section 2.3.1) when considered as H-structures. We denote such classes of $MG_n(H/G_+)$ by $[M^n, g_1, g_2]$.

Assume now that M^n carries a G-structure g. By Lemma 2.3.12, given $\delta \in H/G^0(M^n_+)$, we have two G-structures g and $\delta+g$ on M^n, which agree as H-structures. The induced map $B_g\colon H/G^0(M^n_+) \to MG_n(H/G_+)$, sends δ to $[M^n, g, \delta + g]$ in $MG_n(H/G_+)$.

We then define a map Ψ_g given by the composite

$$H/G^0(M^n_+) \xrightarrow{B_g} MG_n(H/G_+) \xrightarrow{L} MG_n, \qquad (2.3.14)$$

where the map L simply takes $[M^n, g, \delta + g]$ and considers it as a bordism class in MG_n so that

$$\Psi_g(\delta) = [M^n, \delta + g] \in MG_n, \qquad (2.3.15)$$

for any δ in $H/G^0(M^n)$. Hence $\Psi_g(\delta)$ has the effect of changing the G-structure on M^n by the addition of δ, in the sense of Lemma 2.3.12.

Remark 2.3.16. In general Ψ_g is not a homomorphism.

Lemma 2.3.17. *[48, Lemma 2.2] The image of Ψ_g is independent of the G-structure g on M^n.*

Proof. Suppose we have a second G-structure g' on M^n. As a consequence of Lemma 2.3.12 we can express g' as $g + \epsilon$, for some $\epsilon \in H/G^0(M^n_+)$.

We have that $\Psi_g(\delta) = [M^n, \delta + g] = [M^n, \delta + g' - \epsilon] = \Psi_{g'}(\delta - \epsilon)$, and similarly $\Psi_{g'}(\delta) = \Psi_g(\delta + \epsilon)$. So it follows that the images of Ψ_g and $\Psi_{g'}$ on M^n comprise the same subset of MG_n. □

We will therefore write $\Psi(M^n)$ for the image of Ψ_g on $H/G^0(M^n_+)$.

In [48] the authors show that there is an alternative and equivalent definition of Ψ_g, which leads to a reduced map $\widetilde{\Psi}_g$. They introduce an additive homomorphism $D_g\colon MG^0(M^n_+) \to MG_n$, whose composite $D_g J$ with the J-homomorphism (2.3.11) is a map

$$H/G^0(M^n_+) \longrightarrow MG_n, \tag{2.3.18}$$

which is equivalent to Ψ_g [48, Lemma 1.3]. We will study D_g in greater detail later in this section, as it has a simple geometric interpretation as part of a systematic procedure for calculating $\Psi(M^n)$

We will use the new and equivalent definition of Ψ_g to obtain $\widetilde{\Psi}_g$; we first note that the the composition $H/G^0(M^n) \to H/G^0(M^n_+) \to MG^0(M^n_+) \to MG^0(M^n)$, gives a reduced J-homomorphism $\widetilde{J}$, where $MG^0(M^n)$ is given the *circle operation* $x \circ y = x + y + xy$, so that

$$\widetilde{J}(\delta_1 + \delta_2) = \widetilde{J}(\delta_1) + \widetilde{J}(\delta_2) + \widetilde{J}(\delta_1)\widetilde{J}(\delta_2), \tag{2.3.19}$$

for any δ_1, δ_2 in $H/G^0(M^n)$. Then the composite $D_g\widetilde{J}$ is $\widetilde{\Psi}_g\colon H/G^0(M^n) \longrightarrow MG_n$, and we denote the image of $\widetilde{\Psi}_g$ on $H/G^0(M^n)$ by $\widetilde{\Psi}(M^n)$. Again, $\widetilde{\Psi}_g$ is not necessarily a homomorphism. We have introduced the reduced viewpoint as it will be useful for some of our observations in Chapter 7.

In our thesis, we will assume that H/G is connected (that is, $H \leq SO$), thus ensuring that our changes of G-structure do not affect the orientation of M^n. This has various consequences, but in particular we will need the fact [48, Note 3.2] that we can then relate our reduced and unreduced maps by the formula

$$\Psi_g(\delta) = [M^n, g] + \widetilde{\Psi}_g(\delta), \tag{2.3.20}$$

so if g is a bounding G-structure we have $\Psi(M^n) = \widetilde{\Psi}(M^n)$.

We will also require the following definition in Chapter 7.

Definition 2.3.21. A closed, connected G-manifold (M^n, g), with the property that $\Psi_g : H/G(M^n_+) \to MG_n$ is epimorphic, that is the image $\Psi(M^n) = MG_n$, is said to be an *H-universal* G-manifold.

To conclude this chapter we describe a systematic procedure for calculating the image $\Psi(M^n)$, which we will form the basis of our investigations in the final chapter.

In [48] the authors identify the homomorphism D_g with $\langle -, [M^n]_g \rangle$, where $[M^n]_g \in MG_n(M^n_+)$ is the fundamental class of M^n with G-structure g, and $\langle\ ,\ \rangle$ is the Kronecker product for the spectrum MG. Given any x in $MG^*(M^n_+)$ we have that $\langle x, [M^n]_g \rangle = c_*(x \frown [M^n]_g)$, where c collapses M^n to a point, and $\frown$ denotes cap product in MG. Hence $\langle x, [M^n]_g \rangle$ is nothing more than the image under the collapse map of the Poincaré dual of x.

Therefore the process for determining $\Psi_g(\delta)$ breaks into the following steps. Begin by taking the *associated unit* of δ, given by $J(\delta) \in MG^0(M^n_+)$. Then compute the *associated dual*, which is the Poincaré dual $\Gamma_g(\delta) = J(\delta) \frown [M^n]_g \in MG_n(M^n_+)$. Finally, read off the G bordism class of a manifold that represents $\Gamma_g(\delta)$.

2.4 The Borel construction

The Borel construction features throughout our thesis, so we set out the salient details here. A more thorough description and further applications can be found for example in [15] or [22].

Suppose that a topological group G acts freely on a space E by $(g, e) \mapsto ge$, for any $g \in G$ and $e \in E$. Denote the orbit space under this action by B.

Definition 2.4.1. A *principal G-bundle* over B is a bundle $\pi\colon E \to B$, with fibre G.

Now suppose that G acts on a space F by $(g, f) \mapsto gf$, for any $g \in G$ and $f \in F$.

Definition 2.4.2. The quotient space $E \times F/\sim$, under the equivalence relation

$$(e, f) \sim (ge, gf), \tag{2.4.3}$$

is the *Borel construction*, denoted by $E \times_G F$.

For an equivalence class $[e, f] \in E \times_G F$, the map $\omega\colon [e, f] \mapsto \pi(e)$ gives rise to a bundle

$$\omega\colon E \times_G F \longrightarrow B,$$

with fibre F, which is *associated* to the principal bundle $\pi\colon E \longrightarrow B$.

We have a well-known example that illustrates these concepts.

Example 2.4.4. Consider S^{2n+1} embedded in $\mathbb{C}^{n+1}$ as the set $\{z = (z_1, \ldots, z_{n+1}) : |z_1|^2 + \cdots + |z_{n+1}|^2 = 1\}$. Let $t \in S^1$ act diagonally on the sphere S^{2n+1} by $(t, z) \mapsto tz$, yielding the standard principal S^1-bundle $S^{2n+1} \to \mathbb{C}P^n$. Now, given an action of S^1 on $\mathbb{C}$ by $(t, w) \mapsto t^{-1}w$, the associated complex line bundle

$$S^{2n+1} \times_{S^1} \mathbb{C} \longrightarrow \mathbb{C}P^n,$$

is the canonical complex line bundle ζ_1 over $\mathbb{C}P^n$.

2.5 Iterated bundle constructions

To simplify our presentation in Chapters 4 and 6 we will establish here some basic facts about the cohomological properties of iterated bundle constructions, based on the programme detailed in [12, Section 7].

We begin by stating the Leray-Hirsch Theorem for any ring spectrum E.

Theorem 2.5.1. *[56, Theorem 15.47] Let* $F \xrightarrow{\iota} Y \xrightarrow{\pi} X$ *be a fibre bundle in which the base space* X *is* 0*-connected. Suppose there are elements* $y_1, \ldots, y_r \in E^*(Y_+)$*, such that* $\iota^*(y_1), \ldots, \iota^*(y_r) \in E^*(F_+)$ *form a basis of* $E^*(F_+)$ *as a module over* E_**. Then* $E^*(Y_+)$ *is a free* $E^*(X_+)$*-module with basis* $\{y_1, \ldots, y_r\}$*. The module action is given by* $xy = \pi^*(x) \smile y$ *for* $x \in E^*(X_+)$ *and* $y \in E^*(Y_+)$*.*

For the remainder of this section we will work exclusively with complex oriented ring spectra E.

Definition 2.5.2. A *2-generated* connected CW-complex X is one whose integral cohomology ring $H^*(X)$ is generated by a linearly independent set of elements $x_1, \ldots, x_n$ in $H^2(X)$. We say such elements are *2-generators* and n is the *2-rank*.

The first Chern class c_1^H defines an isomorphism between the multiplicative group of complex line bundles over X and $H^2(X;\mathbb{Z}) \cong \mathbb{Z}^n$. Hence there exists line bundles γ_i over X such that $c_1^H(\gamma_i) = x_i$, and the n-tuple $(a(1), \ldots, a(n)) \in \mathbb{Z}^n$ corresponds to the tensor product bundle $\gamma_1^{a(1)} \otimes \cdots \otimes \gamma_n^{a(n)}$ under the isomorphism.

In any complex oriented ring spectrum E, the first Chern class $c_1^E(\gamma_i)$, which we will henceforth denote by y_i^E, lies in $E^2(X)$ for $1 \leq i \leq n$. The Atiyah-Hirzebruch spectral sequence, in which

$$E_2^{i,j} \cong H^i(X; E_j(pt)),$$

converges to $E^*(X)$ (see e.g. [24, Chapter 3]), collapses, since the ordinary cohomology of X is concentrated in even degrees, forcing all differentials to be zero. Therefore $E^*(X)$ is a free E_*-module, spanned by the collection of monomials $\prod_R y_i^E$, where R is any subset of $\{1, \ldots, n\}$; as a free E_*-algebra, $E^*(X)$ is generated by $y_1^E, \ldots, y_n^E$.

Given an n-tuple $(a(1;j), \ldots, a(n;j)) \in \mathbb{Z}^n$, write β_j for the bundle $\gamma_1^{a(1;j)} \otimes \cdots \otimes \gamma_n^{a(n;j)}$ over X. Now, for l_{n+1} such n-tuples, we can define a direct sum bundle

$$\beta := \beta_1 \oplus \cdots \oplus \beta_{l_{n+1}}. \tag{2.5.3}$$

Then let Y denote the total space of the $\mathbb{C}P^{l_{n+1}}$-bundle $\mathbb{C}P(\beta \oplus \mathbb{C})$ over X, given by projectivising the direct sum of β and a trivial complex line bundle $\mathbb{C}$.

Lemma 2.5.4. *The E_*-module $E^*(Y_+)$ is a free module over $E^*(X_+)$, generated by $1, y^E_{n+1}, (y^E_{n+1})^2, \dots, (y^E_{n+1})^{l_{n+1}}$, where y^E_{n+1} is an element of $E^2(Y_+)$. A single relation*

$$(y^E_{n+1})(y^E_{n+1} - c^E_1(\beta_1)) \dots (y^E_{n+1} - c^E_1(\beta_{l_{n+1}})) = 0, \tag{2.5.5}$$

describes the multiplicative structure.

Proof. We will adapt the proof of Proposition 4.2.2. in [10]. We have a fibre bundle

$$\mathbb{C}P^{l_{n+1}} \xrightarrow{\iota} Y \xrightarrow{\pi} X, \tag{2.5.6}$$

and $E^j(\mathbb{C}P^{l_{n+1}})$ is a finitely generated E_*-module for all j. Consider the canonical line bundle γ_{n+1} over Y, whose total space consists of all pairs (λ, v), where λ is a 1-dimensional subspace in the $\mathbb{C}P^{l_{n+1}}$ fibres and v is a point in λ. The pullback $\iota^*\gamma_{n+1}$ is isomorphic to $\zeta_{l_{n+1}}$, the canonical line bundle over $\mathbb{C}P^{l_{n+1}}$. We have that $\{1, \iota^*(c^E_1(\gamma_{n+1})), \dots, \iota^*(c^E_1(\gamma_{n+1})^{l_{n+1}})\}$ forms a basis for $E^*(\mathbb{C}P^{l_{n+1}})$ over E_*. Denote $c^E_1(\gamma_{n+1})$ by y^E_{n+1}.

With these preliminaries in place, we can apply Theorem 2.5.1 and deduce that $E^*(Y_+)$ is a free $E^*(X_+)$-module generated by $y^E_{n+1}, (y^E_{n+1})^2, \dots, (y^E_{n+1})^{l_{n+1}}$, for $y^E_{n+1} \in E^2(Y_+)$. Furthermore, for any $x \in E^*(X_+)$ and $y \in E^*(Y_+)$, the module structure is determined by $xy = \pi^*(x) \smile y$.

The pullback $\pi^*(\beta \oplus \mathbb{C})$ contains γ_{n+1} as a subbundle. Therefore we may use the standard inner product in the fibres of the complex bundle β, to define an l_{n+1}-dimensional bundle $\gamma^\perp_{n+1}$ over Y, such that $\pi^*(\beta \oplus \mathbb{C}) \cong \gamma_{n+1} \oplus \gamma^\perp_{n+1}$. The fibre of $\gamma^\perp_{n+1}$ is given by the orthogonal complement $\lambda^\perp$ in $\beta \oplus \mathbb{C}$.

Now take the total Chern class c^E of $\pi^*(\beta \oplus \mathbb{C}) \cong \pi^*(\beta) \oplus \mathbb{C}$ to obtain

$$\begin{aligned} c^E(\pi^*(\beta)) &= c^E(\gamma_{n+1} \oplus \gamma^\perp_{n+1}) \\ &= (1 + c^E_1(\gamma_{n+1}))c^E(\gamma^\perp_{n+1}) \\ &= (1 + y^E_{n+1})c^E(\gamma^\perp_{n+1}). \end{aligned} \tag{2.5.7}$$

Rearranging we have

$$\begin{aligned} c^E(\gamma_{n+1}^{\perp}) &= c^E(\pi^*(\beta))(1+y_{n+1}^E)^{-1} \\ &= (1+c_1^E(\pi^*\beta_1))\dots(1+c_1^E(\pi^*\beta_{l_{n+1}}))(1+y_{n+1}^E)^{-1} \\ &= (1+c_1^E(\pi^*\beta_1))\dots(1+c_1^E(\pi^*\beta_{l_{n+1}}))(1-y_{n+1}^E+\dots \\ &\dots+(-1)^i(y_{n+1}^E)^i+\dots). \end{aligned}$$

Note that by equating coefficients in degree $2l_{n+1}+2$ of the total Chern class (2.5.7) we have that $y_{n+1}^E c_{l_{n+1}}^E(\gamma_{n+1}^{\perp}) = 0$, because the fibres of β are of dimension $2l_{n+1}$, whence

$$\begin{aligned} &y_{n+1}^E((-1)^{l_{n+1}}(y_{n+1}^E)^{l_{n+1}} + (-1)^{l_{n+1}-1}(y_{n+1}^E)^{l_{n+1}-1}(c_1(\pi^*\beta_1)+\dots \\ &\dots+c_1(\pi^*\beta_{l_{n+1}}))+\dots+c_1(\pi^*\beta_1)\dots c_1(\pi^*\beta_{l_{n+1}})) = 0. \end{aligned}$$

After some simple but tedious manipulation we arrive at

$$(y_{n+1}^E)(y_{n+1}^E - c_1^E(\pi^*\beta_1))\dots(y_{n+1}^E - c_1^E(\pi^*\beta_{l_{n+1}})) = 0, \tag{2.5.8}$$

and our above determination of the module structure of $E^*(Y_+)$ shows that the bundle β_i over Y is simply the pullback $\pi^*(\beta_i)$, for $1 \leq i \leq l_{n+1}$, thus giving our desired cohomology relation. □

Note that Lemma 2.5.4 implies that Y itself is 2-generated with 2-rank $n+1$.

By taking a nonzero vector in the $\mathbb{C}$ summand we have a section ω for the bundle $\pi\colon Y \to X$. The space obtained by the quotient of Y by the image of ω is homeomorphic to the Thom complex of β [54, page 66], which we denote by $T(\beta)$. Label the quotient map by ϑ. The section ω has left inverse π and so the induced cohomology sequence

$$E^*(X) \xleftarrow{\omega^*} E^*(Y) \xleftarrow{\vartheta^*} E^*(T(\beta)), \tag{2.5.9}$$

is split by π^*, thus ensuring it is short exact.

We illustrate Lemma 2.5.4 with a familiar example.

Example 2.5.10. Let $X = \mathbb{C}P^\infty$ and choose β to be ζ_1, the universal complex line bundle over $\mathbb{C}P^\infty$. It follows that $T(\zeta_1) \cong \mathbb{C}P^\infty$ (see e.g. [27, 15.1.7]). Therefore

Y is the projectivisation $\mathbb{C}P(\zeta_1 \oplus \mathbb{C})$, which is homotopy equivalent to $\mathbb{C}P^\infty \vee \mathbb{C}P^\infty$ [51, page 45]. We have that $E^*(Y_+)$ is free over $E^*(\mathbb{C}P^\infty_+) \cong E_*[[x]]$, generated by 1 and $y \in E^2(Y_+)$, with $(y)^2 = xy$. Simplifying the cohomology relation to $y = x$, illustrates the homotopy equivalence between Y and $\mathbb{C}P^\infty \vee \mathbb{C}P^\infty$.

Note that the first Chern class c_1^E in $E^2(\mathbb{C}P^\infty)$ induces a canonical Thom class $t^E \in E^2(T(\zeta_1))$, and gives a Thom isomorphism $E^{*-2}(X_+) \cong E^*(T(\zeta_1))$. This allows us to view y as the pullback $\vartheta^* t^E$.

An alternative proof for Lemma 2.5.4 is given in [12, Lemma 7.2]; a useful fact that arises from this approach is that products of the form $\pi^*(x)y^E_{n+1}$ may be written as $\vartheta^*(xt^E)$, for any element $x \in E^*(X)$.

Chapter 3

Quasitoric manifolds

Quasitoric manifolds can be thought of as a topological analogue of the nonsingular projective toric varieties of algebraic geometry. They were introduced by Davis and Januskiewicz [16], whose work is now regarded as the catalyst for the study of toric topology.

In this chapter we introduce the key definitions and results that we rely upon throughout the rest of our thesis.

We will follow the most recent approach to toric topology, as described by Buchstaber, Panov and Ray in their work [6]; therein the reader will find any omitted proofs, and a wealth of further detail is contained in [5].

We begin by setting out some preliminary details on torus actions and polytopes, which leads to the definition of a quasitoric manifold. The concept of a dicharacteristic function is introduced, which allows a second description of a quasitoric manifold as a quotient of an object known as the moment angle complex. In the next section the ordinary cohomology ring of a quasitoric manifold is computed, and we conclude by considering stably complex and special unitary structures on such manifolds.

3.1 Torus actions and simple polytopes

Coordinatewise multiplication of the n-torus $T^n = (S^1)^n$ on $\mathbb{C}^n$ is called the *standard representation*; the orbit space of this T^n-action is the *nonnegative cone* $\mathbb{R}^n_{\geq}$, which consists of all vectors in $\mathbb{R}^n$ with non-negative coordinates. We call a $2n$-dimensional manifold M^{2n} with an action of T^n a *T^n-manifold*.

Consider open sets $U \subset M^{2n}$ and $V \subset \mathbb{C}^n$, which are closed under the action of T^n, and a homeomorphism $h\colon U \longrightarrow V$.

Definition 3.1.1. Given an automorphism ψ of T^n, we say that a T^n-action on M^{2n} is *locally isomorphic* to the standard representation if $h(tu) = \psi(t)h(u)$, for all $t \in T^n, u \in U$.

Let T_i denote the ith coordinate circle in T^m, for $1 \leq i \leq m$, and given a subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, m\}$, write T_I for a product $\prod_{i \in I} T_i < T^m$. We shall set $T_\emptyset$ equal to $\{1\}$, the trivial subgroup.

Now consider n-dimensional Euclidean space $\mathbb{R}^n$, endowed with the standard orthonormal basis $e_1, \ldots, e_n$, and inner product denoted by $\langle\ ,\ \rangle$. Let $\mathcal{H}$ be the collection of closed half-spaces

$$H_i = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \geq -b_i\}, \tag{3.1.2}$$

for $1 \leq i \leq m$, where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$. The boundary ∂H_i is the *bounding hyperplane* of the half-space H_i; it has an inward pointing normal vector a_i.

Definition 3.1.3. A bounded intersection $\cap_i H_i$ of half-spaces is called an *n dimensional convex polytope*, which we denote by P^n. We assume $\mathcal{H}$ is as small as possible, so that if any H_i is removed from $\mathcal{H}$, the polytope would be enlarged.

A *supporting hyperplane* of P^n is a hyperplane H, which intersects with P^n and contains the polytope in one of the two half-spaces it determines. The intersection $P^n \cap H$ defines a *face* of the polytope, which is itself a convex polytope of dimension $< n$. We will take P^n to be an n-dimensional face of itself, and regard all other faces as *proper faces*. In particular we have *vertices, edges* and *facets* as faces of dimensions 0, 1 and $n-1$ respectively.

Definition 3.1.4. An n-dimensional convex polytope P^n with exactly n facets meeting at each vertex is called *simple*.

Throughout our thesis we will work only with simple polytopes, and we will always use m to denote the number of facets of a polytope.

Example 3.1.5. The *standard n-simplex* Δ^n is the simple polytope given by the intersection of the half-spaces

$$H_i = \begin{cases} \{x : \langle e_i, x\rangle \geq 0\} & \text{for } 1 \leq i \leq n, \\ \{x : \langle a_{n+1}, x\rangle \geq -1\} & \text{for } i = n+1, \end{cases} \tag{3.1.6}$$

in $\mathbb{R}^n$, where $a_{n+1} = (-1, \ldots, -1)$. The vertices of Δ^n are given by the origin 0 and the points $e_1, \ldots, e_n$, while each facet is an $(n-1)$-simplex Δ^{n-1}.

Occasionally it will be helpful to view the n-simplex as the following subset of $\mathbb{R}^{n+1}$

$$\Delta^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \Sigma_i x_i = 1, x_i \geq 0, \text{for all } i\}. \tag{3.1.7}$$

Example 3.1.8. The *standard n-cube* I^n is the simple polytope given by the intersection of the half-spaces

$$\begin{aligned} H_i &= \{x : \langle e_i, x\rangle \geq 0\} \quad \text{for } 1 \leq i \leq n, \\ H_{n+i} &= \{x : -\langle e_i, x\rangle \geq -1\} \text{ for } 1 \leq i \leq n. \end{aligned}$$

in $\mathbb{R}^n$. The vertices of I^n are $(\delta_1, \ldots, \delta_n)$, where $\delta_i = 0$ or 1, for $1 \leq i \leq n$, while each facet is an $(n-1)$-cube I^{n-1}.

Denote by $\mathcal{F}$ the set of facets $\{F_1, \ldots, F_m\}$ of P^n. Every codimension k face of the polytope can be expressed uniquely as

$$F_I = F_{i_1} \cap \cdots \cap F_{i_k}, \tag{3.1.9}$$

where $I = \{i_1, \ldots, i_k\}$ is a subset of $\{1, \ldots, m\}$. We order the F_I lexicographically for each $1 \leq k \leq n$. Moreover, every point $p \in P$ lies in the interior of a unique face F_{I_p}, where $I_p := \{i : p \in F_i\}$. To simplify notation, we will abbreviate F_{I_p} to $F(p)$, and similarly, write $T(p)$ for the subtorus $T_{I_p} < T^m$.

Permuting the order of the facets if necessary, we will assume $F_1 \cap \cdots \cap F_n$ is a vertex $v_\star$. We term such orderings *fine* and name $v_\star$ the *initial vertex* of P^n, the first vertex with respect to the lexicographical ordering of faces (3.1.9).

The fine ordering can be extended to a product of polytopes in the following manner. Let P and P' be finely ordered polytopes of dimensions n and n' and respectively. The set of facets of the product $P \times P'$ may be written as a list

$$F_1 \times P', \dots, F_m \times P', P \times F'_1 \dots, P \times F'_{m'}, \tag{3.1.10}$$

where F_i, for $1 \leq i \leq m$, and F'_i, for $1 \leq i \leq m'$, are the facets of P and P' respectively. We can impose a fine order on $P \times P'$ by moving the facets $P \times F'_1 \dots, P \times F'_{n'}$ into positions $n+1, \dots, n+n'$ in the list (3.1.10). We then have the point $(v_\star, v'_\star) \in \mathbb{R}^{n+n'}$ as the initial vertex of the product. Note that we will have a different ordering if we instead consider $P' \times P$.

We require one final concept before we define a quasitoric manifold.

Definition 3.1.11. A vector $v_1 \in \mathbb{Z}^n$ is called *primitive* if there exists $n-1$ vectors $v_2, \dots, v_{n-1}$ in $\mathbb{Z}^n$, such that $\{v_1, \dots, v_n\}$ forms a basis for $\mathbb{Z}^n$

3.2 Quasitoric manifolds

Definition 3.2.1. Given a simple convex polytope P^n, a T^n-manifold M^{2n} is a *quasitoric manifold over* P^n if

(i) the T^n-action is locally isomorphic to the standard representation,

(ii) there is a projection $\pi\colon M^{2n} \longrightarrow P^n$, mapping every k-dimensional orbit to a point in the interior of a k-dimensional face of P^n, for $k = 0, \dots, n$.

The second condition implies that any points of M^{2n} that are fixed under the action of T^n map to the vertices of P^n, while points at which the action is free project to the interior of the polytope. We define $\pi^{-1}(v_\star)$ to be the *initial fixed point* $x_\star$.

Before we consider examples of quasitoric manifolds, we introduce some further notions.

Definition 3.2.2. The *moment angle complex* $\mathcal{Z}_{P^n}$ associated to a polytope P^n is the quotient space

$$T^m \times P^n / \sim \tag{3.2.3}$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1^{-1} t_2 \in T(p)$.

The free action of T^m on $T^m \times P^n$ induces a T^m-action on $\mathcal{Z}_{P^n}$, with quotient P^n. It is straightforward to check that the isotropy subgroups of the T^m-action are given by $T(p)$ for all $p \in P^n$.

The moment angle complex associated to a product of polytopes splits in the obvious fashion.

Proposition 3.2.4. *[5, Proposition 6.4] If P is a product $P_1 \times P_2$ of simple polytopes P_1, P_2, then $\mathcal{Z}_P = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}$.*

The following fact will also be useful in our thesis.

Lemma 3.2.5. *[5, Construction 6.8] There is a T^m-equivariant embedding $\mathcal{Z}_{P^n} \to \mathbb{C}^m$.*

Moment angle complexes lead to the construction of quasitoric manifolds, which appear as the orbit space of $\mathcal{Z}_{P^n}$ under a certain T^{m-n}-action. To determine that action we make the following definition.

Definition 3.2.6. A *dicharacteristic function* is a homomorphism $l \colon T^m \to T^n$, satisfying the condition that if F_I is a face of codimension k, the map l is monic on T_I.

We denote $l(T(p))$ by $T(F(p))$, so for a vertex v of P^n we have $T(F(v)) = T^n$, while any point w in the interior of the polytope has $T(F(w)) = \{1\}$.

Remark 3.2.7. In [16], Davis and Januskiewicz introduced the notion of a *characteristic function*, which mapped each facet F_i of P^n to a vector λ_i determining the isotropy subgroup of F_i. However, such a vector is only defined up to sign. On introducing the dicharacteristic function, Buchstaber and Ray [9] removed this ambiguity by insisting that each $T(F_i)$ be oriented, therefore choosing the sign of each vector λ_i. In Section 3.4 we will see that this has important consequences for the study of quasitoric manifolds in complex cobordism theory, which is the subject of Chapter 5 of our thesis.

As we insisted on the first n-facets of P^n intersecting in the initial vertex $v_\star$, Definition 3.2.6 implies that the restriction of l to $T_1 \times \cdots \times T_n$ is a monomorphism, and so l is an isomorphism. Hence we can take $T(F_1), \ldots, T(F_n)$ to be a basis for the Lie algebra of T^n. Then the epimorphism of Lie algebras associated to the dicharacteristic l, may be described by a linear transformation $\lambda\colon \mathbb{Z}^m \longrightarrow \mathbb{Z}^n$. The *dicharacteristic matrix* representing the map λ is the $n \times m$ matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & \ldots & 0 & \lambda_{1,n+1} & \ldots & \lambda_{1,m} \\ 0 & 1 & \ldots & 0 & \lambda_{2,n+1} & \ldots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & \lambda_{n,n+1} & \ldots & \lambda_{n,m} \end{pmatrix}. \tag{3.2.8}$$

We can see that the isotropy subgroup associated to each facet F_i is given by

$$T(F_i) = \{(e^{2\pi i\lambda_{1,i}\varphi}, \ldots, e^{2\pi i\lambda_{n,i}\varphi}) \in T^n\}, \tag{3.2.9}$$

where $\varphi \in \mathbb{R}$. The ith column of Λ is a primitive vector $\lambda_i = (\lambda_{1,i}, \ldots, \lambda_{n,i}) \in \mathbb{Z}^n$, called the *facet vector* associated to F_i.

The partition of Λ into $(I_n \mid S)$, where S is an $n\times(m-n)$ submatrix, is known as the *refined form* of the dicharacteristic matrix, and we term S the *refined submatrix*. Given any other vertex $F_{i_1} \cap \cdots \cap F_{i_n}$, Definition 3.2.6 implies that the corresponding columns $\lambda_{i_1}, \ldots \lambda_{i_n}$ form a basis for $\mathbb{Z}^n$ with determinant equal to ± 1.

As a consequence of the condition imposed by Definition 3.2.6, the kernel $K(l)$ of the dicharacteristic l is isomorphic to an $(m-n)$-dimensional subtorus of T^m. Furthermore, for any point $[t,p] \in \mathcal{Z}_{P^n} \cong T^m \times P^n/\sim$, where p belongs to a codimension-k face of P^n, we know that the restriction of l to $T(p) < T^m$ is monic, and fixes an $(m-k)$-dimensional kernel $K(p) \cong T^{m-k} < T^m$. Hence the intersection of $K(p)$ and $T(p)$ is trivial, for every $p \in P^n$. Since such $T(p)$ are the isotropy subgroups of

$\mathcal{Z}_{P^n}$, it follows that $K(l)$ acts freely on $\mathcal{Z}_{P^n}$ and thus defines a principal $K(l)$-bundle $\mathcal{Z}_{P^n} \longrightarrow M^{2n}$ over the quotient space M^{2n}.

We can view the base space M^{2n} as the quotient space,

$$T^n \times P^n / \sim, \tag{3.2.10}$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1^{-1}t_2 \in T(F(p))$. Again, the free action of T^n on $T^n \times P^n$ induces a T^n-action on M^{2n}, with quotient P^n. In [5, Construction 5.12], it is shown that this action is locally standard, and that the projection to P^n behaves as in Definition 3.2.1(ii). Hence the space M^{2n} is a quasitoric manifold.

For an automorphism ψ of T^n, two quasitoric manifolds M_1^{2n} and M_2^{2n} are ψ-*equivariantly diffeomorphic* if there is a diffeomorphism $g\colon M_1^{2n} \to M_2^{2n}$, such that $g(tx) = \psi(t)g(x)$ for all $t \in T^n$, and all $x \in M_1^{2n}$. This leads to the following important result.

Proposition 3.2.11. *[16, Proposition 1.8] Any quasitoric manifold M^{2n} over P^n is ψ-equivariantly diffeomorphic to one of the form (3.2.10).*

As a consequence, we will assume that every quasitoric manifold can be viewed as a quotient of the form (3.2.10).

To illustrate some of the concepts we have introduced in this section, consider the following example.

Example 3.2.12. Consider the n-simplex Δ^n embedded in $\mathbb{R}^{n+1}$, finely ordered as in Example 3.1.5 so that the origin is the initial vertex. The associated moment angle complex $\mathcal{Z}_{\Delta^n}$ is the $(2n+1)$-sphere S^{2n+1} (see e.g. [5, Example 6.7]).

The dicharacteristic function l is chosen so that the associated matrix takes the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix}, \tag{3.2.13}$$

Hence the kernel $K(l)$ is equal to the diagonal subgroup $D := \{(t,\ldots,t)\} < T^{n+1}$. The quotient of $\mathcal{Z}_{\Delta^n}$ under the action of D, is n-dimensional complex projective space $\mathbb{C}P^n$. The n-torus acts on the homogeneous coordinates $[z_1,\ldots,z_{n+1}]$ of $\mathbb{C}P^n$ by

$$(t_1,\ldots,t_n)\cdot[z_1,\ldots,z_{n+1}] = [t_1z_1,\ldots,t_nz_n,z_{n+1}], \tag{3.2.14}$$

with initial fixed point $[0,\ldots,0,1]$.

In Chapter 4 we will consider quasitoric manifolds known as Dobrinskaya towers, the class of which includes the well known examples of Bott towers [12] and bounded flag manifolds [47].

3.3 Cohomology of quasitoric manifolds

Davis and Januskiewicz proved that the integral cohomology ring of a quasitoric manifold is a quotient of the Stanley-Reisner ring of its associated polytope. Their result is described in this section with the aid of the dicharacteristic function. As an illustration, we compute the cohomology ring of complex projective space.

Consider a quasitoric manifold $\pi\colon M^{2n}\to P^n$. Let us assume that we have chosen a dicharacteristic function l, so that the kernel $K(l)$ is an $(m-n)$-torus of the form

$$\{(t_1^{\mu_{1,1}}\ldots t_{m-n}^{\mu_{1,m-n}},\ldots,t_1^{\mu_{m,1}}\ldots t_{m-n}^{\mu_{m,m-n}})\} < T^m, \tag{3.3.1}$$

for fixed integers $\mu_{i,j}$. The map of Lie algebras induced by l gives rise to a short exact sequence

$$0\longrightarrow \mathbb{Z}^{m-n}\xrightarrow{\kappa}\mathbb{Z}^m\xrightarrow{\lambda}\mathbb{Z}^n\longrightarrow 0, \tag{3.3.2}$$

and from (3.3.1) we can see that the $m\times(m-n)$ matrix describing the injection κ is given by

$$\begin{pmatrix} \mu_{1,1} & \cdots & \mu_{1,m-n}\\ \vdots & & \vdots\\ \mu_{m,1} & \cdots & \mu_{m,m-n}\end{pmatrix}. \tag{3.3.3}$$

For every facet of P^n, we can associate to the principal $K(l)$-bundle $\mathcal{Z}_{P^n}\longrightarrow M^{2n}$, a complex line bundle ρ_i given by

$$\mathcal{Z}_{P^n}\times_{K(l)}\mathbb{C}_i\longrightarrow M^{2n}, \tag{3.3.4}$$

where $K(l)$ acts on $\mathbb{C}_i$ by $z_i \mapsto t_1^{\mu_{i,1}} \ldots t_{m-n}^{\mu_{i,m-n}} z_i$, for $1 \leq i \leq m$. These bundles are known as the *facial bundles* of M^{2n}.

By Definition 3.2.1, the preimage of a facet $\pi^{-1}(F_i)$ is a $2(n-1)$-dimensional quasitoric *facial submanifold* M_i over F_i, with isotropy subgroup $T(F_i)$. The normal 2-plane bundle of the embedding $M_i \subset M^{2n}$ is denoted by ν_i, and its total space is isomorphic to $\mathcal{Z}_{F_i} \times_{K(l)} \mathbb{C}_i$ [9, p. 12]. Hence if we restrict ρ_i to M_i we recover ν_i.

Armed with the facial bundles, we can now describe the cohomology ring of the quasitoric manifold, which is generated by $u_1, \ldots, u_m$, where u_i is the first Chern class $c_1(\rho_i) \in H^2(M^{2n})$.

We express the cohomology ring of M^{2n} in terms of the *Stanley-Reisner ring* $\mathbb{Z}[P^n]$ of P^n, which takes the form

$$\mathbb{Z}[P^n] \cong \mathbb{Z}[u_1, \ldots, u_m]/\mathcal{I}, \tag{3.3.5}$$

where $\mathcal{I}$ is the ideal generated by monomials $u_{i_1} \ldots u_{i_j}$, which correspond to empty intersections $F_{i_1} \cap \cdots \cap F_{i_j} = \emptyset$ in P^n. A comprehensive treatment of the Stanley-Reisner ring and its applications in algebra and geometry is given in [52, Chapters II, III].

Theorem 3.3.6. *[16, Theorem 4.14] The integral cohomology ring $H^*(M^{2n})$ is given by*

$$\mathbb{Z}[u_1, \ldots, u_m]/(\mathcal{I} + \mathcal{J}), \tag{3.3.7}$$

where $\mathcal{J}$ is the ideal generated by the image of the elements

$$u_i + \lambda_{i,n+1} u_{n+1} + \cdots + \lambda_{i,m} u_m, \ \textit{for } 1 \leq i \leq n, \tag{3.3.8}$$

in the Stanley-Reisner algebra.

Note that the linear relations (3.3.8) can be ascertained from the rows of the refined submatrix S, and they imply that we only require the elements $u_{n+1}, \ldots, u_{n+m}$ in order to generate $H^*(M^{2n})$ multiplicatively.

Example 3.3.9. Consider $\mathbb{C}P^n$ as in Example 3.2.12. Complex projective space is a quasitoric manifold over the n-simplex Δ^n, which has $n+1$ facets. Hence the cohomology ring $H^*(\mathbb{C}P^n)$ is generated by the elements $u_1, \ldots, u_{n+1}$. The only monomial relation generating $\mathcal{I}$, is given by $u_1 \dots u_{n+1} = 0$, due to the empty intersection of facets $F_1 \cap \cdots \cap F_{n+1}$ in P^n.

With Λ as in (3.2.13), we obtain the linear relations, $u_1 = u_2 = \cdots = u_{n+1}$ in $\mathcal{J}$. On applying Theorem 3.3.6 we recover the well-known (see e.g. [22, Theorem 3.12]) description of the cohomology ring

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[u]/u^{n+1},$$

where we set $u = u_{n+1}$.

3.4 Tangential structures

We now consider quasitoric manifolds with regard to the tangential structures we introduced in Chapter 2. We describe a canonical stably complex structure on an omnioriented quasitoric manifold, again using complex projective space as an illustrative example. In conclusion we give a simple test that determines whether an omnioriented quasitoric manifold carries an SU-structure, and investigate the Chern classes of a quasitoric manifold in terms of its dicharacteristic matrix.

A choice of orientation on each facial bundle ρ_i, or equivalently each facial submanifold M_i, was termed an "omniorientation" of M^{2n} by Buchstaber and Ray [9]. This is of course equivalent to a choice of dicharacteristic function. It is now the convention to also include a choice of orientation on the manifold M^{2n} itself.

Definition 3.4.1. An *omniorientation* of a quasitoric manifold M^{2n} is a choice of orientation on M^{2n} and on each of the facial submanifolds M_i.

It follows that there are 2^{m+1} possible omniorientations on M^{2n}.

The next result will be crucial in our investigation of quasitoric manifolds in cobordism theory.

Proposition 3.4.2. *[6, Proposition 4.5] Any omnioriented quasitoric manifold has a canonical stably complex structure described by an isomorphism*

$$\tau(M^{2n}) \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \cdots \oplus \rho_m. \tag{3.4.3}$$

Note that even though there are 2^{m+1} possible omniorientations, due to the possibility that some may be homotopic to each other, there are not necessarily 2^{m+1} inequivalent stably complex structures on M^{2n}.

Example 3.4.4. We complete our investigation of $\mathbb{C}P^n$ (with omniorientation as specified in Example 3.2.12), by noting that the facial bundle ρ_i over $\mathbb{C}P^n$ is given by

$$S^{2n+1} \times_D \mathbb{C}_i \longrightarrow \mathbb{C}P^n, \tag{3.4.5}$$

Since D acts on $\mathbb{C}_i$ by multiplication by t for all i, by Example 2.4.4 we identify each ρ_i with $\overline{\zeta}_1$, the complex conjugate of the canonical line bundle ζ_1 over $\mathbb{C}P^n$. When $\overline{\zeta}_1$ over $\mathbb{C}P^n$ is restricted to each facial submanifold $\mathbb{C}P^{n-1}$ we recover the appropriate canonical line bundle $\overline{\zeta}_1$ over $\mathbb{C}P^{n-1}$, which is the normal bundle of the embedding $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

Finally, Proposition 3.4.2 gives the well-known (see e.g. [38, p170]) isomorphism,

$$\tau(\mathbb{C}P^n) \oplus \mathbb{R}^2 \cong \overline{\zeta}_1 \oplus \cdots \oplus \overline{\zeta}_1 \cong (n+1)\overline{\zeta}_1. \tag{3.4.6}$$

Proposition 3.4.2 illustrates the importance of the choice of orientation on each facial submanifold, which was determined by the choice of dicharacteristic function. If we wish to study quasitoric manifolds in complex cobordism theory, a choice of omniorientation has to be made so that we can describe the canonical stably complex structure on the manifold. Often in our thesis we will consider the effect of changing the omniorientation on a quasitoric manifold, so we now look at how such changes can be recorded by the machinery we have developed in this chapter.

Changing the orientation of a facial submanifold M_i, or equivalently a facet F_i, has the effect of choosing a different sign for the facet vector λ_i, therefore negating the ith column of the matrix Λ (3.2.8). If one or more of the first n columns of λ are negated, then the refined form of the matrix will be destroyed; we wish to avert this. Suppose we have changed the omniorientation so that columns $i_1, \ldots, i_j$ have been negated, for $1 \leq i_k \leq n$. Denote the resulting matrix Λ'. To restore the refined form, we need to multiply each column of Λ' by the $n \times n$ change of basis matrix that converts the first n columns back to I_n. It is straightforward to check that this has the effect of negating rows $i_1, \ldots, i_j$ of the refined submatrix S.

Henceforth, in order to preserve the refined form of the dicharacteristic matrix, if any of the orientations on a facet F_i are switched, for $1 \leq i \leq n$, the change will be encoded in Λ by negating the ith row of S. For a change of orientation on any other facet, we will continue to negate the appropriate column of S.

In terms of the cohomology ring of a quasitoric manifold M^{2n}, switching the orientation of M_i conjugates the complex line bundle ρ_i, and thus has the effect of negating the cohomology generator $u_i \in H^2(M^{2n})$.

To conclude this chapter, let us investigate the characteristic classes of $\tau(M^{2n})$. Following Proposition 3.4.2 we know that the stable tangent bundle τ^s of M^{2n} splits as a sum of complex line bundles, and so the total Chern class of τ^s is given by $c(\tau^s) = \prod_{i=1}^{m}(1 + u_i)$ in $H^*(M^{2n})$. This leads to the following result.

Corollary 3.4.7. *[6, Corollary 4.8] The ith Chern class $c_i(\tau^s) \in H^{2i}(M^{2n})$ is given by the ith elementary symmetric polynomial in $u_1, \ldots, u_m$, for $1 \leq i \leq n$, and $n < m$.*

Henceforth, we will denote the ith elementary symmetric polynomial in the variables $x_1, \ldots, x_j$ by $\sigma_i(x_1, \ldots, x_j)$.

Given a quasitoric manifold M^{2n}, there is a simple test we can use to check whether an omniorientation induces an SU-structure on M^{2n}.

Lemma 3.4.8. *[7] An omnioriented quasitoric manifold M^{2n} has an SU-structure precisely when every column sum of the refined dicharacteristic matrix Λ equals 1.*

Proof. By Corollary 3.4.7, the first Chern class is given by $c_1(\tau^s) = u_1 + \cdots + u_m$. Using the relations (3.3.8), we can rewrite this first elementary symmetric polynomial as

$$\left(1 - \sum_{i=1}^{n} \lambda_{i,n+1}\right) u_{n+1} + \cdots + \left(1 - \sum_{i=1}^{n} \lambda_{i,m}\right) u_m. \tag{3.4.9}$$

As prescribed by our definition in Section 2.3.2, for M^{2n} to have an SU-structure, we require that the expression (3.4.9) be equal to zero. This happens precisely when each column sum $\sum_{i=1}^{n} \lambda_{i,j}$ of Λ is equal to 1, for $n + 1 \leq j \leq m$. □

To conclude this chapter, we look at what further information can be extracted from the dicharacteristic matrix associated to a quasitoric manifold.

If a vertex $v \in P^n$ is the intersection of facets $F_{i_1}, \dots, F_{i_n}$, then the *sign* $\varepsilon(v)$ of v is the minor formed by the columns $\lambda_{i_1}, \dots, \lambda_{i_n}$ of the dicharacteristic matrix of M^{2n}. Due to the conditions imposed on the dicharacteristic function in Definition 3.2.6, it is straightforward to check that $\varepsilon(v) = \pm 1$.

The notion of the sign of a vertex was central to the work of Dobrinskaya [17], which we will investigate in Chapter 5. However, the idea was first introduced by Panov in [41]. He proved that the nth Chern class of the stable tangent bundle of a quasitoric manifold M^{2n} is given by the sum of the signs of its associated polytope P^n, that is

$$c_n(\tau^s(M^{2n})) = \sum_{v \in P^n} \varepsilon(v). \tag{3.4.10}$$

Remark 3.4.11. Since the first Chern class of a quasitoric manifold can be described by a sum of the columns of Λ (in some sense, a sum of 1×1 minors), and the nth Chern class is a sum of the $n \times n$ minors of Λ, it is natural to ask whether we can realise the other Chern classes of M^{2n} in terms of the matrix Λ. Unfortunately a sum of $i \times i$ minors, for $2 \leq i \leq n-1$, related to codimension-i faces of P^n does not give $c_i(\tau^s)$ in any obvious manner, even though this is analogous to the procedures for $c_1(\tau^s)$ and $c_n(\tau^s)$. Certainly the information that determines all the Chern classes, and hence the Chern numbers, of the quasitoric manifold is bound up in its dicharacteristic matrix, so it would be extremely useful if a simple method of extraction could be found.

Chapter 4

Dobrinskaya towers

In this chapter we study a family of quasitoric manifolds known as Dobrinskaya towers. These spaces are named after N. Dobrinskaya, who introduced them as part of her investigation into the classification of quasitoric manifolds [17]. Choi, Masuda and Suh [11] augmented these results, incorporating Dobrinskaya towers under the name "extended Bott manifolds", and they played an important role in Carter's study of loop spaces on quasitoric manifolds [10].

We have chosen to give a detailed treatment of Dobrinskaya towers for two reasons: they are part of a wider class of manifolds whose SU-structures are studied in Chapter 5, and in Chapter 6 we introduce a quaternionic analogue of the concept, whose role in quaternionic cobordism theory is the subject of Chapter 7.

We begin by constructing Dobrinskaya towers as a sequence of quasitoric manifolds over a product of simplices. Using the methods of Chapter 3, we compute their integral cohomology rings and describe the stably complex structure on an omnioriented tower. We detail a second construction of the tower using iterated bundles, and in conclusion we consider two special subcases of Dobrinskaya towers.

4.1 Constructing the towers

A Dobrinskaya tower can be built out of an iterated sequence of bundles; alternatively, we can realise each manifold in the sequence as a quasitoric manifold over a product of simplices $\Delta^{l_1} \times \cdots \times \Delta^{l_k}$. We begin with the latter viewpoint: let L denote the sum $l_1 + \cdots + l_k$, and write P^L for the product polytope $\Delta^{l_1} \times \cdots \times \Delta^{l_k}$. Each simplex Δ^{l_i} in the product P^L is finely ordered by (3.1.6), so we can use the procedure described in (3.1.10) to finely order P^L itself.

Let the facets of Δ^{l_i} be finely ordered as $F_{i,1}, \ldots, F_{i,l_i+1}$. The intersection of any l_i facets is a vertex of Δ^{l_i}, and the initial vertex $v_i = F_{i,1} \cap \cdots \cap F_{i,l_i}$ is the origin in $\mathbb{R}^{l_i}$.

In the product P^L, the facets are of the form $E_{i,j} := \Delta^{l_1} \times \cdots \times \Delta^{l_i - 1} \times F_{i,j} \times \Delta^{l_i+1} \times \cdots \times \Delta^{l_k}$, and they are finely ordered as

$$\begin{aligned} &E_{1,1}, \ldots, E_{1,l_1}, \ldots, E_{i,1}, \ldots, E_{i,l_i}, \ldots, E_{k,1}, \ldots, E_{k,l_k}, \\ &E_{1,l_1+1}, \ldots, E_{i,l_i+1}, \ldots, E_{k,l_k+1}. \end{aligned} \tag{4.1.1}$$

The first L facets intersect in the initial vertex $(v_1, \ldots, v_k)$, which is the origin in $\mathbb{R}^L$.

To assist any bamboozled readers, we provide an illustration of this ordering of facets in a low dimensional example.

Example 4.1.2. The facets of the polytope $\Delta^2 \times \Delta^1 \times \Delta^2$ are ordered

$$E_{1,1}, E_{1,2}, E_{2,1}, E_{3,1}, E_{3,2}, E_{1,3}, E_{2,2}, E_{3,3},$$

which we can write in full as,

$$\begin{aligned} &F_{1,1} \times \Delta^1 \times \Delta^2, F_{1,2} \times \Delta^1 \times \Delta^2, \Delta^2 \times F_{2,1} \times \Delta^2, \Delta^2 \times \Delta^1 \times F_{3,1}, \Delta^2 \times \Delta^1 \times F_{3,2}, \\ &F_{1,3} \times \Delta^1 \times \Delta^2, \Delta^2 \times F_{2,2} \times \Delta^2, \Delta^2 \times \Delta^1 \times F_{3,3}. \end{aligned}$$

Facets $E_{1,1}, E_{1,2}, E_{2,1}, E_{3,1}, E_{3,2}$ intersect in the initial vertex at the origin in $\mathbb{R}^5$.

We can give a complete description of the vertices of P^L. For any l_i-tuple $\{p_1, \ldots, p_{l_i}\} \subset \{1, \ldots l_i + 1\}$ the intersection of $\bigcap_{j=p_1}^{p_{l_i}} F_{i,j}$ is a vertex of Δ^{l_i}. Therefore in a product of simplices P^L, the vertices are given by the intersection

$$\bigcap_{i=1}^{k} \bigcap_{j=p_1}^{p_{l_i}} E_{i,j}, \tag{4.1.3}$$

as j ranges over all possible l_i-tuples $\{p_1, \ldots, p_{l_i}\} \subset \{1, \ldots l_i + 1\}$, for $1 \leq i \leq k$.

Example 4.1.4. Consider the product of simplices $\Delta^2 \times \Delta^1$. The intersections of any two facets of Δ^2 with any one facet of Δ^1 give the vertices of $\Delta^2 \times \Delta^1$, which we can list as

$$\begin{aligned} &E_{1,1} \cap E_{1,2} \cap E_{2,1}; \quad E_{1,1} \cap E_{1,2} \cap E_{2,2} \\ &E_{1,1} \cap E_{1,3} \cap E_{2,1}; \quad E_{1,1} \cap E_{1,3} \cap E_{2,2} \\ &E_{1,2} \cap E_{1,3} \cap E_{2,1}; \quad E_{1,2} \cap E_{1,3} \cap E_{2,2}. \end{aligned}$$

As usual, the initial vertex is given by $E_{1,1} \cap E_{1,2} \cap E_{2,1}$, the origin in $\mathbb{R}^3$.

In Example 3.2.12 we had that the moment angle complex $\mathcal{Z}_{\Delta^{l_i}}$ is the sphere S^{2l_i+1}. A simple application of Proposition 3.2.4 implies that $\mathcal{Z}_{PL}$ is a product of spheres $S^{2l_1+1} \times \cdots \times S^{2l_k+1}$, which we may embed in $\mathbb{C}^{l_1+1} \times \cdots \times \mathbb{C}^{l_k+1}$ via Lemma 3.2.5.

In order to describe the dicharacteristic function $l \colon T^{L+k} \to T^L$, we now define a *list* $(a_1(l_2), \ldots, a_{k-1}(l_k))$ of integral $l_i(i-1)$-vectors

$$a_{i-1}(l_i) = (a(1,i;1), \ldots, a(i-1,i;1), \ldots, a(1,i;l_i), \ldots, a(i-1,i;l_i)), \tag{4.1.5}$$

for $1 < i \leq k$, which is associated to a sequence $(l_1, \ldots l_k)$ of nonnegative integers.

We choose a dicharacteristic l so that the associated $L \times (L+k)$ dicharacteristic matrix Λ takes the form $(I_L \mid S)$, where I_L is the $L \times L$ identity matrix and S is of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ -a(1,2;1) & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ -a(1,2;l_2) & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ -a(1,i;1) & -a(2,i;1) & \cdots & -a(i-1,i;1) & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ -a(1,i;l_i) & -a(2,i;l_i) & \cdots & -a(i-1,i;l_i) & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ -a(1,k;1) & -a(2,k;1) & \cdots & -a(i-1,k;1) & -a(i,k;1) & \cdots & -a(k-1,k;1) & 1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ -a(1,k;l_k) & -a(2,k;l_k) & \cdots & -a(i-1,k;l_k) & -a(i,k;l_k) & \cdots & -a(k-1,k;l_k) & 1 \end{pmatrix}. \tag{4.1.6}$$

The ith column of Λ is the primitive vector λ_i assigned to the ith facet in the ordering (4.1.1) by the dicharacteristic.

Proposition 4.1.7. *The chosen dicharacteristic function l satisfies the condition imposed by Definition 3.2.6.*

Proof. It will suffice to prove that if the facets $E_{1,i_1}, \ldots, E_{1,i_{l_1}}, \ldots, E_{k,i_1}, \ldots, E_{k,i_{l_k}}$ intersect in a vertex of P^L, then the matrix V comprised of the columns

$$\lambda_{1,i_1}, \ldots, \lambda_{1,i_{l_1}}, \ldots, \lambda_{k,i_1}, \ldots, \lambda_{k,i_{l_k}}, \tag{4.1.8}$$

from Λ, has determinant equal to ± 1.

In the case $k = 1$, the dicharacteristic reduces to that considered in Example 3.2.12, and clearly the condition of Definition 3.2.6 is satisfied.

Our strategy now is to show that the dicharacteristic satisfies the condition of Definition 3.2.6 in the case $k = 2$; the argument is then easily seen to generalise to higher cases $k \geq 2$. For $k = 2$ there are three scenarios to check, which cover all possible situations.

First, let us consider the $(l_1 + l_2) \times (l_1 + l_2 + 2)$ dicharacteristic matrix Λ

$$\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -a(1,2;1) & 1 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & -a(1,2;2) & 1 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & -a(1,2;l_2) & 1
\end{pmatrix}, \tag{4.1.9}$$

in the case when $k = 2$.

According to the ordering (4.1.1) of facets, the first $l_1 + l_2$ columns are the facet vectors $\lambda_{1,1}, \ldots, \lambda_{1,l_1}, \lambda_{2,1}, \ldots, \lambda_{2,l_2}$, which are simply $e_1, \ldots, e_{l_1+l_2}$, where e_i denotes the ith standard basis vector in $\mathbb{R}^{l_1+l_2}$; hence they form an identity matrix $I_{l_1+l_2}$. The final two columns of Λ are the vectors λ_{1,l_1+1} and λ_{2,l_2+1}.

According to our description (4.1.3), the intersection of a choice of any l_1 facets from $E_{1,1}, \ldots, E_{1,l_1+1}$ and any l_2 facets from $E_{2,1}, \ldots, E_{2,l_2+1}$ results in a vertex of $\Delta^{l_1} \times \Delta^{l_2}$. We denote the set containing our chosen facets by

$$C = \{E_{1,i_1}, \ldots, E_{1,i_{l_1}}, E_{2,i_1}, \ldots, E_{2,i_{l_2}}\}.$$

To prove the proposition for all possible sets C, it will suffice to consider 3 cases. In the first instance, if $C = \{E_{1,1}, \ldots, E_{1,l_1}, E_{2,1}, \ldots, E_{2,l_2}\}$ we determine the initial vertex, and the matrix V comprised of the appropriate columns, as described in (4.1.8), is simply the identity matrix. Hence V has determinant equal to 1.

The second case is when the final facet E_{2,l_2+1} is in C. To begin determining a vertex we must now make $l_2 - 1$ further choices for the set C from the facets $E_{2,1}, \ldots, E_{2,l_2}$. If we choose every such facet except E_{2,i_2}, for some $1 \leq i_2 \leq l_2$, we can form a matrix V' whose columns are given by the set

$$\begin{aligned} & \{\lambda_{2,1}, \ldots, \lambda_{2,i_2-1}, \lambda_{2,i_2+1}, \ldots, \lambda_{2,l_2}, \lambda_{2,l_2+1}\} \\ = & \{e_{l_1+1}, \ldots, e_{l_1+i_2-1}, e_{l_1+i_2+1}, \ldots, e_{l_1+l_2}, \lambda_{2,l_2+1}\}. \end{aligned}$$

By (4.1.9) we have $\lambda_{2,l_2+1} = (0, \ldots, 0, 1, \ldots 1)$, so using elementary column operations in V', we can convert the column λ_{2,l_2+1} into $e_{l_1+i_2}$.

To finally determine our vertex, we must now make l_1 choices from $E_{1,1}, \ldots, E_{1,l_1+1}$ to complete the set C; there are two further subcases to consider.

If $E_{1,l_1+1} \notin C$, the corresponding matrix V is simply the identity matrix $I_{l_1+l_2}$, with the column vector $e_{l_1+i_2}$ switched to the final column. Hence V has determinant equal to -1.

If $E_{1,l_1+1} \in C$, then the matrix V consists of columns

$$\begin{aligned} & \{\lambda_{1,1}, \ldots, \lambda_{1,i_1-1}, \lambda_{1,i_1+1}, \ldots, \lambda_{1,l_1}, \lambda_{2,1}, \ldots, \lambda_{2,i_2-1}, \lambda_{2,i_2+1}, \ldots, \lambda_{2,l_2}, \lambda_{1,l_1+1}, e_{l_1+i_2}\} \\ = & \{e_1, \ldots, e_{i_1-1}, e_{i_1+1}, \ldots, e_{l_1}, e_{l_1+1}, \ldots, e_{l_1+i_2-1}, e_{l_1+i_2+1}, \ldots, e_{l_1+l_2}, \lambda_{1,l_1+1}, e_{l_1+i_2}\}, \end{aligned}$$

for some $1 \leq i_1 \leq l_1$. Again, it is clear that we can use elementary column operations to convert the vector λ_{1,l_1+1} into e_{i_1}. Then V is simply the identity matrix $I_{l_1+l_2}$, with the column vectors $e_{i_1}, e_{l_1+i_2}$ switched to the final two columns respectively, hence V has determinant equal to 1.

The final case is for a vertex determined by a set C such that $E_{1,l_1+1} \in C$, but $E_{2,l_1+1} \notin C$. Using similar reasoning to the previous case, we can show that the matrix V must be formed from columns

$$\{e_1, \ldots, e_{i_1-1}, e_{i_1+1}, \ldots, e_{l_1}, e_{l_1+1}, \ldots, \ldots, e_{l_1+l_2}, \lambda_{1,l_1+1}\}, \tag{4.1.10}$$

and using elementary column operations we can convert λ_{1,l_1+1} into e_{i_1}. Then V is $I_{l_1+l_2}$ with the column vector e_{i_1} shifted to the final column, hence V has determinant equal to -1.

For $k \geq 2$, we can follow a similar procedure. To determine a vertex of $P^L = \Delta^{l_1} \times \cdots \times \Delta^{l_k}$ we begin by making l_k choices from the columns of the dicharacteristic matrix (4.1.6), which correspond to facets $E_{k,1}, \ldots, E_{k,l_k+1}$. Again, elementary column operations can be used to reduce the chosen vectors to a collection $S_k = \{e_{k,i_1}, \ldots, e_{k,i_{l_k}}\}$ of distinct standard basis vectors $e_{k,i_j} \in \mathbb{R}^L$. We continue as in the $k = 2$ case, making l_j choices from $E_{j,1}, \ldots, E_{j,l_j+1}$, for $j = k-1, \ldots, 1$ in turn, to determine a vertex in P^L. After each choice of l_j vectors is made, elementary column operations can be used to reduce the vectors to a collection of distinct standard basis vectors $S_j = \{e_{j,i_1}, \ldots, e_{j,i_{l_j}}\}$; note that each element of the set S_i is also distinct from every element in the set S_j, for all i and all j. Therefore any choice of vertex in P^L ultimately determines a set C of $L = l_1 + \cdots + l_k$ distinct standard basis vectors in $\mathbb{R}^L$, and so the matrix whose columns are comprised of the elements of the set C has determinant ± 1, as required. □

Given our choice of dicharacteristic l, the k-dimensional kernel $K(l)$ is equal to

$$\begin{aligned}\{(t_1, \ldots, t_1, t_1^{-a(1,2;1)} t_2, \ldots, t_1^{-a(1,2;l_1)} t_2, \ldots, t_1^{-a(1,i;1)} \ldots t_{i-1}^{-a(i-1,i;1)} t_i, \ldots \\ \ldots, t_1^{-a(1,i;l_i)} \ldots t_{i-1}^{-a(i-1,i;l_i)} t_i, \ldots, t_1^{-a(1,k;1)} \ldots t_{k-1}^{-a(k-1,k;1)} t_k, \ldots \\ \ldots, t_1^{-a(1,k;l_k)} \ldots t_{k-1}^{-a(k-1,k;l_k)} t_k, t_1^{-1}, \ldots, t_k^{-1}) : t_i \in T, 1 \leq i \leq k\} < T^{L+k},\end{aligned} \tag{4.1.11}$$

and the quotient of $\mathcal{Z}_{P^L} = S^{2l_1+1} \times \cdots \times S^{2l_k+1}$ under the free, effective action of $K(l)$ is a $2L$-dimensional manifold DO_k. Since by Proposition 4.1.7 our dicharacteristic function satisfies the conditions of Definition 3.2.6, it follows from Proposition 3.2.11 that DO_k is a quasitoric manifold.

Definition 4.1.12. The sequence of quasitoric manifolds $(DO_k : k \leq n)$ arising from a list $(a_1(l_2), \ldots, a_{n-1}(l_n))$ associated to $(l_1, \ldots, l_n)$, is a *Dobrinskaya tower of height* n (note that n may be infinite). The individual quasitoric manifold DO_k is known as the *kth stage* of the tower, for $1 \leq k \leq n$.

For any two adjacent stages DO_k and DO_{k-1} in the tower, we have a projection map $\pi_k\colon DO_k \to DO_{k-1}$, defined as follows. Given a point of DO_k, which is the equivalence class

$$[z_{1,1}, \dots, z_{1,l_1}, \dots, z_{k,1}, \dots, z_{k,l_k}, z_{1,l_1+1}, \dots z_{k,l_k+1}],$$

of a point in $\mathcal{Z}_{P^L}$ under the action of the kernel (4.1.11), the map π_k is given by projecting onto the equivalence class

$$[z_{1,1}, \dots, z_{1,l_1}, \dots, z_{k-1,1}, \dots, z_{k-1,l_{k-1}}, z_{1,l_1+1}, \dots z_{k-1,l_{k-1}+1}],$$

of a point in DO_{k-1}.

Finally we note that $(t_1, \dots, t_L) \in T^{L+k}/K(l) \cong T^L$ acts on the equivalence class $[z_{1,1} \dots, z_{k,l_k}, z_{1,l_1+1}, \dots, z_{k,l_k+1}]$ of DO_k by

$$(t_1, \dots, t_L) \cdot [z_{1,1} \dots, z_{k,l_k}, z_{1,l_1+1}, \dots, z_{k,l_k+1}] = [t_1 z_{1,1} \dots, t_L z_{k,l_k}, z_{1,l_1+1}, \dots, z_{k,l_k+1}], \tag{4.1.13}$$

and the initial fixed point of DO_k is $[0, \dots, 0, 1, \dots, 1]$. Label this T^L-action on DO_k by α_k.

4.2 Cohomology of Dobrinskaya towers

As described in Section 3.3, the integral cohomology ring $H^*(DO_k)$ is determined by two sets of linear and monomial relations. We begin here by computing the monomial relations that arise from the Stanley-Reisner ring $\mathbb{Z}[P^L]$. Let $u_{i,j} \in H^2(DO_k)$ be the first Chern class $c_1^H(\rho_{i,j})$ of the facial bundle $\rho_{i,j}$ associated to the facet $E_{i,j}$. In a simplex Δ^{l_i}, we have that $\{F_{i,1}, \dots, F_{i,l_i+1}\}$ is the only collection of nonintersecting facets. It follows that in P^L, we have

$$E_{i,1} \cap \dots \cap E_{i,l_i+1} = \emptyset,$$

for $1 \le i \le k$. Therefore the Stanley-Reisner ring $\mathbb{Z}[P^L]$ is isomorphic to

$$\mathbb{Z}[u_{1,1}, \dots, u_{1,l_1}, \dots, u_{k,1}, \dots, u_{k,l_k}, u_{1,l_1+1}, \dots, u_{k,l_k+1}]/\mathcal{I}_k, \tag{4.2.1}$$

where $\mathcal{I}_k$ is the ideal

$$(u_{i,1}\dots u_{i,l_i+i} : 1 \le i \le k).$$

The second set of relations in the cohomology ring are linear and arise from the dicharacteristic function. We can use the matrix (4.1.6) to read off the following equations

$$u_{i,j} = a(1,i;j)u_{1,l_1+1} + \cdots + a(i-1,i;j)u_{i-1,l_{i-1}+1} - u_{i,l_i+1}, \tag{4.2.2}$$

in $H^*(DO_k)$, for $1 \le j \le l_i$, and for $1 \le i \le k$. Hence the elements $u_{1,l_1+1}, \dots, u_{k,l_k+1}$ suffice to generate the cohomology ring multiplicatively; let us relabel these elements $v_1, \dots, v_k$, respectively.

Proposition 4.2.3. *The integral cohomology ring $H^*(DO_k)$ is isomorphic to*

$$\mathbb{Z}[v_1, \dots, v_k]/(\mathcal{I}_k + \mathcal{J}_k), \tag{4.2.4}$$

where $\mathcal{I}_k + \mathcal{J}_k$ is the ideal generated by

$$\begin{aligned} v_i(a(1,i;1)v_1 + &\cdots + a(i-1,i;1)v_{i-1} - v_i)\dots \\ &\dots(a(1,i;l_i)v_1 + \cdots + a(i-1,i;l_i)v_{i-1} - v_i), \text{ for } 1 \le i \le k. \end{aligned} \tag{4.2.5}$$

Proof. Apply Theorem 3.3.6. The product $u_{i,1}\dots u_{i,l_i}$ of linear relations (4.2.2) becomes zero when multiplied by $u_{i,l_i+1} := v_i$ in $\mathbb{Z}[P^L]$, yielding the required relations (4.2.5). □

4.3 Stably complex structure

To allow us to describe the facial bundles of DO_k, we first define complex line bundles

$$\mu_i \colon S^{2l_1+1} \times \cdots \times S^{2l_k+1} \times_{K(l)} \mathbb{C} \longrightarrow DO_k, \tag{4.3.1}$$

for $1 \le i \le k$, where the action of the kernel $K(l)$, as described in (4.1.11), on $\mathbb{C}$ is defined by $z \mapsto t_i^{-1}z$.

The action of $K(l)$ on the moment angle complex $\mathcal{Z}_{P^L} \cong S^{2l_1+1} \times \cdots \times S^{2l_k+1}$ is equivariantly diffeomorphic to that of $(\mathbb{C}_\times)^k$ on $(\mathbb{C}^{l_1+1} \setminus 0) \times \cdots \times (\mathbb{C}^{l_k+1} \setminus 0)$ by

$$\begin{aligned}
&(t_1, \ldots, t_k) \cdot (z_{1,1}, \ldots, z_{1,l_1}, \ldots z_{k,1}, \ldots, z_{k,l_k}, z_{1,l_1+1} \ldots z_{k,l_k+1}) = \\
&(t_1 z_{1,1}, \ldots, t_1 z_{1,l_1}, t_1^{-a(1,2;1)} t_2 z_{2,1}, \ldots, t_1^{-a(1,2;l_2)} t_2 z_{2,l_2}, \ldots \\
&\ldots, t_1^{-a(1,k;1)} \ldots t_{k-1}^{-a(k-1,k;1)} t_k z_{k,1}, \ldots, t_1^{-a(1,k;l_k)} \ldots t_{k-1}^{-a(k-1,k;l_k)} t_k z_{k,l_k}, \\
&t_1^{-1} z_{1,l_1+1}, \ldots, t_k^{-1} z_{k,l_k+1}),
\end{aligned} \tag{4.3.2}$$

where each complex coordinate $z_{i,j}$ is associated to a facet $E_{i,j}$ of P^L. Following our definition in (3.3.4), we can now identify the facial bundles $\rho_{i,j}$ associated to each facet $E_{i,j}$ as

$$\begin{aligned}
\rho_{i,j} &= \mu_1^{a(1,i;j)} \otimes \cdots \otimes \mu_{i-1}^{a(i-1,i;j)} \otimes \overline{\mu}_i, \qquad (4.3.3)\\
\rho_{i,l_i+1} &= \mu_i,
\end{aligned}$$

for $1 \leq j \leq l_i$ and $1 \leq i \leq k$. To simplify these expressions, let μ_0 be the trivial complex line bundle $\mathbb{C}$, and define

$$\alpha_{i-1,j} := \mu_1^{a(1,i;j)} \otimes \cdots \otimes \mu_{i-1}^{a(i-1,i;j)}.$$

After applying Proposition 3.4.2 we arrive at the following.

Proposition 4.3.4. *The stably complex structure on DO_k induced by the chosen omniorientation is described by an isomorphism*

$$\tau(DO_k) \oplus \mathbb{R}^{2k} \cong \bigoplus_{i=1}^{k} \left(\mu_i \oplus (\overline{\mu}_i \otimes (\bigoplus_{j=1}^{l_i} \alpha_{i-1,j})) \right). \tag{4.3.5}$$

4.4 Dobrinskaya towers as iterated bundles

In this section we consider the second construction of a Dobrinskaya tower, built out of an iterated sequence of bundles. Using the results of Section 2.5 we compute the E-cohomology of each stage in the tower. Finally we check that the two constructions agree.

Suppose we have a list $(a_1(l_2), \ldots, a_{n-1}(l_n))$ associated to a sequence of integers $(l_1, \ldots l_n)$. Our alternative construction of the Dobrinskaya tower is inductive, beginning with the assumption that for any integer $k \geq 1$ we have constructed a $(k-1)$th stage DO'_{k-1} as a smooth $2(l_1 + \cdots + l_{k-1})$-dimensional manifold, 2-generated by $y_i^H \in H^2(DO'_{k-1})$, and carrying line bundles γ_i such that $c_1^H(\gamma_i) = y_i^H$, for $1 \leq i \leq k-1$. Denote by $\beta_{i-1,j}$ the tensor product bundle,

$$\gamma_1^{a(1,i;j)} \otimes \cdots \otimes \gamma_{i-1}^{a(i-1,i;j)}, \tag{4.4.1}$$

for $1 \leq j \leq l_i$ and $1 \leq i \leq k$, and define the *ith bundle* of the construction

$$\beta(a_{i-1}) := \beta_{i-1,1} \oplus \cdots \oplus \beta_{i-1,l_i}. \tag{4.4.2}$$

Then DO'_k is defined to be the total space of $\mathbb{C}P(\beta(a_{k-1}) \oplus \mathbb{C})$, the projectivisation of the direct sum of $\beta(a_{k-1})$ and a trivial complex line bundle $\mathbb{C}$.

We define DO'_0 to be the space consisting of a single point so that the first bundle is trivial; it follows that the next stage DO'_1 is $\mathbb{C}P^{l_1}$. Lemma 2.5.4 implies that DO'_k has 2-generators y_i^H for $1 \leq i \leq k$, where y_k^H is the first Chern class $c_1^H(\gamma_k)$ of the canonical line bundle γ_k over DO'_k; moreover, this result provides a description of the cohomology ring $E^*(DO'_{k+})$, which is compatible with Proposition 4.2.3 when E is the integral Eilenberg-Mac Lane spectrum H.

Proposition 4.4.3. *The E_*-algebra $E^*(DO'_{k+})$ is isomorphic to*

$$E_*[y_1^E, \ldots, y_k^E]/\mathcal{K}_k \tag{4.4.4}$$

where $\mathcal{K}_k$ is the ideal

$$((y_i^E)(y_i^E - c_1^E(\beta_{i-1,1})) \ldots (y_i^E - c_1^E(\beta_{l_{i-1,l_i}})) : 1 \leq i \leq k). \tag{4.4.5}$$

As a consequence of the discussion that followed Lemma 2.5.4, we have projections $\pi_k \colon DO'_k \to DO'_{k-1}$, sections $\omega_k \colon DO'_{k-1} \to DO'_k$ and quotient maps $\vartheta_k \colon DO'_k \to T(\beta(a_{k-1}))$.

We claim that DO'_k is in fact the kth stage of a Dobrinskaya tower.

Proposition 4.4.6. *[10, Prop. 4.2.3] Given a Dobrinskaya tower $(DO_k : k \leq n)$, there exists a diffeomorphism $\theta_k \colon DO_k \to DO'_k$, for any $1 \leq k \leq n$, which pulls back γ_i to μ_i for all $1 \leq i \leq k$.*

Proof. To simplify the algebra in what follows, we will change the omniorientation on DO_k by negating the first L columns of its dicharacteristic matrix Λ. It is straightforward to check that this has the effect of switching the last k coordinates of the kernel $K(l)$ described by (4.1.11) from $(t_1^{-1}, \ldots, t_k^{-1})$ to $(t_1, \ldots, t_k)$. Such a change is permissible, since any change in omniorientation leaves the underlying smooth structure of a quasitoric manifold unaffected.

The proof of the proposition is by induction. For the base case, simply take $k = 0$; both DO_0 and DO'_0 are a point, and then μ_0 and γ_0 are simply trivial complex line bundles $\mathbb{C}$.

For $k \geq 0$ we will assume that we have the required diffeomorphism θ_k. Our strategy is to pullback the individual line bundles $\beta_{k,j}$ along θ_k to allow us to identify the pullback of their direct sum $\theta^*(\beta(a_k)) = \theta^*(\beta_{k,1} \oplus \cdots \oplus \beta_{k,l_k})$. This allows a detailed description of the fibres of $\mathbb{C}P(\theta_k^*\beta(a_k) \oplus \mathbb{C})$ as the total space of a projective bundle, which we then identify with DO_{k+1} by studying the action of $K(l)$ on $\mathcal{Z}_{PL}$.

To begin, the inductive hypothesis implies that $\theta_k^*(\beta_{k,j})$ is the complex line bundle

$$S^{2l_1+1} \times \cdots \times S^{2l_k+1} \times_{K(l)} \mathbb{C} \longrightarrow DO_k,$$

where $K(l)$ acts on $\mathbb{C}$ by $z \mapsto t_1^{-a(1,k+1;j)} \ldots t_k^{-a(k,k+1;j)} z$. It follows that in the $\mathbb{C}^{l_{k+1}+1}$ fibre of the direct sum bundle $\theta_k^*\beta(a_k) \oplus \mathbb{C}$, we identify $(z_1, \ldots, z_{l_{k+1}+1}) \in \mathbb{C}^{l_{k+1}+1}$ with

$$\begin{aligned}&(t_1^{-a(1,k+1;1)} \ldots t_k^{-a(k,k+1;1)} z_1, \ldots,\\ &t_1^{-a(1,k+1;l_k)} \ldots t_k^{-a(k,k+1;l_k)} z_{l_{k+1}}, z_{l_{k+1}+1}).\end{aligned}$$

Taking lines in each of the $\mathbb{C}^{l_{k+1}+1}$ fibres, that is, identifying

$$(z_1, \ldots, z_{l_{k+1}+1}) \sim z(z_1, \ldots, z_{l_{k+1}+1}) \qquad (4.4.7)$$

for some $z \in \mathbb{C}$, we obtain the projectivisation $\mathbb{C}P(\theta_k^*\beta(a_k) \oplus \mathbb{C}) \cong \theta_k^*(DO'_{k+1})$.

We aim to show that this projectivisation is DO_{k+1}. In (4.3.2) we described the $K(l)$-action on $\mathcal{Z}_{PL} = S^{2l_1+1} \times \cdots \times S^{2l_{k+1}+1}$, whose quotient is DO_{k+1}. Taking account of the change of omniorientation, $K(l)$ acts on $(z'_1, \ldots, z'_{l_{k+1}+1})$ in $S^{2l_{k+1}+1} \cong$

$(\mathbb{C}^{l_{k+1}+1} \setminus 0)$ by

$$(t_1^{-a(1,k+1;1)} \dots t_k^{-a(k,k+1;1)} t_{k+1} z'_1, \dots,$$
$$t_1^{-a(1,k+1;l_k)} \dots t_k^{-a(k,k+1;l_k)} t_{k+1} z'_{l_{k+1}}, t_{k+1} z'_{l_{k+1}+1}).$$

Setting $z = t_{k+1} \in \mathbb{C}$ in (4.4.7), we see that our above description of $\theta_k^*(DO'_{k+1})$ as $\mathbb{C}P(\theta_k^*\beta(a_k) \oplus \mathbb{C})$ is precisely DO_{k+1}. Therefore DO_{k+1} is the pullback of DO'_{k+1} along θ_k, and the required diffeomorphism θ_{k+1} is given by the resulting map between the total spaces of these projective bundles. Finally we have that μ_i is the pull back $\theta_{k+1}^*(\gamma_i)$ for $1 \leq i \leq k+1$. □

As a consequence of Proposition 4.4.6 we will denote the kth stage of a Dobrinskaya tower by DO_k, regardless of how it has been constructed.

4.5 Special cases

To conclude this chapter, we focus on two special subfamilies of Dobrinskaya tower that will feature throughout the rest of our thesis: the Bott tower and the bounded flag manifold. We use the results of Section 2.3.3 to describe stably complex structures on such manifolds.

Assume we have the sequence of manifolds $(DO_k : k \leq n)$, arising from a list $(a_1(l_2), \dots, a_{n-1}(l_n))$ associated to $(l_1, \dots, l_n)$.

Definition 4.5.1. A *Bott tower of height* n is a Dobrinskaya tower in which $l_i = 1$ for all $1 \leq i \leq n$.

To distinguish the Bott tower, we will denote the kth stage by B^k. Again for convenience we set B^0 to be the one-point space, and B^1 is $\mathbb{C}P^1$. At the second stage $B^2 = \mathbb{C}P(\gamma^{a(1,2;1)} \oplus \mathbb{C})$, if $a(1,2;1)$ is even, B^2 is $\mathbb{C}P^1 \times \mathbb{C}P^1$, while if $a(1,2;1)$ is odd, B^2 is a Hirzebruch surface $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, that is, a connected sum of complex projective planes with opposite orientations (see [5, Example 5.64], [12, page 10]).

Bott towers were first introduced in the context of algebraic geometry by Grossberg and Karshon [21]. The construction was translated into algebraic topology by

Civan and Ray [12], who calculated the real K-theory of particular families of Bott towers, and enumerated their possible stably complex structures. More recently Masuda and Panov [34] have studied Bott towers that admit certain interesting circle actions.

Each stage $B^k = \mathbb{C}P(\beta_{k-1,1} \oplus \mathbb{C})$ is a $\mathbb{C}P^1$ bundle over B^{k-1}. Since $\mathbb{C}P^1$ is homeomorphic to S^2, we may equally consider the kth stage of the tower as the total space of a 2-sphere bundle $S(\beta_{k-1,1} \oplus \mathbb{R})$.

Proposition 4.5.2. *There is an isomorphism*

$$\tau(B^k) \oplus \mathbb{R} \cong \mathbb{R} \oplus \left(\bigoplus_{i=1}^{k} \beta_{i-1,1} \right), \tag{4.5.3}$$

which determines a bounding stably complex structure on B^k.

Proof. The isomorphism follows from Proposition 2.3.7. The structure bounds since it extends to the 3-disc bundle $D(\beta_{k-1,1} \oplus \mathbb{R})$. □

It turns out that this stably complex structure is isomorphic to the structure on DO_k described in Proposition 4.3.4, which was induced by our chosen omniorientation. To see this, begin by setting $l_i = 1$, for all i, in the stably complex structure (4.3.5). We obtain

$$\tau(B^k) \oplus \mathbb{R}^{2k} \cong \bigoplus_{i=1}^{k} \left(\mu_i \oplus (\overline{\mu}_i \otimes (\alpha_{i-1,1}))\right). \tag{4.5.4}$$

Now, Civan and Ray [12, p. 31] exhibit an isomorphism

$$\mathbb{C} \oplus \beta_{i-1,1} \cong \gamma_{i+1} \oplus (\overline{\gamma}_{i+1} \otimes \beta_{i-1,1}),$$

which, coupled with the diffeomorphism $\theta_k \colon \alpha_{i-1,1} \mapsto \beta_{i-1,1}$ of Proposition 4.4.6, reduces (4.5.4) to

$$\tau(B^k) \oplus \mathbb{R}^{2k} \cong \mathbb{C}^k \oplus \left(\bigoplus_{i=1}^{k} \beta_{i-1,1} \right). \tag{4.5.5}$$

Clearly this is isomorphic to the stably complex structure induced by the isomorphism (4.5.3). Note that this is a special case: choosing a different omniorientation on B^k can lead to a stably complex structure on the Bott tower, which is not isomorphic to the bounding structure induced by Proposition 2.3.7.

In addition to the Bott tower, we can specialise further.

Definition 4.5.6. Consider the Bott tower $(B^k : k \leq n)$ arising from the list $(a_1(1), \ldots, a_{n-1}(1))$ in which $a_i(1) = (0, \ldots, 0, 1)$ for all i. Then each stage B^k is a *bounded flag manifold*.

The bounded flag manifold B^n is the space of all *bounded flags* V in $\mathbb{C}^{n+1}$; that is, the set of all sequences $V = \{V_1 \subset V_2 \subset \cdots \subset V_n \subset V_{n+1} = \mathbb{C}^{n+1}\}$, in which V_i contains the coordinate subspace $\mathbb{C}^{i-1}$ spanned by the first $i-1$ standard basis vectors of $\mathbb{C}^{n+1}$, for $2 \leq i \leq n$ (see e.g. [5, Example 5.36]). In our construction, these manifolds are of the form $B^k = \mathbb{C}P(\gamma_{k-1} \oplus \mathbb{C})$. In this guise they were studied by Ray [47], who proved that bounded flag manifolds, and their quaternionic analogues, play an important role in cobordism theory. We will return to the latter case in Chapter 7, where we extend Ray's study of the quaternionic analogue of $\mathbb{C}P(\gamma_{k-1} \oplus \mathbb{C})$ in quaternionic cobordism theory.

Chapter 5

Quasitoric manifolds with SU-structure

Suppose we have a stably complex manifold M^n. Then there is an isomorphism $\tau(M^n) \oplus \mathbb{R}^{2k-n} \cong \theta$, for some k-dimensional complex vector bundle θ, as prescribed by Definition 2.3.1. We will write $[M^n, \theta]$ for the equivalence class of a stably complex manifold M^n under the bordism relation. Then the set of all such classes, equipped with the two operations of disjoint union and Cartesian product, forms the *complex cobordism ring* MU_*, the coefficient ring described in Example 2.2.4.

By virtue of Proposition 3.4.2, every omnioriented quasitoric manifold is stably complex, so such manifolds are perfect candidates for investigation via the methods of complex cobordism.

In Chapter 1 we discussed one of the first results that arose from this research: the discovery by Buchstaber and Ray [8] that every $2n$-dimensional stably complex manifold is complex cobordant to a disjoint union of products of quasitoric manifolds. By developing the notion of a connected sum of omnioriented quasitoric manifolds, this result was refined to give a quasitoric basis for MU_* in dimensions > 2.

Theorem 5.0.1. *[6, Theorem 5.9] In dimensions > 2, every complex cobordism class contains a quasitoric manifold M^{2n}, which is necessarily connected. The stably complex structure on M^{2n} is induced by an omniorientation, ensuring it is compatible with the action of the torus.*

More recently Buchstaber, Panov and Ray have made the following conjecture.

Conjecture 5.0.2. *[7] A quasitoric manifold M^{2n} whose omniorientation induces an SU-structure on the stable tangent bundle of M^{2n}, is complex cobordant to zero in MU_*, for all SU-structures induced by omniorientations on M^{2n}.*

If the omniorientation on a quasitoric manifold M^{2n} induces such an SU-structure, we will refer to M^{2n} as a *quasitoric SU-manifold*.

To put Conjecture 5.0.2 into context, we first note that any quasitoric SU-manifold M^{2n} must lie in the image of the map $F\colon MSU_* \to MU_*$, which simply forgets the SU-structure on M^{2n}. We wish to investigate whether M^{2n} bounds in the *complex* cobordism ring, which is not the same as addressing the question of whether M^{2n} bounds in MSU_*. This is due to the fact that the kernel of F is comprised of the torsion elements of MSU_*, which appear in $MSU_{8n+1} \cong MSU_{8n+2}$ for all n; this torsion is comprised of p copies of $\mathbb{Z}/2$, where p is the number of partitions of n [13]. Therefore any $M^{2n} \in \mathit{Tors}MSU_*$ is mapped by F to zero in MU_*, regardless of whether it is zero in MSU_*. Of course, in such cases the conjecture is trivially true.

On the other hand, the image of the injection $MSU_*/\mathit{Tors} \to MU_*$ realises a significant portion of MU_*. Conner and Floyd described this image in [13, Theorem 19.1], by first introducing the subgroup $W_{2n} \subset MU_{2n}$ of stably complex manifolds whose Chern numbers vanish if they are divisible by the square of their first Chern class $c_1^2(\tau^s)$. Coupled with a boundary operator $\partial\colon W_{2n} \to W_{2n-2}$, they obtain a chain complex (W_*, ∂). The image of $MSU_{2n}/\mathit{Tors} \to MU_{2n}$ is given by the cycle group $Z(W_{2n}, \partial)$, when $n \neq 8i+4$, while the image is described by the boundary group $B(W_{2n}, \partial)$, when $n = 8i+4$. In particular, the former group $Z(W_{2n}, \partial)$ is precisely those $2n$-dimensional stably complex manifolds whose Chern numbers vanish if they are divisible by their first Chern class $c_1(\tau^s)$ [13, (6.4)]. Therefore in the vast majority of cases, the conjecture is far from trivial. Note though, that when $n \neq 8k+2$ for some k, the problem is in fact equivalent to deciding whether M^{2n} bounds in MSU_* or not.

Now let us consider the evidence supporting Conjecture 5.0.2. For some ring R, the generalised elliptic genus $T_{ell_*}\colon MU_* \to R$, is a particular type of Hirzebruch genus (see e.g. [25]; for applications in toric topology see [5, Section 5.4]). In [7] the authors prove that T_{ell_*} is such that for any quasitoric SU-manifold M^{2n}

$$T_{ell_*}([M^{2n}, \theta]) = 0. \tag{5.0.3}$$

Furthermore, T_{ell_*} is an isomorphism on cobordism classes $[M^{2n}, \theta]$ when $n < 5$. It follows that Conjecture 5.0.2 is true in dimensions < 10. However, the genus T_{ell_*} is only an epimorphism for $n \geq 5$, and so (5.0.3) alone will not suffice to prove the conjecture, since $[M^{2n}, \theta]$ could lie in the kernel of T_{ell_*}.

Nevertheless, we have sufficient evidence to make an investigation of the conjecture worthwhile.

Now suppose we are able to view a quasitoric manifold M^{2n} as the total space of an equivariant bundle, which has a quasitoric base space and quasitoric fibre. In these circumstances, we follow the lead of Dobrinskaya [17] and term such M^{2n} *reducible*; this notion will be made precise in Definition 5.1.5 below.

In this chapter, we will prove Conjecture 5.0.2 for complex projective space $\mathbb{C}P^n$, and for all reducible quasitoric manifolds N^{2n} with fibre $\mathbb{C}P^1$. This latter class includes, for example, any stage DO_k in a Dobrinskaya tower (4.1.12) with $l_k = 1$, and any stage B^k in a Bott tower (4.5.1).

We begin by introducing reducible quasitoric manifolds and we consider some illuminating examples. Then we determine some cohomological properties of N^{2n}, which will be central to our proof of the conjecture in this case. Next we incorporate a collection of signs into the dicharacteristic matrix Λ, in order to encode the chosen orientation on each facet of the polytope associated to N^{2n}. In the following section we use our modified Λ to make a crucial observation about the Chern classes of the stable tangent bundle $\tau^s(N^{2n})$, which ultimately allows us to prove that when N^{2n} has an SU-structure, it is cobordant to zero in MU_*.

In the final part of this chapter, which is mostly independent of the earlier sections, we study SU-structures induced by omniorientations on complex projective space $\mathbb{C}P^n$, and confirm Conjecture 5.0.2 for this important class of spaces. Related results on Dobrinskaya towers and other quasitoric manifolds are given, and we conclude with a discussion of potential strategies for a full proof of the conjecture.

We would like to express our gratitude to Dmitry Leykin, with whom the concept of the signed dicharacteristic matrix of Section 5.2 was developed.

5.1 Reducible quasitoric manifolds

In the first half of this section we define reducible quasitoric manifolds and consider examples and results that link the concept with ideas from the earlier chapters of our thesis. The rest of the section is devoted to describing the cohomology ring of a particular class of such spaces, in terms of the cohomology ring of their base space and fibre. We begin by obtaining a partial description of the dicharacteristic matrix associated to this class, which gives an insight into the linear relations in their cohomology ring. Following this, we study their associated polytopes to garner information on the monomial cohomology relations, which arise from the Stanley-Reisner algebra of the polytope.

Suppose we have an (n_1+n_2)-torus $T^{n_1+n_2}$, and a homomorphism

$$\psi\colon T^{n_1+n_2} \to T^{n_2},$$

for some positive integers n_1, n_2.

Given a space E with a $T^{n_1+n_2}$-action g_1, and a space B with a T^{n_2}-action g_2, a bundle $\pi : E \to B$ with fibre F is said to be *ψ-equivariant* if the diagram

$$\begin{array}{ccc} T^{n_1+n_2} \times E & \xrightarrow{g_1} & E \\ \Big\downarrow{\scriptstyle \psi\times\pi} & & \Big\downarrow{\scriptstyle \pi} \\ T^{n_2} \times B & \xrightarrow[g_2]{} & B, \end{array} \tag{5.1.1}$$

commutes.

Consider some simple polytopes P^{n_B} and P^{n_F}, which have m_B and m_F facets respectively. Let B^{2n_B}, F^{2n_F} and $E^{2(n_B+n_F)}$ be quasitoric manifolds over P^{n_B}, P^{n_F} and $P^{n_B} \times P^{n_F}$ respectively. Assume these manifolds B^{2n_B}, F^{2n_F} and $E^{2(n_B+n_F)}$ are constructed as the quotients of moment angle complexes $\mathcal{Z}_{P^{n_B}}, \mathcal{Z}_{P^{n_F}}$ and $\mathcal{Z}_{P^{n_B}\times P^{n_F}}$ by the kernels of respective dicharacteristic functions l_B, l_F and l_E. By Proposition 3.2.4 we have that $\mathcal{Z}_{P^{n_B}\times P^{n_F}} = \mathcal{Z}_{P^{n_B}} \times \mathcal{Z}_{P^{n_F}}$.

Now suppose that we have a bundle

$$F^{2n_F} \longrightarrow E^{2(n_B+n_F)} \longrightarrow B^{2n_B}, \tag{5.1.2}$$

with quasitoric base space, fibre and total space. We term a bundle of the form (5.1.2) a *quasitoric bundle*.

Let $K(l_X)$ denote the kernel of the dicharacteristic function l_X, for $X = B, F$ or E, and assume that the diagram

$$\begin{array}{ccccc} K(l_F) & \xrightarrow{\iota} & K(l_E) & \xrightarrow{\pi} & K(l_B) \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\ T^{m_F} & \xrightarrow{\iota} & T^{m_B}\times T^{m_F} & \xrightarrow{\pi} & T^{m_B}, \end{array} \tag{5.1.3}$$

commutes, where maps ι are inclusions and maps π are projections. Then by taking quotients we have a short exact sequence

$$1 \to T^{m_F}/K(l_F) \to (T^{m_B}\times T^{m_F})/K(l_E) \xrightarrow{\psi} T^{m_B}/K(l_B) \to 1. \tag{5.1.4}$$

With quasitoric manifolds B^{2n_B}, F^{2n_F} and $E^{2(n_B+n_F)}$ as defined above, we can make the following definition.

Definition 5.1.5. If the quasitoric bundle $F^{2n_F} \to E^{2(n_B+n_F)} \to B^{2n_B}$, is such that the associated diagram (5.1.3) commutes, then the total space $E^{2(n_B+n_F)}$ is a *reducible* quasitoric manifold over $P^{n_B}\times P^{n_F}$.

Note that in these circumstances, the quasitoric bundle (5.1.2) is a ψ-equivariant bundle, with respect to the homomorphism $\psi\colon (T^{m_B}\times T^{m_F})/K(l_E) \to T^{m_B}/K(l_B)$ that features in the short exact sequence (5.1.4)

If a quasitoric manifold does not satisfy the conditions of Definition 5.1.5, we may refer to it as *irreducible*.

The notion of reducibility is due to Dobrinskaya, [17, Section 4]; however, the definition that is given therein is somewhat difficult to follow, so we have decided to set out the full details to avoid any confusion in our thesis.

Before we consider some examples, we note that our quasitoric bundle with reducible total space fits into the following diagram

$$\begin{array}{ccccc}
\mathcal{Z}_{P^{n_F}} & \xrightarrow{\ \iota\ } & \mathcal{Z}_{P^{n_B}} \times \mathcal{Z}_{P^{n_F}} & \xrightarrow{\ \pi\ } & \mathcal{Z}_{P^{n_B}} \\
\downarrow{\scriptstyle K(l_F)} & & \downarrow{\scriptstyle K(l_E)} & & \downarrow{\scriptstyle K(l_B)} \\
F^{2n_F} & \xrightarrow{\ \iota\ } & E^{2(n_B+n_F)} & \xrightarrow{\ \pi\ } & B^{2n_B} \\
\downarrow{\scriptstyle T^{m_F}/K(l_F)} & & \downarrow{\scriptstyle (T^{m_B}\times T^{m_F})/K(l_E)} & & \downarrow{\scriptstyle T^{m_B}/K(l_B)} \\
P^{n_F} & \xrightarrow{\ \iota\ } & P^{n_B} \times P^{n_F} & \xrightarrow{\ \pi\ } & P^{n_B},
\end{array}$$

which commutes since the corresponding diagram (5.1.3) commutes. The maps ι are inclusions and maps π are projections, and vertical maps $X \to Y$ in the diagram are labelled by the groups acting on X to give orbit space Y.

The following familiar example should elucidate several of the ideas discussed above.

Example 5.1.6. Consider the quasitoric manifold given by the third stage B^3 in a Bott tower $(B^k : k \leq 3)$, as defined in Section 4.5. The simple polytope associated to B^3 is the 3-cube I^3. With dicharacteristic l_3 as specified by the matrix (4.1.6), a point $[z_1, \dots, z_6]$ in B^3 is an equivalence class

$$(z_1, \dots, z_6) \sim (t_1 z_1, t_1^{-a(1,2;1)} t_2 z_2, t_1^{-a(1,3;1)} t_2^{-a(2,3;1)} t_3 z_3, t_1^{-1} z_4, t_2^{-1} z_5, t_3^{-1} z_6), \qquad (5.1.7)$$

where $(z_1, \dots, z_6) \in \mathbb{C}^6$, and $(t_1, t_2, t_3) \in T^3$.

By definition, B^3 is a bundle with fibre the quasitoric manifold $\mathbb{C}P^1$ over I^1, and base space the quasitoric manifold B^2 over I^2, where B^2 is the second stage in the same Bott tower $(B^k : k \leq 3)$. Since $I^3 = I^2 \times I^1$, it follows that $\mathbb{C}P^1 \to B^3 \to B^2$ is a quasitoric bundle.

It is straightforward to check from the descriptions of the dicharacteristic functions involved, that the appropriate diagram of the form (5.1.3) commutes, and so B^3 is a reducible quasitoric manifold with fibre $\mathbb{C}P^1$ and base space B^2.

There is a canonical $T^6/K(l_3) \cong T^3$-action α_3 on B^3 given by $[z_1, \ldots, z_6] \mapsto [s_1z_1, s_2z_2, s_3z_3, z_4, z_5, z_6]$ for $(s_1, s_2, s_3) \in T^3$, as described in (4.1.13). We also have the homomorphism $\psi\colon T^3 \to T^2$, such that $(t_1, t_2, t_3) \mapsto (t_1, t_2)$, and the projection

$$\pi_3\colon B^3 \longrightarrow B^2, \tag{5.1.8}$$

which acts by $[z_1, \ldots, z_6] \mapsto [z_1, z_2, z_4, z_5]$. Then the corresponding diagram of the form (5.1.1) commutes as

$$\begin{aligned}
\pi_3 \cdot \alpha_3((s_1, s_2, s_3), [z_1, \ldots, z_6]) &= \pi_3[s_1z_1, s_2z_2, s_3z_3, z_4, z_5, z_6] \\
&= [s_1z_1, s_2z_2, z_4, z_5] \\
&= \alpha_2((s_1, s_2), [z_1, z_2, z_4, z_5]) \\
&= \alpha_2(\psi \times \pi_3)((s_1, s_2, s_3), [z_1, \ldots, z_6]),
\end{aligned}$$

where α_2 is the canonical $T^4/K(l_2) \cong T^2$-action on B^2. This confirms that our quasitoric bundle is indeed ψ-equivariant.

Note that in a similar manner to the above example, any stage B^k in a Bott tower $(B^k : k \leq n)$ is a reducible quasitoric manifold with fibre $\mathbb{C}P^1$ and base space B^{k-1}. Further examples of reducible quasitoric manifolds include the stages DO_k in a Dobrinskaya tower $(DO_k : k \leq n)$, where the fibre of the corresponding quasitoric bundle is $\mathbb{C}P^{l_k}$ and the base space is DO_{k-1}.

Example 5.1.9. The basis for MU_* given by Theorem 5.0.1, is in terms of quasitoric manifolds $B_{i,j}$, with $i \leq j$, over $I^i \times \Delta^{j-1}$. Each $B_{i,j}$ is the total space of a quasitoric bundle with fibre $\mathbb{C}P^{j-1}$, and base space the bounded flag manifold B^i. A thorough description of these quasitoric manifolds is given in [6, Example 3.13], and it is straightforward to use the details therein to check that each $B_{i,j}$ is in fact a reducible quasitoric manifold over $I^i \times \Delta^{j-1}$.

Remark 5.1.10. In defining reducible quasitoric manifolds, we insisted that the polytope associated to the total space $E^{2(n_B+n_F)}$ was the product of the polytopes

P^{n_B} and P^{n_F} that are associated to the base space B^{2n_B} and fibre F^{2n_F} respectively. Note that conversely a quasitoric manifold M^{2n} over some product of polytopes is not necessarily reducible; however, Dobrinskaya gave conditions on the dicharacteristic function, in terms of the signs of each vertex, as defined at the close of Chapter 3, to determine when this is the case [17, Theorem 6].

Let us put this work in context with some of the other themes in our thesis. Suppose we have a quasitoric manifold M^{2L} over a product of simplices $P^L = \Delta^{l_1} \times \cdots \times \Delta^{l_k}$.

Lemma 5.1.11. *[17, Section 4] If the sign of every vertex of P^L is positive, then M^{2L} is the kth stage of a Dobrinskaya tower.*

Note that on the other hand, if some of the signs are negative, then M^{2L} could still be the kth stage in a Dobrinskaya tower, or some other reducible quasitoric manifold, or it could be an irreducible quasitoric manifold.

Reducible quasitoric manifolds certainly merit further study. The additional knowledge of how such manifolds fibre over their base space should make them a useful tool for studying problems in toric topology. It is precisely this information that we will now utilise when studying Conjecture 5.0.2 in the second half of this chapter.

To get underway, let N^{2n+2} denote a reducible quasitoric manifold over some polytope P^{n+1}, with base space a quasitoric manifold M^{2n} over some P^n, and fibre $\mathbb{C}P^1$. The latter is a quasitoric manifold over the 1-simplex Δ^1, which is homeomorphic to the interval $I := [0, 1]$, so we have that

$$P^{n+1} = P^n \times I. \tag{5.1.12}$$

The facets of I are the end points of the interval, finely ordered as F'_1, F'_2.

Before considering the cohomology ring of N^{2n+2} we must establish some preliminary information on M^{2n}.

As usual we will assume that P^n has m facets, and the facets of the polytope are finely ordered as $F_1, \dots, F_m$. Furthermore, we will assume that a dicharacteristic function $l_n\colon T^m \to T^n$ has been chosen, and the kernel of l_n is denoted by $K(l_n)$. Then, as in Section 3.2, the dicharacteristic l_n gives rise to a short exact sequence

$$0 \longrightarrow \mathbb{Z}^{m-n} \xrightarrow{\kappa_n} \mathbb{Z}^m \xrightarrow{\lambda_n} \mathbb{Z}^n \longrightarrow 0, \tag{5.1.13}$$

describing the map of Lie algebras induced by l_n. The $n \times m$ dicharacteristic matrix Λ_n describing the map λ_n is written in refined form as

$$\Lambda_n = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix}. \tag{5.1.14}$$

Remark 5.1.15. Note that our quasitoric manifold M^{2n}, which depends only on the choice of dicharacteristic l_n, may have a dicharacteristic matrix Λ_n of a different form to (5.1.14) if we choose a different basis for the Lie algebra of T^n. Similarly, if we change the order of the facets of P^n, then the order of the columns in Λ_n will change, to give a different dicharacteristic matrix associated to M^{2n}. However, as is made clear in Section 3.2, we can always choose the basis for the Lie algebra of T^n in such a way that we can write Λ_n in refined form. Furthermore, if we change the order of the facets of P^n, this change does not affect the omniorientation of M^{2n}. Therefore we wish to make clear here that the main results of this chapter will not be dependent on displaying Λ_n in the particular form (5.1.14). We will further emphasise this point in Section 5.4.

The $m \times (m-n)$ matrix describing the injection κ_n will be written as

$$K_n = \begin{pmatrix} \mu_{1,1} & \dots & \mu_{1,m-n} \\ \vdots & & \vdots \\ \mu_{m,1} & \dots & \mu_{m,m-n} \end{pmatrix}, \tag{5.1.16}$$

for $1 \le i \le m$, and some fixed $\mu_{i,j} \in \mathbb{Z}$, for $1 \le j \le m-n$. By Lemma 3.2.5, we can embed the moment angle complex $\mathcal{Z}_{P^n}$ in $\mathbb{C}^m$, so that the action of $K(l_n)$ on the complex coordinates of $\mathcal{Z}_{P^n}$ is given by

$$z_i \mapsto t_1^{\mu_{i,1}} \dots t_{m-n}^{\mu_{i,m-n}} z_i,$$

for $z_i \in \mathbb{C}_i$, with $1 \leq i \leq m$. Here $\mathbb{C}_i$ denotes the ith copy of $\mathbb{C}$ in the product $\mathbb{C}^m$. We will use this information in our description of the cohomology ring of N^{2n+2}.

The polytope $P^{n+1} = P^n \times I$ associated to N^{2n+2} has $m+2$ facets. Using the procedure described in (3.1.10) to finely order a product of finely ordered polytopes, the facets of P^{n+1} should be ordered as

$$F_1 \times I, \dots, F_n \times I, P^n \times F_1', F_{n+1} \times I, \dots, F_m \times I, P^n \times F_2'. \tag{5.1.17}$$

By Proposition 3.2.4, the moment angle complex associated with P^{n+1} is described by

$$\mathcal{Z}_{P^{n+1}} = \mathcal{Z}_{P^n} \times S^3, \tag{5.1.18}$$

where S^3 is the 3-sphere $\{(z_1, z_2) \mid z_1\overline{z_1} + z_2\overline{z_2} = 1\} \subset \mathbb{C}^2$.

If the dicharacteristic function of N^{2n+2} is denoted by l_{n+1}, we have a kernel $K(l_{n+1})$, such that the quotient of $\mathcal{Z}_{P^{n+1}}$ by $K(l_{n+1})$ is N^{2n+2}. Following our splitting (5.1.18), the kernel $K(l_{n+1})$ must act on $(z_1, z_2) \in S^3$ by

$$(z_1, z_2) \mapsto (t_1^{\alpha_1} \dots t_{m-n+1}^{\alpha_{m-n+1}} z_1, t_1^{\beta_1} \dots t_{m-n+1}^{\beta_{m-n+1}} z_2), \tag{5.1.19}$$

for some $\alpha_i, \beta_j \in \mathbb{Z}$, for all i, j, while $K(l_{n+1})$ acts on $\mathcal{Z}_{P^n}$ in the same manner as the action of $K(l_n)$ on $\mathcal{Z}_{P^n}$, because N^{2n+2} is reducible.

Example 5.1.20. If $\alpha_1 = \dots = \alpha_{m-n} = \beta_1 = \dots = \beta_{m-n} = 0$, $\alpha_{m-n+1} = \pm 1$ and $\beta_{m-n+1} = \pm 1$, then N^{2n+2} is simply the product $M^{2n} \times \mathbb{C}P^1$. In terms of reducible quasitoric manifolds, we can view N^{2n+2} as a trivial ψ-equivariant $\mathbb{C}P^1$-bundle over M^{2n}, where $\psi\colon T^{m+2}/K(l_{n+1}) \to T^m/K(l_n)$ is defined as in the short exact sequence (5.1.4).

Associated to l_{n+1} we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}^{m-n+1} \xrightarrow{\kappa_{n+1}} \mathbb{Z}^{m+2} \xrightarrow{\lambda_{n+1}} \mathbb{Z}^{n+1} \longrightarrow 0, \tag{5.1.21}$$

and taking into account the fine ordering (5.1.17), the matrix describing the map

κ_{n+1} is given by

$$K_{n+1} = \begin{pmatrix} \mu_{1,1} & \dots & \mu_{1,m-n} & 0 \\ \vdots & & \vdots & \vdots \\ \mu_{n,1} & \dots & \mu_{n,m-n} & 0 \\ \alpha_1 & \dots & \alpha_{m-n} & \alpha_{m-n+1} \\ \mu_{n+1,1} & \dots & \mu_{n+1,m-n} & 0 \\ \vdots & & \vdots & \vdots \\ \mu_{m,1} & \dots & \mu_{m,m-n} & 0 \\ \beta_1 & \dots & \beta_{m-n} & \beta_{m-n+1} \end{pmatrix}. \tag{5.1.22}$$

If we denote the dicharacteristic matrix associated to l_{n+1} by Λ_{n+1}, then by (5.1.21) we have

$$\Lambda_{n+1} K_{n+1} v_{n+1}^t = 0, \tag{5.1.23}$$

where v_{n+1}^t is the transpose of an arbitrary vector $(v_1, \dots, v_{m-n+1}) \in \mathbb{Z}^{m-n+1}$.

Lemma 5.1.24. *The final column of Λ_{n+1} is given by a vector $(0, \dots, 0, a)$, for some integer a.*

Proof. Let us write Λ_{n+1} as

$$\begin{pmatrix} 1 & 0 & \dots & 0 & a_{1,n+2} & \dots & a_{1,m+2} \\ 0 & 1 & \dots & 0 & a_{2,n+2} & \dots & a_{2,m+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a_{n+1,n+2} & \dots & a_{n+1,m+2} \end{pmatrix}, \tag{5.1.25}$$

for some $a_{i,j} \in \mathbb{Z}$, with columns corresponding to the ordering of facets (5.1.17).

The equation (5.1.23) provides $n+1$ linear equations in $v_1, \dots, v_{m-n+1}$, the ith equation being given by setting the product of the ith row of Λ_{n+1} with $K_{n+1} v_{n+1}^t$ equal to zero. It is a matter of simple linear algebra to collect the terms in v_{m-n+1} in the $(n+1)$th equation, to ascertain that $a_{n+1,m+2} = -\alpha_{m-n+1}/\beta_{m-n+1}$.

Collecting terms in v_{m-n+1} in the remaining n linear equations, it is easy to deduce that $a_{i,m+2} = 0$, for all $1 \leq i \leq n$.

We established in Section 3.2 that every column of an $n \times m$ dicharacteristic matrix must be a primitive vector in $\mathbb{Z}^n$. Since the final column of Λ_{n+1} must meet this criterion, it follows that $-\alpha_{m-n+1}/\beta_{m-n+1} = \pm 1$, which completes the proof. □

If $(a_{1,m+2}, \ldots, a_{n,m+2}, a_{n+1,m+2}) = (0, \ldots, 0, \pm 1)$ in Λ_{n+1}, then the first n linear equations arising from (5.1.23) reduce to the linear equations

$$\Lambda_n K_n v_n^t = 0, \tag{5.1.26}$$

where $v_n = (v_1, \ldots, v_n) \in \mathbb{Z}^n$, which arise from the short exact sequence (5.1.13). This observation is enough to prove the following result.

Proposition 5.1.27. *The $(n+1) \times (m+2)$ dicharacteristic matrix Λ_{n+1} takes the form*

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \lambda_{1,n+1} & \ldots & \lambda_{1,m} & 0 \\ 0 & 1 & \ldots & 0 & 0 & \lambda_{2,n+1} & \ldots & \lambda_{2,m} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \lambda_{n,n+1} & \ldots & \lambda_{n,m} & 0 \\ 0 & 0 & \ldots & 0 & 1 & a_{n+1,n+2} & \ldots & a_{n+1,m+1} & a \end{pmatrix}, \tag{5.1.28}$$

where $a = \pm 1$.

We can see the dicharacteristic matrix of M^{2n} inside the dicharacteristic matrix of N^{2n+2} as the columns $1, \ldots, n, n+2, \ldots, m+1$, in rows $1, \ldots, n$. The dicharacteristic matrix of the fibre $\mathbb{C}P^1$ comprises columns $n+1, m+2$ in row $n+1$. Furthermore, it is straightforward to check that $N^{2n+2} = M^{2n} \times \mathbb{C}P^1$ as in Example 5.1.20, precisely when $a_{n+1,n+2} = \cdots = a_{n+1,m+1} = 0$.

Readers may find it helpful to keep in mind the following example, which illustrates each of these aspects of Λ_{n+1}.

Example 5.1.29. By Example 5.1.6, the 3rd stage of a Bott tower B^3 over $I^3 = I^2 \times I$, is reducible with base space B^2 and fibre $\mathbb{C}P^1$. From (4.1.6), the dicharacteristic matrix for B^3 is given by

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a(1,2;1) & 1 & 0 \\ 0 & 0 & 1 & -a(1,3;1) & -a(2,3;1) & 1 \end{pmatrix},$$

where columns $1, 2, 4, 5$ in rows 1 and 2 comprise the dicharacteristic matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -a(1,2;1) & 1 \end{pmatrix},$$

for the base space B^2 over I^2.

Columns 3 and 6 in row 3 of the dicharacteristic matrix of B^3 give the dicharacteristic matrix $(1\ 1)$ for the fibre $\mathbb{C}P^1$ over I^1. Note that the omniorientation of $\mathbb{C}P^1$ here differs from that of Example 3.2.12.

The third stage B^3 of a Bott tower is the total space of the projective bundle $\mathbb{C}P((\gamma_1^{a(1,3;1)} \otimes \gamma_2^{a(2,3;1)}) \oplus \mathbb{C})$ over B^2. For the trivial case when $a(1,3;1) = a(2,3;1) = 0$, we see that B^3 becomes $B^2 \times \mathbb{C}P^1$, the total space of $\mathbb{C}P(\mathbb{C} \oplus \mathbb{C})$ over B^2.

Our description (5.1.28) of Λ_{n+1} provides us with information about the linear relations in the cohomology ring $H^*(N^{2n+2})$. To fully understand this ring, we must now shed light on the monomial relations in $H^*(N^{2n+2})$, which arise from the Stanley-Reisner ring $\mathbb{Z}[P^{n+1}]$ of the polytope P^{n+1}.

With the facets of N^{2n+2} ordered finely as in (5.1.17), let $u_i \in H^2(N^{2n+2})$ be the generator associated to the ith facet in the ordering.

The polytope $P^{n+1} = P^n \times I$ is, in some sense, an $(n+1)$-dimensional analogue of a prism, in that it is composed of two copies of P^n, viewed as $P^n \times F_1'$ and $P^n \times F_2'$, joined through $F_i \times I$ for $1 \leq i \leq m$. Hence we can immediately deduce that $P^n \times F_1' \cap P^n \times F_2' = \emptyset$, and so we have a relation $u_{n+1}u_{m+2} = 0$ in $\mathbb{Z}[P^{n+1}]$.

Now consider the quasitoric manifold M^{2n} over P^n, the base space of our reducible quasitoric manifold N^{2n+2}. Assuming that the Stanley-Reisner ring of P^n is given by

$$\mathbb{Z}[P^n] \cong \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}_n,$$

then the remaining relations in $\mathbb{Z}[P^{n+1}]$ are given by the image of the ideal $\mathcal{I}_n$ under the map $r\colon \mathbb{Z}[P^n] \to \mathbb{Z}[P^{n+1}]$, which relabels the elements v_i by

$$\begin{aligned} v_i &\mapsto u_i \quad \text{for} \quad 1 \leq i \leq n, \\ v_i &\mapsto u_{i+1} \quad \text{for} \quad n+1 \leq i \leq m, \end{aligned}$$

to take account of the ordering (5.1.17). This follows from the fact that if $F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset$ in P^n, then

$$F_{i_1} \times I \cap \cdots \cap F_{i_k} \times I = \emptyset,$$

in P^{n+1}.

Denoting the image of $\mathcal{I}_n$ under r by $r(\mathcal{I}_n)$ we can collect our observations together as follows.

Proposition 5.1.30. *The Stanley-Reisner ring of P^{n+1} is described by*

$$\mathbb{Z}[P^{n+1}] \cong \mathbb{Z}[u_1, \ldots, u_{m+2}]/\mathcal{I}_{n+1}, \tag{5.1.31}$$

where $\mathcal{I}_{n+1}$ is $r(\mathcal{I}_n) \cup \{u_{n+1}u_{m+2}\}$.

We have now attained adequate information on the cohomology ring of N^{2n+2} to conclude this section with an observation that we will rely upon in Section 5.4.

Suppose we have a relation $u_{i_1} \ldots u_{i_j} = 0$ in $r(\mathcal{I}_n) \subset \mathcal{I}_{n+1}$, where $u_{i_1}, \ldots, u_{i_l}$ are such that $1 \leq i_k \leq n$, and $u_{i_{l+1}}, \ldots, u_{i_j}$ are such that $n+2 \leq i_k \leq m+1$. Then using the linear cohomology relations, our monomial $u_{i_1} \ldots u_{i_j}$ can be rewritten to give the following relation in the cohomology ring $H^*(N^{2n+2})$

$$(-\lambda_{i_1,n+1}u_{n+2} - \cdots - \lambda_{i_1,m}u_{m+1}) \ldots (-\lambda_{i_l,n+1}u_{n+2} - \cdots - \lambda_{i_l,m}u_{m+1})u_{i_{l+1}} \ldots u_{i_j} = 0. \tag{5.1.32}$$

By multiplying out the above expression (5.1.32), and rearranging as necessary, we are able to deduce the following result.

Lemma 5.1.33. *The relations in the cohomology ring of N^{2n+2} are such that the cohomology class u_i^l can be expressed as a polynomial p over $\mathbb{Z}$ in $u_{n+2}, \ldots, u_{m+1}$, for some integer l, and for $n+2 \leq i \leq m+1$.*

So there is no relation in $H^*(N^{2n+2})$ that allows us to rewrite a product of Chern classes $u_{i_1}, \ldots, u_{i_j}$, for $i_k \neq n+1, m+2$, so that it contains a term in u_{m+2}, since any such relation $u_{i_1}, \ldots, u_{i_j}$ must be of the form (5.1.32), and clearly we cannot rearrange this expression so that terms in u_{m+2} appear.

5.2 Encoding omniorientations in Λ

A quasitoric manifold M^{2n} has 2^{m+1} possible omniorientations; if one such induces an SU-structure which bounds in MU_*, then we cannot conclude that Conjecture 5.0.2 is true for M^{2n}, as one of the 2^m alternative omniorientations might induce a nonbounding SU-structure on the manifold. In this section we introduce the notion of a *signed* dicharacteristic matrix that allows us to deal with this problem.

Following the definition of the dicharacteristic matrix Λ (3.2.8), we discussed the effect on Λ of changing omniorientation. To preserve the refined form of the matrix after such a change, if a facet F_i, for $1 \leq i \leq n$ is switched, then we negate the ith row of the refined submatrix S. While if the orientation on a facet F_i, for $n+1 \leq i \leq m$ is switched, we negate the ith column of S. We wish to encode this data in Λ.

Let $\epsilon(m) := (\epsilon_1, \ldots, \epsilon_m)$ be a *list of signs* ϵ_i, which can take values ± 1 for $1 \leq i \leq m$.

Definition 5.2.1. Given a dicharacteristic matrix Λ, and a list of signs $\epsilon(m)$, the *signed dicharacteristic matrix* $\Lambda_{\epsilon(m)}$ is the $n \times m$ matrix $(I_n \mid S_{\epsilon(m)})$, where $S_{\epsilon(m)}$ is given by

$$\begin{pmatrix} \epsilon_1\epsilon_{n+1}\lambda_{1,n+1} & \ldots & \epsilon_1\epsilon_m\lambda_{1,m} \\ \epsilon_2\epsilon_{n+1}\lambda_{2,n+1} & \ldots & \epsilon_2\epsilon_m\lambda_{2,m} \\ \vdots & \ddots & \vdots \\ \epsilon_n\epsilon_{n+1}\lambda_{n,n+1} & \ldots & \epsilon_n\epsilon_m\lambda_{n,m} \end{pmatrix}. \tag{5.2.2}$$

The sign ϵ_i represents the choice of omniorientation on the facet F_i and so, on the facial bundle ρ_i. The *initial omniorientation* is given by $\epsilon_i = 1$, for all $1 \leq i \leq m$, as in this case $\Lambda_{\epsilon(m)}$ reverts to the original Λ.

We encode the effect in a change in omniorientation as follows. Begin with the initial omniorientation $\epsilon(m) := (1, \ldots, 1)$; if the complex line bundles $\rho_{i_1}, \ldots, \rho_{i_j}$ are conjugated, then map $\epsilon_k \mapsto -\epsilon_k$ for $i_1 \leq k \leq i_j$. This ensures that the appropriate rows and columns in $S_{\epsilon(m)}$ are negated to give the correct representation in the dicharacteristic matrix.

Incorporating the ϵ_i allows us to work with a completely general omniorientation, but it will also be easy to consider special cases by substituting in particular values for ϵ_i if necessary.

5.3 Chern classes

In Section 3.3 we observed that $u_{n+2}, \ldots, u_{m+2}$ suffice to generate $H^*(N^{2n+2})$ multiplicatively. This is due to the fact that, referring to the signed dicharacteristic matrix of N^{2n+2}, we can write

$$u_i = -\epsilon_i \epsilon_{n+2} \lambda_{i,n+1} u_{n+2} - \cdots - \epsilon_i \epsilon_{m+1} \lambda_{i,m} u_{m+1},$$

for $1 \leq i \leq n$, and

$$u_{n+1} = -\epsilon_{n+1}\epsilon_{n+2} a_{n+1,n+2} u_{n+2} - \cdots - \epsilon_{n+1}\epsilon_{m+1} a_{n+1,m+1} u_{m+1} - \epsilon_{n+1}\epsilon_{m+2} a u_{m+2}, \tag{5.3.1}$$

where $a = \pm 1$. Note that the element u_{m+2} only appears in two of the expressions for the first Chern classes u_i of the facial bundles ρ_i; namely $c_1(\rho_{n+1}) = u_{n+1}$ as in (5.3.1), and $c_1(\rho_{m+2}) = u_{m+2}$. This observation will be important in the proof of our main theorem in the penultimate section.

As noted in Remark 3.4.11, we do not have a simple method that will extract the Chern classes of the stable tangent bundle $\tau^s(M^{2n})$ of a quasitoric manifold M^{2n} directly from its dicharacteristic matrix. However, for our manifolds N^{2n+2} we have the following result, which gives us enough information on the Chern classes to help to determine Chern numbers in the next section.

From the introductory discussion above, it is clear that the Chern classes of the stable tangent bundle of N^{2n+2} are polynomials over $\mathbb{Z}$ in $u_{n+2}, \ldots u_{m+2}$.

Proposition 5.3.2. *Whenever the element u_{m+2} appears as a term in a Chern class of the stable tangent bundle of N^{2n+2}, its coefficient contains a factor of $(1 - \epsilon_{n+1}\epsilon_{m+2}a)$.*

To illustrate Proposition 5.3.2, consider the Chern classes of B^3, the third stage of a Bott tower considered in Example 5.1.6. The generators of $H^*(B^3)$ are u_4, u_5

and $u_6 \in H^2(B^3)$, and in this case $a = 1$.

$$\begin{aligned}
c_1(\tau^s) &= u_4(1 - \epsilon_1\epsilon_4 + \epsilon_2\epsilon_4 a(1,2;1) + \epsilon_3\epsilon_4 a(1,3;1)) + \\
&\quad u_5(1 - \epsilon_2\epsilon_5 + \epsilon_3\epsilon_5 a(2,3;1)) + u_6(1 - \epsilon_3\epsilon_6), \\
c_2(\tau^s) &= u_4u_5(1 - \epsilon_1\epsilon_4 - \epsilon_2\epsilon_5 + \epsilon_1\epsilon_2\epsilon_4\epsilon_5 + \epsilon_3\epsilon_4 a(1,3;1) - \\
&\quad \epsilon_2\epsilon_3\epsilon_4\epsilon_5 a(1,3;1) + \epsilon_3\epsilon_5 a(2,3;1) - \epsilon_1\epsilon_3\epsilon_4\epsilon_5 a(2,3;1) + \\
&\quad \epsilon_3\epsilon_4 a(1,2;1)a(2,3;1)) + \\
&\quad -u_4u_6(1 - \epsilon_3\epsilon_6)(\epsilon_1\epsilon_4 - \epsilon_2\epsilon_4 a(1,2;1) - 1) + \\
&\quad -u_5u_6(\epsilon_2\epsilon_5 - 1)(1 - \epsilon_3\epsilon_6), \\
c_3(\tau^s) &= -u_4u_5u_6(\epsilon_1\epsilon_4 - 1)(\epsilon_2\epsilon_5 - 1)(1 - \epsilon_3\epsilon_6).
\end{aligned}$$

With each occurrence of u_6, the coefficient contains a factor of $(1 - \epsilon_3\epsilon_6)$.

Proof of Proposition 5.3.2. Denote the stable tangent bundle of N^{2n+2} by τ^s. By Corollary 3.4.7 the ith Chern class of τ^s is given by $c_i(\tau^s) = \sigma_i(u_1, \ldots, u_{m+2})$.

We noted above that the element u_{m+2} only appears in $c_1(\rho_{n+1})$ and $c_1(\rho_{m+2})$, so when we evaluate this ith elementary symmetric polynomial the only terms that contain u_{m+2} are monomials of the form $u_{j_1} \ldots u_{j_{i-2}} u_{n+1} u_{m+2}$, for $i \geq 2$, where $u_{j_k} \neq u_{n+1}, u_{m+2}$, or sums of monomials

$$u_{l_1} \ldots u_{l_{i-1}} u_{n+1} + u_{l_1} \ldots u_{l_{i-1}} u_{m+2}, \tag{5.3.3}$$

for $i \geq 1$, where $u_{l_k} \neq u_{n+1}, u_{m+2}$. By the Stanley-Reisner relations described in Proposition 5.1.30 , the former are zero since $u_{n+1}u_{m+2} = 0$, and if the relations cancel out one of the monomials in the sum (5.3.3), then clearly they will cancel out the other. So we can assume that after applying the relations, if any terms in u_{m+2} remain, we are left only with terms of the form (5.3.3).

Using the linear relations in $H^*(N^{2n+2})$, we can rewrite the ith Chern class as

$$\begin{aligned}
c_i(\tau^s) = &\sigma_i(-\epsilon_1\epsilon_{n+2}\lambda_{1,n+1}u_{n+2} - \cdots - \epsilon_1\epsilon_{m+1}\lambda_{1,m}u_{m+1}, \ldots, \\
&- \epsilon_{n+1}\epsilon_{n+2}a_{n+1,n+2}u_{n+2} - \cdots - \epsilon_{n+1}\epsilon_{m+1}a_{n+1,m+1}u_{m+1} - \epsilon_{n+1}\epsilon_{m+2}au_{m+2}, \\
&u_{n+2}, \ldots, u_{m+2}).
\end{aligned}$$

The coefficient of u_{m+2} in $c_1(\rho_{n+1}) = u_{n+1}$ is $-\epsilon_{n+1}\epsilon_{m+2}a$ and the coefficient of u_{m+2} in $c_1(\rho_{m+2}) = u_{m+2}$ is just 1. So collecting together the terms in u_{m+2} we have the following term in our Chern class

$$(1 - \epsilon_{n+1}\epsilon_{m+2}a)u_{m+2}p,$$

for some polynomial p that does not contain any terms in u_{m+2}. Hence we can write

$$c_i(\tau^s) = (1 - \epsilon_{n+1}\epsilon_{m+2}a)u_{m+2}p + q,$$

where p and q are polynomials over $\mathbb{Z}$ in $u_{n+2}, \ldots, u_{m+1}$. □

5.4 Characteristic numbers

In this section we present our proof of Conjecture 5.0.2 for reducible quasitoric manifolds N^{2n+2}.

We follow the description in [38], that for each partition $Q = q_1, \ldots, q_k$ of n, the *Qth Chern number* of an n-dimensional manifold M^n, $c_Q[M^n] = c_{q_1} \ldots c_{q_k}[M^n]$ is, by definition, the integer

$$< c_{q_1}(\tau^s) \ldots c_{q_k}(\tau^s), [M^n] >, \tag{5.4.1}$$

where $[M^n] \in H_n(M^n; \mathbb{Z})$ is the fundamental homology class of the manifold, and τ^s is its stable tangent bundle. For any partition Q of n, the element $c_{q_1}(\tau^s) \ldots c_{q_k}(\tau^s)$ lies in the top dimension of the cohomology ring of M^n.

Note that we are working with *tangential* characteristic numbers, rather than the more commonly used *normal* characteristic numbers. In the latter case $c_{q_j}(\tau^s)$ is replaced by $c_{q_j}(\nu^s)$ in (5.4.1), where ν^s is the stable normal bundle of M^n.

We can make the following observation about the Chern classes of the stable tangent bundle of N^{2n}.

Lemma 5.4.2. *Given a partition $q_1, \ldots, q_k$ of $n+1$, the product $c_{q_1}(\tau^s) \ldots c_{q_k}(\tau^s)$ of Chern classes in $H^{2n+2}(N^{2n+2})$ is equal to a sum of terms that each contain a factor of the cohomology class $u_{m+2} \in H^2(N^{2n+2})$.*

Proof. A product $c_{q_1}(\tau^s)\dots c_{q_k}(\tau^s) \in H^{2n+2}(N^{2n+2})$ can be written as a sum of terms of the form $u_{n+2}^{i_{n+2}}\dots u_{m+2}^{i_{m+2}}$, for integers $i_j \geq 0$, with $i_{n+2}+\dots+i_{m+2} = n+1$.

Now suppose that our product $c_{q_1}(\tau^s)\dots c_{q_k}(\tau^s)$ does not contain any terms with factors of u_{m+2}. It follows that the product must be comprised of a sum of terms of the form $u_{n+2}^{i_{n+2}}\dots u_{m+1}^{i_{m+1}}$, with $i_{n+2}+\dots+i_{m+1} = n+1$.

The elements $u_{n+2},\dots,u_{m+1}$ are in the image of the map $r\colon \mathbb{Z}[P^n] \to \mathbb{Z}[P^{n+1}]$ of Stanley-Reisner rings, which was introduced at the close of Section 5.1. We have that $r(v_{n+1}^{i_{n+2}}\dots v_m^{i_{m+1}}) = u_{n+2}^{i_{n+2}}\dots u_{m+1}^{i_{m+1}}$, where the element $v_{n+1}^{i_{n+2}}\dots v_m^{i_{m+1}}$ lies in the cohomology ring $H^*(M^{2n})$ of the base space of the quasitoric bundle $\mathbb{C}P^1 \to N^{2n+2} \to M^{2n}$. We have that $v_{n+1}^{i_{n+2}}\dots v_m^{i_{m+1}} = 0$, since this element lies in the cohomology group $H^{2n+2}(M^{2n})$, which is zero for dimensional reasons. Therefore we can deduce that $u_{n+2}^{i_{n+2}}\dots u_{m+1}^{i_{m+1}} = 0$ in $H^{2n+2}(N^{2n+2})$. Hence to ensure that the product $c_{q_1}(\tau^s)\dots c_{q_k}(\tau^s)$ of Chern classes lies in $H^{2n+2}(N^{2n+2})$, we require that it is comprised of sums of terms of the form $u_{n+2}^{i_{n+2}}\dots u_{m+2}^{i_{m+2}}$, with $i_{n+2}+\dots+i_{m+2} = n+1$, and $i_{m+2} \neq 0$. □

By combining Lemma 3.4.8 and Proposition 5.1.27 we obtain the following fact.

Corollary 5.4.3. *The quasitoric manifold N^{2n+2} has an SU-structure induced by an omniorientation on its stable tangent bundle, only if $\epsilon_{n+1}\epsilon_{m+2}a = 1$.*

Now everything is in place for us to prove our main theorem.

Theorem 5.4.4. *If a reducible quasitoric manifold N^{2n+2} with fibre $\mathbb{C}P^1$ has an SU-structure induced by an omniorientation, then N^{2n+2} is cobordant to zero in the complex cobordism ring MU_*.*

Proof. Let $q_1,\dots,q_k$ by a partition of $2n+2$ so that $c_{q_1}\dots c_{q_k}[N^{2n+2}]$ is a Chern number given by

$$< c_{q_1}(\tau^s)\dots c_{q_k}(\tau^s), [N^{2n+2}] >, \tag{5.4.5}$$

where $[N^{2n+2}]$ is the fundamental class of N^{2n+2} in the top dimension of its homology; it is determined by the chosen omniorientation on N^{2n+2}.

By Lemma 5.1.33, the relations in the cohomology ring of N^{2n+2} imply that any element $u_i^j \in H^{2j}(N^{2n+2})$, for $n+2 \leq i \leq m+1$, can be rewritten in terms of $u_{n+2}, \ldots, u_{m+1}$, but never in terms of u_{m+2}. For example, at the third stage B^3 of a Bott tower we have

$$u_5^3 = \left(\frac{\epsilon_4 a(1,2;1) u_4}{\epsilon_5} \right)^2 u_5, \tag{5.4.6}$$

and it is not possible for u_6 to arise from a combination of powers of u_4 and u_5.

It follows that for a product of Chern classes $c_{q_1}(\tau^s) \ldots c_{q_k}(\tau^s)$ to contain the term u_{m+2}, which Lemma 5.4.2 implies is necessary if it is to lie in the top dimension of $H^*(N^{2n+2})$, at least one of the terms $c_{q_l}(\tau^s)$ in the product has to have a term in u_{m+2} in its polynomial expression.

By Proposition 5.3.2, if a Chern class $c_{q_l}(\tau^s)$ contains a term in u_{m+2}, then that term has a factor of $(1 - \epsilon_{n+1}\epsilon_{m+2}a)$ in its coefficient. Hence the product of Chern classes $c_{q_1}(\tau^s) \ldots c_{q_k}(\tau^s)$ has at least one factor of $(1 - \epsilon_{n+1}\epsilon_{m+2}a)$, and so must the Chern number $c_{q_1} \ldots c_{q_k}[N^{2n+2}]$. By Corollary 5.4.3, if N^{2n+2} is an SU-manifold then $\epsilon_{n+1}\epsilon_{m+2}a = 1$. It follows that the Chern number $c_Q[N^{2n+2}] = c_{q_1} \ldots c_{q_k}[N^{2n+2}] = 0$ for any partition Q of $2n+2$, and so N^{2n+2} is cobordant to zero in MU_*. □

Remark 5.4.7. In light of Remark 5.1.15 we emphasise that Theorem 5.4.4 holds for the reducible quasitoric SU-manifold N^{2n+2}, regardless of the ordering of the facets of its associated polytope P^{n+1}. While our proof of Theorem 5.4.4 did require us to use a particular ordering of the facets of P^{n+1}, since we can always reorder the facets accordingly without affecting the omniorientation of N^{2n+2}, and thus its complex cobordism class, then Theorem 5.4.4 holds for N^{2n+2} with any choice of ordering for the facets of P^{n+1}.

5.5 Complex projective space

One of the most natural examples of quasitoric manifolds are the complex projective spaces $\mathbb{C}P^n$, which we introduced in Example 3.2.12. If we could prove Conjecture 5.0.2 in the case of $\mathbb{C}P^n$, then since these spaces are so fundamental in toric topology, it would offer compelling evidence to support the full conjecture. This is the main aim of our final section.

We begin by studying the case when $n = 2k$ is even and show that there are no SU-structures on $\mathbb{C}P^{2k}$ that are induced by omniorientations. Following this we prove Conjecture 5.0.2 for $\mathbb{C}P^n$ in the case when $n = 2k+1$ is odd. In conclusion, we speculate on several possible approaches to a complete proof of the conjecture.

To reiterate our comments from Chapter 1, we feel we should make clear that some of the results in this section follow from what are probably well-known observations about $\mathbb{C}P^n$, though we do not always have specific references to call upon. However, our interest is in reinterpreting these results in the new context of toric topology.

To begin, we note that for $\mathbb{C}P^n$, we need only verify Conjecture 5.0.2 in cases when n is odd, courtesy of the following observation.

Proposition 5.5.1. *There is no omniorientation on $\mathbb{C}P^{2n}$ that induces an SU-structure on the stable tangent bundle $\tau^s(\mathbb{C}P^{2n})$.*

Proof. By Example 3.2.12, the signed refined submatrix of an omnioriented $\mathbb{C}P^{2n}$ is the column vector $v = (-\epsilon_1\epsilon_{2n+1}, \dots, -\epsilon_{2n}\epsilon_{2n+1})$ in $\mathbb{Z}^{2n}$. By Lemma 3.4.8, the omniorientation on $\mathbb{C}P^{2n}$ induces an SU-structure on its stable tangent bundle if and only if the column sum satisfies

$$-\epsilon_1\epsilon_{2n+1} - \epsilon_2\epsilon_{2n+1} - \cdots - \epsilon_{2n}\epsilon_{2n+1} = 1. \tag{5.5.2}$$

Suppose that $\mathbb{C}P^{2n}$ is omnioriented so that a of the entries $-\epsilon_{i_1}\epsilon_{2n+1}, \dots, -\epsilon_{i_a}\epsilon_{2n+1}$ of v become $+1$. Then the remaining $2n-a$ entries $-\epsilon_i\epsilon_{2n+1}$ of v, where $i \neq i_1, \dots, i_a$, must be equal to -1. There are two subcases to consider:

If $a = n$, then the column sum (5.5.2) is zero. Therefore such an omniorientation does not induce an SU-structure on $\mathbb{C}P^{2n}$.

If $a \neq n$, then the column sum (5.5.2) is $2a - 2n$. Since both a and n are integers, $2(a-n) \neq 1$, and hence such an omniorientation does not induce an SU-structure on $\mathbb{C}P^{2n}$. □

Let us consider some of the implications of this result. In Section 5.1 we saw that we can view the reducible quasitoric manifold DO_k, the kth stage of a Dobrinskaya tower $(DO_k : k \leq n)$, as the total space of a quasitoric bundle $\mathbb{C}P^{l_k} \to DO_k \to DO_{k-1}$.

Lemma 5.5.3. *The quasitoric manifold DO_k has an SU-structure induced by an omniorientation, only if, when viewed as a quasitoric bundle, the fibre $\mathbb{C}P^{l_k}$ has an SU-structure induced by an omniorientation.*

Proof. This result is easily seen to be true by looking at the final column of the dicharacteristic matrix (4.1.6) for DO_k. □

In particular, if l_k is even so that the fibre of DO_k is some $\mathbb{C}P^{2j}$, then it follows from Proposition 5.5.1 and Lemma 5.5.3 that there is no omniorientation on DO_k that induces an SU-structure on its stable tangent bundle.

Following Proposition 5.5.1, confirmation of Conjecture 5.0.2 for complex projective space $\mathbb{C}P^n$ reduces to verifying the conjecture in the cases when n is odd; in such cases the signed refined submatrix induced by an omniorientation on $\mathbb{C}P^{2n+1}$ is the column vector

$$v = (-\epsilon_1\epsilon_{2n+2}, \ldots, -\epsilon_{2n+1}\epsilon_{2n+2}) \in \mathbb{Z}^{2n+1}.$$

In a similar fashion to the even dimensional case above, the omniorientation induces an SU-structure on $\mathbb{C}P^{2n+1}$ if and only if the column sum satisfies

$$-\epsilon_1\epsilon_{2n+2} - \epsilon_2\epsilon_{2n+1} - \cdots - \epsilon_{2n+1}\epsilon_{2n+2} = 1. \tag{5.5.4}$$

Only if $\mathbb{C}P^{2n+1}$ is omnioriented so that $n+1$ of the entries in v become $+1$, and the remaining n of the entries in v become -1, is equation (5.5.4) satisfied, inducing an SU-structure on $\mathbb{C}P^{2n+1}$.

Assume now that we have satisfied the condition (5.5.4) so that the omniorientation has induced an SU-structure on $\mathbb{C}P^{2n+1}$. Without loss of generality, we may then assume that v is the vector $(1, \ldots, 1, -1, \ldots, -1) \in \mathbb{Z}^{2n+1}$, in which the first $n+1$ entries are 1; this explicit choice is justified in Remark 5.5.7 below. Label by t the SU-structure induced by this choice of v. If the cohomology generators of $H^*(\mathbb{C}P^{2n+1})$ are given by elements $u_1, \ldots, u_{2n+2}$ in $H^2(\mathbb{C}P^{2n+1})$, then the linear relations arising from associated dicharacteristic matrix, with refined submatrix v, are as follows

$$\begin{aligned} u_1 = u_2 = \cdots = u_{n+1} &= -u_{2n+2}, \\ u_{n+2} = u_{n+3} = \cdots = u_{2n+1} &= u_{2n+2}. \end{aligned} \tag{5.5.5}$$

To simplify notation, we will denote $u_{2n+2} \in H^2(\mathbb{C}P^{2n+1})$ by x. Then by Corollary 3.4.7, the pth Chern class of the stable tangent bundle of $\mathbb{C}P^{2n+1}$ is given by

$$c_p(\tau^s(\mathbb{C}P^{2n+1})) = \sigma_p(x, \ldots, x, -x, \ldots, -x), \tag{5.5.6}$$

that is, the nth elementary symmetric polynomial in $n+1$ copies of x and $n+1$ copies of $-x$.

Remark 5.5.7. We are able to make our choice $(1, \ldots, 1, -1, \ldots, -1)$ for v without loss of generality, because for the remainder of this section we are interested only in the cohomology generators $u_1, \ldots, u_{2n+2} \in H^2(\mathbb{C}P^{2n+1})$. If $\mathbb{C}P^{2n+1}$ has an SU-structure then we know that $n+1$ of the entries in v must be $+1$, and the remaining n of the entries in v must be -1. So whatever the choice of omniorientation that determines the vector v explicitly, we know that the cohomology generators will always be of the form (5.5.5), albeit in a different order to that shown above. This will be sufficient for our purposes below.

Following Remark 5.5.7, we will assume that in the remainder of this chapter, $\mathbb{C}P^{2n+1}$ is equipped with the SU-structure t. We have the following result on the odd Chern classes of the stable tangent bundle $\tau^s(\mathbb{C}P^{2n+1})$ of the quasitoric SU-manifold $\mathbb{C}P^{2n+1}$.

Lemma 5.5.8. *If an omniorientation on $\mathbb{C}P^{2n+1}$ induces an SU-structure on its stable tangent bundle, then the Chern classes $c_{2k+1}(\tau^s(\mathbb{C}P^{2n+1}))$ in $H^{4k+2}(\mathbb{C}P^{2n+1})$ are zero, for all $k \geq 0$.*

Proof. Since an omniorientation has induced the SU-structure t on $\mathbb{C}P^{2n+1}$, we know that the Chern classes of $\tau^s(\mathbb{C}P^{2n+1})$ are given by elementary symmetric polynomials of the form (5.5.6). As is well-known (see e.g. [38, p. 189]), this is equivalent to writing the total Chern class $c(\tau^s(\mathbb{C}P^{2n+1})) \in H^*(\mathbb{C}P^{2n+1})$ as

$$\begin{aligned} c(\tau^s(\mathbb{C}P^{2n+1})) &= (1+x)^n(1-x)^n \\ &= (1-x^2)^n. \end{aligned}$$

By expanding $(1-x^2)^n$ it is clear that the total Chern class has no terms in x^{2k+1}, so it follows that the Chern classes $c_{2k+1}(\tau^s(\mathbb{C}P^{2n+1}))$ in $H^{4k+2}(\mathbb{C}P^{2n+1})$ are zero, for all $k \geq 0$. □

Lemma 5.5.8 is the foundation for the following result, which confirms Conjecture 5.0.2 in the case of odd dimensional complex projective space.

Theorem 5.5.9. *If the quasitoric manifold $\mathbb{C}P^{2n+1}$ has an SU-structure induced by an omniorientation, then $\mathbb{C}P^{2n+1}$ it is cobordant to zero in the complex cobordism ring MU_*.*

Proof. Let $q_1, \dots, q_k$ by a partition of $2n+1$ so that $c_{q_1} \dots c_{q_k}[\mathbb{C}P^{2n+1}]$ is a Chern number given by

$$< c_{q_1}(\tau^s) \dots c_{q_k}(\tau^s), [\mathbb{C}P^{2n+1}] >, \tag{5.5.10}$$

where $[\mathbb{C}P^{2n+1}]$ is the fundamental class of $\mathbb{C}P^{2n+1}$ in the top dimension of its homology; it is determined by the chosen omniorientation on $\mathbb{C}P^{2n+1}$.

Every partition $q_1, \dots, q_k$ of $2n+1$ must be such that at least one q_i is odd. For if $q_1, \dots, q_k$ were all even, then their sum would be even, which cannot be true since by definition we have $q_1 + \dots + q_k = 2n+1$. So it follows that at least one of the Chern classes $c_{q_i}(\tau^s)$ that contributes to the Chern number $c_{q_1} \dots c_{q_k}[\mathbb{C}P^{2n+1}]$, must be such that $q_i = 2j+1$.

Then since $\mathbb{C}P^{2n+1}$ has an SU-structure, by Lemma 5.5.8 the Chern number $c_{q_1} \dots c_{q_k}[\mathbb{C}P^{2n+1}]$ is zero, for all possible partitions $q_1, \dots, q_k$ of $2n+1$. Hence we have that $\mathbb{C}P^{2n+1}$, with SU-structure induced by an omniorientation, is a boundary in MU_*. □

It is worth noting that we can prove Theorem 5.5.9 geometrically by constructing the quasitoric SU-manifold $\mathbb{C}P^{2n+1}$ as the boundary of a $(4n+3)$-dimensional stably complex manifold. If $\mathbb{C}P^{2n+1}$ has SU-structure t, which is induced by an omniorientation, then it follows from our description (5.5.5) of the cohomology generators $u_i \in H^2(\mathbb{C}P^{2n+1})$, for $1 \leq i \leq 2n+2$, and Example 3.4.4 that there is an isomorphism

$$\tau(\mathbb{C}P^{2n+1}) \oplus \mathbb{R}^2 \cong \bar{\zeta}_1 \oplus \dots \oplus \bar{\zeta}_1 \oplus \zeta_1 \oplus \dots \oplus \zeta_1 \cong (n+1)\bar{\zeta}_1 \oplus (n+1)\zeta_1, \tag{5.5.11}$$

on the stable tangent bundle of $\mathbb{C}P^{2n+1}$.

Given ξ_1, the universal quaternionic line bundle over $\mathbb{H}P^n$, it is well-known (see e.g. [3] or [4]) that there is an isomorphism of real bundles

$$\xi_1 \otimes_{\mathbb{H}} \overline{\xi}_1 \cong L \oplus \mathbb{R}, \tag{5.5.12}$$

for some real 3-plane bundle L over $\mathbb{H}P^n$. In [4, Section 2], the authors show that the 2-sphere bundle $\pi\colon S(L) \to \mathbb{H}P^n$, is such that $S(L) = \mathbb{C}P^{2n+1}$; moreover, they give an isomorphism of complex bundles

$$\pi^*(\xi_1) \cong \overline{\zeta}_1 \oplus \zeta_1. \tag{5.5.13}$$

By Proposition 2.3.7, and using the isomorphism (5.5.12), we have

$$\begin{aligned} \tau(S(L)) \oplus \mathbb{R}^2 &\cong \pi^*(L \oplus \mathbb{R} \oplus \tau(\mathbb{H}P^n)) \\ &\cong \pi^*((\xi_1 \otimes_{\mathbb{H}} \overline{\xi}_1) \oplus \tau(\mathbb{H}P^n)). \end{aligned} \tag{5.5.14}$$

In Chapter 6 we will see that there is an isomorphism on the tangent bundle of $\mathbb{H}P^{2n}$ given by

$$\tau(\mathbb{H}P^n) \oplus (\xi_1 \otimes_{\mathbb{H}} \overline{\xi}_1) \cong (n+1)\xi_1,$$

so pairing this with the isomorphism (5.5.13) we reduce (5.5.14) to

$$\tau(S(L)) \oplus \mathbb{R}^2 \cong (n+1)\overline{\zeta}_1 \oplus (n+1)\zeta_1. \tag{5.5.15}$$

This isomorphism is precisely the isomorphism (5.5.11). Hence if $\mathbb{C}P^{2n+1}$ is a quasitoric SU-manifold, we may consider it as the 2-sphere bundle $S(L)$ over $\mathbb{H}P^{2n}$. Then $\mathbb{C}P^{2n+1}$ is cobordant to zero in the complex cobordism ring, because the stably complex structure given by the isomorphism (5.5.11) bounds, as it extends via (2.3.9) to the 3-disc bundle $D(L)$ over $\mathbb{H}P^{2n}$. This illustrates the geometry underlying Theorem 5.5.9.

Further work should of course be devoted to verifying Conjecture 5.0.2.

Let $F\colon MSU_* \to MU_*$ be the map that simply forgets the special unitary structure on a manifold, and let R be a subring of MSU_* generated by the projective spaces $\mathbb{C}P^{2n+1}$ and manifolds of the form N^{2n}. If the image $\mathrm{Im}(F) = F(R)$, then Conjecture

5.0.2 is true, because it follows from Theorems 5.4.4 and 5.5.9 that $F(M^{2n}) = 0$, for any $M^{2n} \in R$. In other words, any quasitoric SU-manifold M^{2n} is cobordant in MU_* to a combination of products and connected sums of SU-manifolds of the form N^{2n} and $\mathbb{C}P^{2n+1}$, which necessarily bound in MU_*. Of course, it is almost certainly not the case that $\mathrm{Im}(F) = F(R)$, but we could conceivably prove the conjecture for a larger class of manifolds than $\mathbb{C}P^{2n+1}$ and the N^{2n} alone if we knew how much of the image of F is comprised by $F(R)$. The memoir of Conner and Floyd [13], where the image of F is studied in detail, would be a good starting point for any such investigations.

It is unclear how to extend Theorem 5.4.4 to reducible quasitoric manifolds M with a general quasitoric fibre F. The proof of our result relied on the fact that the final column of Λ was of the form $(0, \dots, 0, a)$. If, for example, M was the kth stage DO_k of a Dobrinskaya tower, so that $F = \mathbb{C}P^{l_k}$; if $l_k \neq 1$, then the proof would fail as the final column of Λ would be of the form $(0, \dots, 0, a_1, \dots, a_{l_k})$. However, given Theorem 5.5.9, perhaps a first step in this direction would be a proof of the conjecture for any reducible quasitoric manifold with fibre the quasitoric SU-manifold $\mathbb{C}P^{2n+1}$.

Suppose we relax the condition of reducibility on M. Then even if its associated polytope reduces to $P^n \times I$, the final column of Λ can take a form that differs from $(0, \dots, 0, a)$, and so an attempt to extend Theorem 5.4.4 to such manifolds M would also founder.

In Example 5.1.9 we considered the reducible quasitoric manifolds $B_{i,j}$ over $I^i \times \Delta^{j-1}$, with $i \leq j$, which comprised the basis for MU_* given by Theorem 5.0.1. Since $B_{i,j}$ is the total space of a quasitoric bundle with base space the bounded flag manifold B_i and fibre $\mathbb{C}P^{j-1}$, in light of Theorems 5.4.4 and 5.5.9 it would be natural to hope that we could go on to prove that every $B_{i,j}$ with SU-structure bounds in MU_*. However, we have the following observation.

Lemma 5.5.16. *There is no omniorientation on the reducible quasitoric manifold $B_{i,j}$ that induces an SU-structure on its stable tangent bundle.*

Proof. In [9, Example 4.5], the dicharacteristic matrix of $B_{i,j}$ is shown to contain a column of the form $v = (0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{i+j-1}$, which has only two nonzero entries. No matter how we manipulate the omniorientation of $B_{i,j}$, it is impossible to make the entries in the vector v sum to 1. It follows from Lemma 3.4.8 that $B_{i,j}$ is never a quasitoric SU-manifold. □

Consequently it seems unlikely that calculations with the $B_{i,j}$ manifolds will lead to a full solution to Conjecture 5.0.2.

Suppose instead that we could construct an alternative basis for MU_* comprised of quasitoric manifolds V^{2n}, which could admit SU-structures induced by omniorientations. If we could prove the conjecture for any quasitoric SU-manifold V^{2n}, then a proof of the full conjecture would require confirmation that every quasitoric SU-manifold is complex cobordant to a connected sum of products of V^{2n} with SU-structure, which would then necessarily be a boundary. This is by no means straightforward because if M_1 has an SU-structure, and $[M_1, \theta_1] \sim [M_2, \theta_2]$ in MU_*, it does not follow that M_2 should necessarily have an SU-structure.

Chapter 6

Quaternionic towers

In this chapter we define a quaternionic analogue of the Dobrinskaya towers of Chapter 4. Due to the noncommutativity of the quaternions $\mathbb{H}$, the process is not entirely straightforward, but by refining our definitions where necessary, we are able to carry over many aspects of the Dobrinskaya tower to the new setting.

The first problem we encounter is that the tensor product over $\mathbb{H}$ of two quaternionic line bundles is not itself a quaternionic bundle. To overcome this difficulty, in the first section we describe an operation on quaternionic bundles, which will play the role of the tensor product when we construct our quaternionic towers. We then verify that the preliminary results of Section 2.5 still hold when reformulated in the quaternionic milieu. This is the foundation for the second section, in which we define the quaternionic tower using an iterated bundle construction, determine its F-cohomology ring and consider stably quaternionic structures on the tower. We will find that unlike Dobrinskaya towers, in which each stage is a stably complex manifold, the manifolds comprising a quaternionic tower are not in general stably quaternionic. In the next section we concentrate on some exceptional cases of quaternionic tower that do admit stably quaternionic structures.

In addition to the iterated bundle construction, Dobrinskaya towers have a second description as quasitoric manifolds. Therefore we devote the final section to considering the possibility of constructing a quaternionic analogue of a quasitoric manifold to describe our quaternionic towers.

Throughout this chapter, F will denote a quaternionic oriented ring spectrum, as defined in Section 2.2.

6.1 Quaternionic line bundles

Quaternionic line bundles over a space X are classified by homotopy classes of maps $X \to \mathbb{H}P^\infty$. However, $\mathbb{H}P^\infty$ is not an Eilenberg-Mac Lane space, so in contrast with complex line bundles, there is no isomorphism between the multiplicative group of quaternionic line bundles over X and the cohomology group $H^4(X)$.

As a consequence, we will have much less freedom when building our quaternionic towers. Each stage DO_k in a Dobrinskaya tower is determined by a vector $a_{k-1}(l_k)$ of integers (4.1.5), which represents the tensor powers over $\mathbb{C}$ of the complex line bundles used to build each stage. We cannot carry over this idea to the quaternionic construction, as the tensor product over $\mathbb{H}$ of two quaternionic line bundles is not itself a quaternionic line bundle, it is only a real bundle. This follows from the fact that if $\mathbb{H}$ and $\overline{\mathbb{H}}$ denote the quaternions considered as right and left $\mathbb{H}$-modules respectively, then we have the well-known isomorphism $\mathbb{H} \otimes_{\mathbb{H}} \overline{\mathbb{H}} \cong \mathbb{R}^4$.

In this section, we consider an operation on quaternionic line bundles, which we will use in place of the tensor product when we construct our quaternionic towers in Section 6.2. We will focus on using *self-maps* $\mathbb{H}P^n \to \mathbb{H}P^n$ of quaternionic projective space, using the paper of Granja [20] as our primary reference.

With $p_1^H(\xi_1) \in H^4(\mathbb{H}P^n)$ denoting the first quaternionic Pontryagin class of the canonical quaternionic line bundle ξ_1 over $\mathbb{H}P^n$, we have the following notion.

Definition 6.1.1. The *degree* $d \in \mathbb{Z}$ of a self-map $g\colon \mathbb{H}P^n \longrightarrow \mathbb{H}P^n$ (n may be infinite), is such that $g^* p_1^H(\xi_1) = d p_1^H(\xi_1)$.

This allows us to classify self-maps of $\mathbb{H}P^\infty$, courtesy of the following result by Mislin.

Theorem 6.1.2. *[39] Self-maps of infinite quaternionic projective space $\mathbb{H}P^\infty$ are classified up to homotopy by their degree.*

Moreover, in [18, Theorem 1.2] Feder and Gitler show that there exists a self-map g of $\mathbb{H}P^\infty$ if and only if the degree of g is zero or an odd square integer $(2r+1)^2$, for $r \in \mathbb{Z}$. Sullivan constructed explicit maps with such degrees [55, Corollary 5.10].

Self-maps of $\mathbb{H}P^1$ are also classified up to homotopy by their degree, and it is well known that a self-map $\mathbb{H}P^1 \to \mathbb{H}P^1$ of degree d exists for any integer d, but for general n, self-maps of $\mathbb{H}P^n$ are not classified by degree [19], [33].

Remark 6.1.3. The permissible degrees of self-maps $\mathbb{H}P^n \to \mathbb{H}P^n$ for general n are unknown, but their determination is the subject of the Feder-Gitler Conjecture, a summary of which, and details of its confirmation in degrees $n \leq 5$, can be found in [20].

Though the picture is far from complete, we now have enough information to use self-maps of quaternionic projective space to define an operation on quaternionic line bundles.

Any quaternionic line bundle θ over a space X is the pullback along the classifying map $\theta\colon X \to \mathbb{H}P^\infty$ of the universal quaternionic line bundle ξ_1. Hence, for a degree d self-map g_d of $\mathbb{H}P^\infty$, we can define $\theta^{[d]}$ to be the pullback of ξ_1 along $g_d\theta$. By Theorem 6.1.2, self-maps of $\mathbb{H}P^\infty$ are classified up to homotopy by their degree, so the bundles $\theta^{[d]}$ are distinct for each d; moreover, by Definition 6.1.1, we have that $p_1^F(\theta^{[d]}) = dp_1^F(\theta)$ in $F^4(X)$.

By Feder and Gitler's result on the permissible degrees of self-maps $\mathbb{H}P^\infty \to \mathbb{H}P^\infty$, we have the following result.

Proposition 6.1.4. *Given a quaternionic line bundle θ over a space X, there exists distinct quaternionic line bundles $\theta^{[d]}$ over X, where $d = (2r+1)^2$, for $r \in \mathbb{Z}$, or $d = 0$.*

When $d = 1$, the bundle $\theta^{[1]}$ is θ itself, and we interpret $\theta^{[0]}$ as the trivial quaternionic line bundle $\mathbb{H}$. We use the notation $\theta^{[d]}$ to emphasise that our operation is different to the tensor product of d copies of θ, which is usually written as θ^d.

Remark 6.1.5. Since the Feder-Gitler conjecture has been verified in dimensions $n \leq 5$, we could be bolder and consider a larger pool of bundles $\theta^{[d]}$ over n-dimensional spaces, for all the known degree d self-maps of $\mathbb{H}P^n$. However, for simplicity and consistency, we will limit ourselves to maps of odd square or zero degree throughout our thesis.

Thus we have shown that we can replace the ill-behaved tensor product of quaternionic line bundles, with our operation built out of self-maps of quaternionic projective space. This will be central to our construction of quaternionic analogues of the Dobrinskaya tower in the remainder of this chapter. As a first step towards that goal, we now confirm that the preliminary results of Section 2.5 hold when translated to the quaternionic setting.

Definition 6.1.6. A *4-generated* connected CW-complex X is one whose integral cohomology ring $H^*(X)$ is generated by a linearly independent set of elements $x_1, \ldots, x_n$ in $H^4(X)$. We say such elements are *4-generators* and n is the *4-rank.*

Suppose there exist quaternionic line bundles χ_i over X, such that the first quaternionic Pontryagin class $p_1^H(\chi_i) \in H^4(X)$ is the 4-generator x_i. For any quaternionic oriented ring spectrum F, the quaternionic Pontryagin class $p_1^F(\chi_i)$ also lies in $F^4(X)$, for $1 \leq i \leq n$. Henceforth, we will denote $p_1^F(\chi_i)$ by y_i^F.

As in the complex scenario, the Atiyah-Hirzebruch spectral sequence converging to $F^*(X)$ collapses, since the ordinary cohomology of X is again concentrated in even degrees, which forces all differentials to be zero. It follows that $F^*(X)$ is a free F_*-module, spanned by the collection of monomials $\prod_R y_i^F$, where R is any subset of $\{1, \ldots, n\}$; as a free F_*-algebra, $F^*(X)$ is generated by $y_1^F, \ldots, y_n^F$.

In contrast to complex line bundles, the tensor product of two quaternionic line bundles is not itself quaternionic, as we discussed earlier in this section. To take account of this, we now make our first serious deviation from mimicking the programme of Section 2.5. Rather than working with a tensor product bundle $\chi_1^{a(1)} \otimes \cdots \otimes \chi_n^{a(n)}$ for integers $a(i)$, we must choose only one of the bundles, χ_i say, from $\chi_1, \ldots, \chi_n$. Now we are able to form $\chi_i^{[a(i)]}$, where $a(i)$ is $(2r+1)^2$, for some $r \in \mathbb{Z}$, or $a(i) = 0$, as prescribed by Proposition 6.1.4.

We now make l choices of bundles $\chi_{i_1}, \dots, \chi_{i_l}$ from $\chi_1, \dots, \chi_n$, allowing the possibility of choosing any of the bundles more than once, and define a direct sum bundle

$$\chi := \chi_{i_1}^{[a(1)]} \oplus \cdots \oplus \chi_{i_l}^{[a(l)]}, \tag{6.1.7}$$

This bundle will play the role of (2.5.3) in our quaternionic setting.

Given our 4-generated space X, which carries the n quaternionic line bundles χ_i, for $1 \leq i \leq n$, let Y denote the total space of the $\mathbb{H}P^l$-bundle $\mathbb{H}P(\chi \oplus \mathbb{H})$ over X.

Lemma 6.1.8. *The F_*-module $F^*(Y_+)$ is a free module over $F^*(X_+)$, generated by $1, y_{n+1}^F, (y_{n+1}^F)^2, \dots, (y_{n+1}^F)^{l_{n+1}}$, where y_{n+1}^F is an element of $F^4(Y_+)$. A single relation*

$$(y_{n+1}^F)(y_{n+1}^F - p_1^F(\chi_{i_1}^{[a(1)]})) \dots (y_{n+1}^F - p_1^F(\chi_{i_l}^{[a(l)]})) = 0,$$

describes the multiplicative structure.

Proof. We need only note that the Leray-Hirsch Theorem (2.5.1) holds for a quaternionic oriented cohomology theory. Then the proof of Lemma 2.5.4, with obvious adjustments to take account of the quaternionic setting, will suffice. □

Lemma 6.1.8 implies that Y itself is 4-generated with 4-rank $n+1$.

Taking a nonzero vector in the $\mathbb{H}$ summand yields a section ω for the bundle $\pi \colon Y \to X$. As in the complex case, the space obtained by the quotient of Y by the image of ω is homeomorphic to the Thom complex $T(\chi)$ of χ. Label the quotient map by ϑ. The section ω has left inverse π and so the induced cohomology sequence

$$F^*(X) \xleftarrow{\omega^*} F^*(Y) \xleftarrow{\vartheta^*} F^*(T(\chi)), \tag{6.1.9}$$

is split by π^*, thus ensuring it is short exact.

We record here a particular case of Lemma 6.1.8, which we will call upon in Chapter 7.

Example 6.1.10. Let $X = \mathbb{H}P^\infty$ and choose χ to be ξ_1, the universal quaternionic line bundle over $\mathbb{H}P^\infty$. It follows that $T(\xi_1) \cong \mathbb{H}P^\infty$. Therefore Y is the projectivisation $\mathbb{H}P(\xi_1 \oplus \mathbb{H})$, and it is straightforward to extend Segal's reasoning [51, page 45] to the quaternionic case to show that Y is homotopy equivalent to $\mathbb{H}P^\infty \vee \mathbb{H}P^\infty$.

We have that $F^*(Y_+)$ is a free module over $F^*(\mathbb{H}P^\infty_+) \cong F_*[[x]]$, generated by 1 and $y \in F^4(Y_+)$, with $(y)^2 = xy$, and simplifying the cohomology relation to $y = x$ illustrates the homotopy equivalence between Y and $\mathbb{H}P^\infty \vee \mathbb{H}P^\infty$.

Note that the first quaternionic Pontryagin class $p_1^F \in F^4(\mathbb{H}P^\infty)$ induces a canonical Thom class $t^F \in F^4(T(\xi_1))$, and so we have a Thom isomorphism $F^{*-4}(X_+) \cong F^*(T(\xi_1))$. This allows us to view y as the pullback $\vartheta^* t^F$.

As in the complex case, products of the form $\pi^*(x)y^F_{n+1}$ may be written as $\vartheta^*(xt^F)$, for any $x \in F^*(X)$.

6.2 Constructing the towers

Having established the fundamentals, we are able to define quaternionic towers using a construction analogous to that of the Dobrinskaya towers of Section 4.4. Following this, we calculate the F-cohomology ring for each stage in the tower and consider their stably quaternionic structures.

Suppose we are given a sequence of nonnegative integers $(l_1, \ldots, l_n)$, to which we associate a *bundle list* $(j(l_1), \ldots, j(l_n))$, where $j(l_i)$ is an l_i-vector $(j_{i,1}, \ldots, j_{i,l_i})$ with entries taken from the set $\{1, \ldots, i-1\}$. Note that we allow repetition of choices, so that if, say $l_3 = 5$, then $(1, 1, 2, 1, 1)$ or $(2, 1, 2, 1, 2)$ are both valid examples of the vector $j(l_3)$. The quaternionic Dobrinskaya tower will be constructed inductively: the $(k-1)$th stage carries $k-1$ quaternionic line bundles, and we will choose l_k of them to construct the kth stage. The vector $(j(l_k))$ records the chosen bundles.

Furthermore, to $(l_1, \ldots, l_n)$ we associate a *quaternionic list* $(a_1(l_2), \ldots, a_{n-1}(l_n))$ of l_i-vectors

$$a_{i-1}(l_i) = (a(i, 1), \ldots, a(i, l_i)), \tag{6.2.1}$$

with entries $a(i, j) = (2r + 1)^2$, for some $r \in \mathbb{Z}$, or $a(i, j) = 0$.

For the inductive construction of the tower, assume we have already built the $(k-1)$th stage QT_{k-1} as a smooth $4(l_1 + \cdots + l_{k-1})$-dimensional manifold, 4-generated by y_i^H and carrying quaternionic line bundles χ_i, such that $p_1^H(\chi_i) = y_i^H$, for $1 \leq i \leq$

$k-1$. Define the *ith bundle* of the construction to be

$$\chi(a_{i-1}) := \chi_{j_{i,1}}^{[a(i,1)]} \oplus \cdots \oplus \chi_{j_{i,l_i}}^{[a(i,l_i)]}. \qquad (6.2.2)$$

Then QT_k is defined to be the total space of $\mathbb{H}P(\chi(a_{k-1})\oplus\mathbb{H})$, the projectivisation of the direct sum of $\chi(a_{k-1})$ and a trivial quaternionic line bundle $\mathbb{H}$.

We define QT_0 to be the space consisting of a single point so that the first bundle is trivial, then the next stage QT_1 is $\mathbb{H}P^{l_1}$. Lemma 6.1.8 implies that QT_k has 4-generators y_i^H for $1 \leq i \leq k$, where y_k^H is the first Pontryagin class $p_1^H(\chi_k)$ of the canonical line bundle χ_k over QT_k.

Definition 6.2.3. Given the nonnegative integers $(l_1, \ldots, l_n)$, the sequence of manifolds $(QT_k : k \leq n)$, arising from a bundle list $(j(l_1), \ldots, j(l_n))$ and a quaternionic list $(a_1(l_2), \ldots, a_{n-1}(l_n))$, is a *quaternionic tower of height* n (which may be infinite). The individual manifold QT_k is known as the *kth stage* of the tower, for $1 \leq k \leq n$.

As a consequence of the discussion that followed Lemma 6.1.8, we have projections $\pi_k \colon QT_k \to QT_{k-1}$, sections $\omega_k \colon QT_{k-1} \to QT_k$ and quotient maps $\vartheta_k \colon QT_k \to T(\chi(a_{k-1}))$.

From our preliminary investigation of 4-generated projective bundles, we can determine the F-cohomology ring of any stage QT_k.

Proposition 6.2.4. *For any quaternionic oriented ring spectrum F, the F_*-algebra $F^*(QT_{k+})$ is isomorphic to*

$$F_*[y_1^F, \ldots, y_k^F]/\mathcal{L}_k$$

where $\mathcal{L}_k$ is the ideal

$$((y_i^F)(y_i^F - p_1^F(\chi_{j_{i,1}}^{[a(i,1)]})) \ldots (y_i^F - p_1^F(\chi_{j_{i,l_i}}^{[a(i,l_i)]})) : 1 \leq i \leq k). \qquad (6.2.5)$$

Proof. The result follows from $k-1$ applications of Lemma 6.1.8. □

Note further that when $a(i,j) = 0$ for all $1 \leq j \leq l_i$, so that $QT_i \cong \mathbb{H}P^{l_i}$, we recover from (6.2.5) the usual relation $(y_i^F)^{l_i+1} = 0$, on $F^*(QT_{i+}) \cong F^*(\mathbb{H}P^{l_i}_+)$.

In general, each stage QT_k in a quaternionic tower does not carry a stably quaternionic structure. Essentially this is because each QT_k is a $\mathbb{H}P^{l_k}$-bundle over QT_{k-1}, and $\mathbb{H}P^{l_k}$ does not have a stably quaternionic structure when $l_k \neq 1$. This is due to the fact (see e.g. [26, Theorem 1.1]) that its tangent bundle admits an isomorphism

$$\tau(\mathbb{H}P^{l_1}) \oplus (\xi_1 \otimes_{\mathbb{H}} \overline{\xi}_1) \cong \xi_1 \oplus \cdots \oplus \xi_1 \cong (l_k + 1)\xi_1, \tag{6.2.6}$$

and $\xi_1 \otimes_{\mathbb{H}} \overline{\xi}_1$ is not quaternionic. In the exceptional case when $l_k = 1$, Proposition 2.3.7 implies that $\mathbb{H}P^1$ carries a trivial stably quaternionic structure, and there is an isomorphism $\xi_1 \otimes_{\mathbb{H}} \overline{\xi}_1 \cong \xi_1 \oplus \xi_1$ of real bundles over $\mathbb{H}P^1$ (for more details see [3] and [26]). In the following section, we pay particular attention to this exceptional case, to define quaternionic towers in which each stage does carry a stably quaternionic structure.

6.3 Special cases

In this section we concentrate on two particular subfamilies of quaternionic tower. We begin by defining the quaternionic analogue of a Bott tower, and describe a quaternionic structure on the stable tangent bundle of each stage. Then we specialise further to define a bounded quaternionic flag manifold.

Suppose we have a quaternionic tower $(QT_k : k \leq n)$, arising from the sequence of integers $(l_1, \ldots, l_n)$, as in Definition 6.2.3.

Definition 6.3.1. A *quaternionic Bott tower of height n* is a quaternionic tower in which $l_i = 1$ for all $1 \leq i \leq n$.

To distinguish the quaternionic Bott tower, we will denote the kth stage by Q^k. Again for convenience we set Q^0 to be the one-point space, while Q^1 is $\mathbb{H}P^1$. When $a(2,1) = 0$, the second stage $Q^2 = \mathbb{H}P(\chi_{j_{2,1}}^{[a(2,1)]} \oplus \mathbb{H})$ takes the form $\mathbb{H}P^1 \times \mathbb{H}P^1$, while for $a(2,1) = 1$ we can interpret Q^2 as $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^{\,2}$, that is, a connected sum of quaternionic projective planes with opposite orientations.

Remark 6.3.2. Unlike the situation for Bott towers, which we documented in Section 4.5, the isomorphism class of the direct sum bundle $\chi_{j_{2,1}}^{[a(2,1)]} \oplus \mathbb{H}$ does not depend only on the parity of $a(2,1)$. Therefore unless $a(2,1) = 0$ or 1, we cannot necessarily view Q^2 as either $\mathbb{H}P^1 \times \mathbb{H}P^1$ or $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^{\,2}$.

At the close of Section 6.2, we saw that $\mathbb{H}P^1$ carries a stably quaternionic structure. This fact is especially relevant for the quaternionic Bott tower since each stage $Q^k = \mathbb{H}P(\chi_{j_{k,1}}^{[a(k,1)]} \oplus \mathbb{H})$ is an $\mathbb{H}P^1$-bundle over Q^{k-1}. Since $\mathbb{H}P^1$ is homeomorphic to the 4-sphere S^4, we may equally consider the kth stage of the tower as the total space of a 4-sphere bundle $S(\chi_{j_{k,1}}^{a(k,1)} \oplus \mathbb{R})$. We prefer this viewpoint because it allows us to easily describe a stably quaternionic structure on Q^k.

Proposition 6.3.3. *The tangent bundle of Q^k admits an isomorphism*

$$\tau(Q^k) \oplus \mathbb{R} \cong \mathbb{R} \oplus \left(\bigoplus_{i=1}^{k} \chi(a_i) \right), \tag{6.3.4}$$

where $\chi(a_i) = \chi_{j_{i,1}}^{[a(i,1)]}$. The isomorphism determines a bounding stably quaternionic structure on Q^k.

Proof. The isomorphism follows from Proposition 2.3.7. The structure bounds since it extends to the 5-disc bundle $D(\chi_{j_{k,1}}^{[a(k,1)]} \oplus \mathbb{R})$. □

We also have the quaternionic analogue of Civan and Ray's result [12, Proposition 3.3], which gives a decomposition of the suspension ΣQ^k as a wedge of suspensions of Thom complexes.

Proposition 6.3.5. *Given a quaternionic Bott tower $(Q^k : k \leq n)$, there is a homotopy equivalence*

$$h_k \colon \Sigma Q^k \to \Sigma S^4 \vee \Sigma T(\chi(a_1)) \vee \cdots \vee \Sigma T(\chi(a_{k-1})), \tag{6.3.6}$$

for each $1 \leq k \leq n$.

Proof. First consider the map $h_i \colon \Sigma Q^i \to \Sigma Q^{i-1} \vee \Sigma T(\chi(a_{i-1}))$, given by the sum $\Sigma\pi_i + \Sigma\vartheta_i$. Now for the homotopy inverse, take the cofibre sequence

$$Q^i = S(\chi(a_{i-1}) \oplus \mathbb{R}) \xrightarrow{\pi_i} Q^{i-1} \xrightarrow{\iota_i} T(\chi(a_{i-1}) \oplus \mathbb{R}) \xrightarrow{\varrho_i} \Sigma Q^i \longrightarrow \dots,$$

in which ι_i is the inclusion of the zero section, and the map ϱ_i collapses this copy of Q^{i-1} in $T(\chi(a_i) \oplus \mathbb{R})$. Since $T(\chi(a_i) \oplus \mathbb{R}) \cong \Sigma T(\chi(a_i))$ [27, Corollary 15.1.6], and the section ω_i is π_i^{-1}, our inverse is given by $\Sigma\omega_i \vee \varrho_i$.

Having established the homotopy equivalence h_i, a simple inductive argument, with the base case $k = 1$ provided by taking $T(\chi(a_0)) = S^4$, completes the proof. □

We can interpret this result in terms of the F-cohomology of Q^k, again following the lead of Civan and Ray. From Proposition 6.2.4, we have that $F^{4i}(Q^k_+)$ is the F_*-module generated by monomials $\prod_R y_i^F$, for subsets $R \subseteq \{1, \dots, k\}$ of cardinality i. The homotopy equivalence h_k induces an isomorphism in F-cohomology, which desuspends to give an additive splitting of the F_*-module $F^*(Q^k)$ as

$$F^*(Q^k) \cong \langle y^F_{\leq 1} \rangle \oplus \cdots \oplus \langle y^F_{\leq k} \rangle , \tag{6.3.7}$$

where $\langle y^F_{\leq i} \rangle$ denotes the free F_*-submodule of $F^*(Q^k)$, generated by monomials $\prod_R y_i^F$ with $R \subseteq \{1, \dots, i\}$ and $i \in R$.

We can specialise further to define a subfamily of quaternionic Bott towers.

Definition 6.3.8. Consider the quaternionic Bott tower $(Q^k : k \leq n)$ arising from a bundle list $(j(l_1), \dots, j(l_n))$, in which each $j(l_i) = (i-1)$, and a quaternionic list $(a_1(l_2), \dots, a_{n-1}(l_n))$, in which each $a_{i-1}(l_i) = (1)$ for all i. Then each stage Q^k is a *bounded quaternionic flag manifold*.

The bounded quaternionic flag manifold Q^n is the space of all *bounded quaternionic flags* V in $\mathbb{H}^{n+1}$; that is, the set of all sequences $V = \{V_1 \subset V_2 \subset \cdots \subset V_n \subset V_{n+1} = \mathbb{H}^{n+1}\}$, in which V_i contains the coordinate subspace $\mathbb{H}^{i-1}$ spanned by the first $i-1$ standard basis vectors of $\mathbb{H}^{n+1}$, for $2 \leq i \leq n$. In our construction, these manifolds are of the form $Q^k = \mathbb{H}P(\chi_{k-1} \oplus \mathbb{H})$, or equivalently $S(\chi_{k-1} \oplus \mathbb{R})$. As in the complex case, Ray [47] studied such manifolds in cobordism theory, and the final chapter of our thesis will be concerned with extending this work.

6.4 Quaternionic quasitoric manifolds

Each stage in a Dobrinskaya tower can be realised as a quasitoric manifold, therefore it is natural to try to construct each stage QT_k of our quaternionic tower as a quaternionic analogue of a quasitoric manifold. The possibility of such a construction is the subject of this final section. We want to take the fundamental ideas of toric topology, as outlined in Chapter 3, and explore to what extent we are able to formulate a quaternionic analogue of the theory. In conclusion, we briefly survey other authors' approaches to quaternionic toric topology.

The unit sphere in the quaternions is the 3-sphere S^3. Though S^3 is not commutative, we would like the *quaternionic n-torus* $(S^3)^n$ to play the role of the n-torus T^n in our construction of a quaternionic theory of toric topology. Coordinatewise left multiplication of $(S^3)^n$ on $\mathbb{H}^n$ is called the *standard representation* of $(S^3)^n$ and a $4n$-dimensional manifold M^{4n} with an action of $(S^3)^n$ is termed an $(S^3)^n$*-manifold*. Then we have the notion of an $(S^3)^n$ -action on M^{4n} being *locally isomorphic* to the standard representation in the obvious manner, similar to Definition 3.1.1.

Complex projective space $\mathbb{C}P^n$, viewed as a quasitoric manifold over the n-simplex Δ^n, is one of the fundamental constructions in toric topology. Before we consider the more complicated case of a quaternionic tower, let us construct quaternionic projective space $\mathbb{H}P^n$ by analogy with Example 3.2.12.

Example 6.4.1. In place of the moment angle complex $\mathcal{Z}_{\Delta^n} = S^{2n+1}$ we will take the $(4n+3)$-sphere S^{4n+3}; such a choice will be valid if the orbit space of S^{4n+3} under the standard left action of $(S^3)^{n+1}$ is the n-simplex Δ^n.

Let us consider the case when $n = 2$. We view S^{11} as the set $\{(a_1, a_2, a_3) \in \mathbb{H}^3 : |a_1|^2 + |a_2|^2 + |a_3|^2 = 1\}$, and a point in $(S^3)^3$ is denoted by (v_1, v_2, v_3). Our quaternionic torus $(S^3)^3$ acts freely on the left of S^{11} by coordinatewise multiplication. Then the orbit of a point (a_1, a_2, a_3) under this action of $(S^3)^3$ is given by $\{(v_1a_1, v_2a_2, v_3a_3) : v_i \in S^3\}$, which we can rewrite as

$$\left\{\left(\frac{v_1a_1|a_1|}{|a_1|}, \frac{v_2a_2|a_2|}{|a_2|}, \frac{v_3a_3|a_3|}{|a_3|}\right)\right\} = \{(v_1', v_2', v_3') \cdot (|a_1|, |a_2|, |a_3|) : v_i' \in S^3\},$$

where $v_i' = v_ia_i/|a_i|$. So the orbit space of S^{11} under the free action of $(S^3)^3$ is homeomorphic to the space of points $(b_1, b_2, b_3) \in \mathbb{R}^3$, where $b_i = |a_i|$. Now, since $b_i \geq 0$ and $b_1+b_2+b_3 = 1$, we have that $\{(b_1, b_2, b_3)\}$ is exactly Δ^2 as in the description (3.1.7).

In a similar manner, if we define a diagonal subgroup $D_{\mathbb{H}}$ to be $\{v, v, v\} < (S^3)^3$, the orbit space of S^{11} under the action of $D_{\mathbb{H}}$ can be seen to be $\mathbb{H}P^2$.

It is straightforward to generalise these arguments to show that we can view both $\mathbb{H}P^n$, and the associated polytope Δ^n, as quotients of S^{4n+3} by appropriate actions of $D_{\mathbb{H}} < (S^3)^{n+1}$ and $(S^3)^{n+1}$ respectively.

Now we can turn to the quaternionic tower. Continuing our analogy with the complex scenario, the polytope associated with QT_k is the product of simplices $\Delta^{l_1} \times \cdots \times \Delta^{l_k}$, which we earlier denoted by P^L, where $L = l_1 + \cdots + l_k$. Following Example 6.4.1, we replace the moment angle complex $\mathcal{Z}_{P^L} = S^{2l_1+1} \times \cdots \times S^{2l_k+1}$ by $S^{4l_1+3} \times \cdots \times S^{4l_k+3}$, which we denote by $\mathcal{Q}_{P^L}$.

As with the Dobrinskaya tower, we would like to realise each stage QT_k of the quaternionic tower as a quotient of $\mathcal{Q}_{P^L}$ by the action of a group of the form

$$\{(v_1, \ldots, v_1, v_{j_{1,1}}^{-a(1,1)} v_2, \ldots, v_{j_{1,l_1}}^{-a(1,l_1)} v_2, \ldots, v_{j_{i,1}}^{-a(i,1)} v_i, \ldots, v_{j_{i,l_i}}^{-a(i,l_i)} v_i, \ldots$$
$$\ldots, v_{j_{k,1}}^{-a(k,1)} v_k, \ldots, v_{j_{k,l_k}}^{-a(k,l_k)} v_k, v_1^{-1}, \ldots, v_k^{-1}) : v_j \in S^3, 1 \leq j \leq k\} < (S^3)^{L+k}.$$

However, due to the noncommutativity of the quaternions the above subspace does not form a group under multiplication, unless $a(i,j) = 0$ for all i, j. In light of Example 6.4.1, it is easy to see that this exceptional case corresponds to a tower in which each stage QT_i is the product

$$\mathbb{H}P^{l_1} \times \cdots \times \mathbb{H}P^{l_i}.$$

Otherwise, without a suitable group to act on $\mathcal{Q}_{P^L}$, we cannot continue to mimic the quasitoric manifold construction of the Dobrinskaya tower for our quaternionic case.

A naïve transfer of the ideas of Chapter 3 to the realm of the quaternions therefore fails. We now consider two alternative approaches, giving a brief outline of each, and explaining the implications for our goal of realising QT_k as a quaternionic analogue of a quasitoric manifold.

Mazaud [35] [36] studies closed, orientable 8-dimensional manifolds M^8, which are equipped with an action of $S^3 \times S^3$, and whose orbit space M^2 is a surface with boundary. Assume that the interior of the orbit space consists entirely of free orbits, while points of M^8 with nontrivial isotropy subgroups occur over the boundary of M^2. In this situation Mazaud obtains a complete equivariant classification of $(S^3 \times S^3)$-actions on such manifolds M^8.

If M^2 is viewed as a 2-dimensional polytope P^2, such that fixed points under the $(S^3 \times S^3)$-action project to the vertices of P^2 and points with 3-dimensional isotropy subgroups project to the facets of P^2, then M^8 is a quaternionic analogue of a quasitoric manifold. Mazaud identifies the manifolds that arise in precisely this situation, and he views them, in similar fashion to construction (3.2.10), as

$$S^3 \times S^3 \times P^2 / \sim, \tag{6.4.2}$$

where the identification $\sim$ depends on the isotropy subgroups of the $(S^3 \times S^3)$-action.

We record here some possible examples of M^8.

Example 6.4.3. [35, Examples 2.4]

(i) When P^2 is the 2-simplex Δ^2, the manifold M^8 is either $\mathbb{H}P^2$ or $\overline{\mathbb{H}P}^2$, that is, quaternionic projective planes with opposite orientations.

(ii) When P^2 is the square I^2, the manifold M^8 is either $S^4 \times S^4$ or the connected sum $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$.

(iii) In general, for polytopes P^2 with 4 or more vertices, M^8 is a connected sum of copies of $S^4 \times S^4$ or a connected sum of copies of $\mathbb{H}P^2$ and $\overline{\mathbb{H}P}^2$.

As we saw in Section 6.3, the manifolds $S^4 \times S^4$ and $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$ are the second stage of a Bott tower $Q^2 = \mathbb{H}P(\chi_{j_{2,1}}^{[a(2,1)]} \oplus \mathbb{H})$, when $a(1,2) = 0$ and 1, respectively. However, as Remark 6.3.2 indicates, it is unclear how to represent any other Q^2 with $a(1,2) \neq 0, 1$ in the form (6.4.2). Nevertheless, Example 6.4.3 does show that some special cases of our quaternionic towers can be viewed as a quaternionic analogue of a quasitoric manifold.

Remark 6.4.4. Notice that since $\chi_{j_{2,1}} = \chi_1$, the connected sum $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$ is in fact the bounded quaternionic flag manifold $\mathbb{H}P(\chi_1 \oplus \mathbb{H})$. Since a sequence of bounded quaternionic flag manifolds is the simplest nontrivial quaternionic tower, extending Mazaud's analysis to higher dimensions to include $\mathbb{H}P(\chi_k \oplus \mathbb{H})$, for $k > 1$, would be a good starting point for any further investigations. Unfortunately, such an extension does not follow in any obvious manner from the results of [35] and [36].

A second approach to a quaternionic version of toric topology is due to Scott [50], who constructs spaces M called *quaternionic toric varieties*, which are inspired by Davis and Januskiewicz's quasitoric manifolds. They are of the form

$$(S^3)^n \times P^n / \sim, \tag{6.4.5}$$

where P^n is a simple polytope, and the equivalence relation, which is distinct from that of construction (6.4.2), depends on certain self-diffeomorphisms of $(S^3)^n$. This identification does not always preserve the left $(S^3)^n$-action on $(S^3)^n \times P^n$ due to the noncommutativity of the unit quaternions S^3. However, every unit quaternion u gives rise to an inner automorphism on S^3, given by $a_u : v \mapsto uvu^{-1}$, for $v \in S^3$. Since we have $a_u = a_{-u}$, the group of such inner automorphisms on S^3 is exactly $SO(3)$. Therefore, in general Scott's spaces do not have a full $(S^3)^n$-action, but only a diagonal action of $SO(3)$, given by $u \cdot [v_1, \ldots, v_n, p] \mapsto [uv_1u^{-1}, \ldots, uv_nu^{-1}, p]$, for $u \in SO(3)$ and $[v_1, \ldots, v_n, p] \in M$ (see [50, Property 2.3]).

Though the methods of Mazaud and Scott are closely related to those of toric topology, we are no closer to constructing each stage QT_k of a quaternionic tower as a quaternionic analogue of a quasitoric manifold. Aside from trivial cases, we were only able to represent $Q^2 = \mathbb{H}P(\chi_1 \oplus \mathbb{H})$ in the form (6.4.2), but only because we knew beforehand that this Q^2 was homeomorphic to $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$. In general, even if we were able to define some QT'_k as an $(S^3)^L$-manifold using an identification of the form (6.4.2) (or as a quaternionic toric variety with an $SO(3)$-action using (6.4.5)), it is unclear how we would prove that this manifold is diffeomorphic to QT_k. Simply modifying the proof of Proposition 4.4.6 to take account of the quaternions is inadequate, since this relies on viewing QT_k as a quotient of $\mathcal{Q}_{PL}$, and as we discovered above, such an approach is unavailable to us.

Clearly there is scope to further develop the ideas discussed in this section, perhaps starting with the suggestion of Remark 6.4.4. We emphasise that while our attempt to mimic the programme of Chapters 3 and 4 to construct a quaternionic analogue of a quasitoric manifold failed, we do not rule out the possibility that our manifolds QT_k could yet fit into the schemes developed by Mazaud and Scott.

Chapter 7

Quaternionic towers and cobordism theory

Quaternionic towers play an important part in the search for geometrical representatives of elements in the quaternionic cobordism ring MSp_*. Ray [47] utilised the subfamily of bounded quaternionic flag manifolds to help describe the torsion elements $\varphi_m \in MSp_{8m-3}$, and he claimed that a collection of simply connected manifolds Y^{4n+1}, which are related to the Dobrinskaya towers, could also serve as representatives for φ_m.

In this chapter we will verify Ray's conjecture by constructing the manifolds Y^{4n+1} and proving that they can indeed represent φ_m, when $n = 2m-1$. We achieve this by applying the machinery of Ray, Switzer and Taylor [48], which evaluates all the possible stably quaternionic structures on our manifolds by computing the image $\Psi(Y^{4n+1})$ of the map $\Psi_g\colon SO/Sp^0(Y^{4n+1}_+) \longrightarrow MSp_{4n+1}$, which we introduced in Section 2.3.4.

We begin with some preliminaries on quaternionic cobordism, defining the torsion elements φ_m and obtaining their original geometrical representatives. In the next section we consider the geometry of the manifolds Y^{4n+1} and calculate their F-cohomology rings, for any quaternionic oriented ring spectrum F. The remainder of the chapter is devoted to computing $\Psi(Y^{4n+1})$, beginning with descriptions of the groups $U/Sp^0(Y^{4n+1})$ and $SO/Sp^0(Y^{4n+1})$. This allows us to determine the associated units of Y^{4n+1} in the following section. Then we obtain fundamental classes in $MSp_*(Y^{4n+1})$, which leads finally to our calculation of $\Psi(Y^{4n+1})$.

Throughout this chapter F will be a quaternionic oriented ring spectrum, but note that in order to simplify our presentation, when the context is clear we frequently drop the superscript F from our notation for characteristic classes and F-cohomology generators.

7.1 Torsion in MSp_*

In this section we consider the role of a particular subfamily of quaternionic tower in MSp_*. We state some well-known results in quaternionic cobordism theory and give the various definitions that we will need for our work in the remainder of this chapter. Our main objective is to define the torsion elements φ_m, and study Ray's original geometrical representatives.

In Definition 6.3.8 we described a quaternionic tower in which each stage $Q^k = \mathbb{H}P(\chi_{k-1} \oplus \mathbb{H})$ was a bounded quaternionic flag manifold, carrying quaternionic line bundles $\chi_0, \ldots, \chi_k$. Since $\mathbb{H}P^1 \cong S^4$, we may equally view Q^k as a 4-sphere bundle $S(\chi_{k-1} \oplus \mathbb{R})$ over Q^{k-1}.

Applying Proposition 6.3.3 in the case of Q^k, we obtain an isomorphism

$$\tau(Q^k) \oplus \mathbb{R} \cong \mathbb{R} \oplus \left(\bigoplus_{i=0}^{k-1} \chi_i \right). \tag{7.1.1}$$

To describe $\nu^s(Q^k)$, the stable normal bundle of Q^k, we introduce a quaternionic bundle $\chi_k^\perp$ over Q^k, which satisfies

$$\chi_{k-1} \oplus \mathbb{H} \cong \chi_k^\perp \oplus \chi_k.$$

Over each point $l \in Q^k$, which is a quaternionic line in $\chi_{k-1} \oplus \mathbb{H}$, the orthogonal compliment $l^\perp$ in $\chi_{k-1} \oplus \mathbb{H}$ is taken as the fibre of $\chi_k^\perp$; in fact, we can pull back similarly defined orthogonal bundles from each lower stage Q^i to define $\chi_i^\perp$ over Q^k, such that

$$\chi_{i-1} \oplus \mathbb{H} \cong \chi_i^\perp \oplus \chi_i,$$

for $i = 1, \ldots, k-1$. By [47, Proposition 1.4], this allows us to describe an isomorphism on $\nu^s(Q^k)$, the stable normal bundle of Q^k, which is given by

$$\nu^s(Q^k) \cong \bigoplus_{i=1}^{k-1} (k-i)\chi_i^{\perp}. \tag{7.1.2}$$

The following properties of BSp, the classifying space of the infinite symplectic group Sp, are well-known (see e.g. [56, (16.34)]).

The quaternionic cobordism ring of BSp is isomorphic to the algebra of formal power series

$$MSp^*(BSp_+) \cong MSp_*[[p_1, p_2, \ldots]], \tag{7.1.3}$$

generated by the universal Pontryagin classes $p_i = p_i(\xi) \in MSp^{4i}(BSp_+)$, for $i \geq 1$.

We now introduce a particular collection of elements in $MSp^*(BSp)$, which will be useful throughout this chapter. To begin, suppose we have a Hopf algebra H, equipped with a coproduct $f\colon H \to H \otimes H$.

Definition 7.1.4. An element x of a Hopf algebra H is *primitive* if $f(x) = x \otimes 1 + 1 \otimes x$.

Primitive elements are important, for if a Hopf algebra H is generated as an algebra by a collection of such elements, then the coproduct f is uniquely determined by the product $H \otimes H \to H$.

Since BSp is a loop space, it has an H-space structure, and so $MSp^*(BSp)$ is a Hopf algebra (see e.g. [22, Section 3C]). The coproduct $f\colon MSp^*(BSp) \to MSp^*(BSp) \otimes MSp^*(BSp)$ is induced by the Whitney sum operation on quaternionic bundles. In [42] the collection of primitive elements of $MSp^*(BSp)$ is described as having one additive generator $P_i \in MSp^{4i}(BSp)$ for all i. To express these generators as polynomials in quaternionic Pontryagin classes $p_i \in MSp^{4i}(BSp)$, with $P_1 = p_1$, we have an inductive formula

$$P_i = \sum_{j=1}^{i-1} (-1)^{j+1} p_j P_{i-j} + (-1)^{i+1} i p_i. \tag{7.1.5}$$

In low dimensions this yields

$$\begin{aligned}
P_1 &= p_1,\\
P_2 &= p_1^2 - 2p_2,\\
P_3 &= p_1^3 - 3p_1p_2 + 3p_3,\\
P_4 &= p_1^4 - 4p_1^2p_2 + 2p_2^2 + 4p_1p_3 - 4p_4.
\end{aligned}$$

In this chapter we often work with quaternionic line bundles. If θ denotes a quaternionic line bundle over a space X, then θ is classified by a map $X \to \mathbb{H}P^\infty$. Restricting (7.1.3) to $BSp(1) \cong \mathbb{H}P^\infty$, we have that the quaternionic cobordism ring of $\mathbb{H}P^\infty$ is isomorphic to the algebra of formal power series

$$MSp^*(\mathbb{H}P^\infty_+) \cong MSp_*[[p_1]],$$

generated by the first universal Pontryagin class $p_1 = p_1(\xi_1) \in MSp^4(\mathbb{H}P^\infty_+)$. The dual quaternionic bordism module $MSp_*(\mathbb{H}P^\infty_+)$ is described as follows.

Proposition 7.1.6. *[56, (16.34)] The free MSp_*-module $MSp_*(\mathbb{H}P^\infty_+)$ is generated by $q_0, q_1, q_2, \dots,$ where $q_k \in MSp_{4k}(\mathbb{H}P^\infty_+)$, and $q_0 = 1 \in MSp_0(\mathbb{H}P^\infty_+)$.*

The usual properties of duality imply that the cap product of q_n in $MSp_{4n}(\mathbb{H}P^\infty_+)$ with the first universal quaternionic Pontryagin class p_1 gives

$$p_1 \frown q_n = q_{n-1}, \text{ for } n = 1, 2, \dots, \tag{7.1.7}$$

so that $p_1^i \in MSp^{4i}(\mathbb{H}P^\infty_+)$ is dual to $q_i \in MSp_{4i}(\mathbb{H}P^\infty_+)$, for all i.

The bounded quaternionic flag manifolds Q^k now come to prominence as geometrical representatives for the generators of $MSp_*(\mathbb{H}P^\infty)$.

Proposition 7.1.8. *[47, Proposition 2.2] The generators $q_k \in MSp_{4k}(\mathbb{H}P^\infty)$ are represented by the manifolds*

$$\chi_k \colon Q^k \longrightarrow \mathbb{H}P^\infty, \tag{7.1.9}$$

where χ_k is the classifying map of the canonical quaternionic line bundle χ_k over Q^k.

Proposition 7.1.8 will allow us to fabricate geometrical representatives for the torsion elements φ_m in MSp_{8m-3}.

First we must define the φ_m. Let $g_n^F \in F^n(S^n)$ be a generator of the cohomology ring $F^*(S^n)$, and take η to be the real Hopf line bundle over S^1. The tensor product of η with the universal quaternionic line bundle ξ_1 over $\mathbb{H}P^\infty$, yields a new quaternionic bundle $\eta \otimes_{\mathbb{R}} \xi_1$ over $S^1 \times \mathbb{H}P^\infty$.

Given the isomorphism

$$MSp^*(S^1 \times \mathbb{H}P^\infty) \cong MSp_*[[p_1, g_1]]/g_1^2, \tag{7.1.10}$$

we can expand $p_1(\eta \otimes_{\mathbb{R}} \xi_1) \in MSp^4(S^1 \times \mathbb{H}P^\infty)$ to give

$$p_1(\eta \otimes_{\mathbb{R}} \xi_1) = p_1 + \sum_{k>0} g_1 \theta_k p_1^k, \tag{7.1.11}$$

where $\theta_k \in MSp_{4k-3}$ for $k = 1, 2, \ldots$.

The element θ_1, represented by the circle with suitable stably quaternionic structure, generates MSp_1 [44, Lemma 4.1]; Roush [49] proved that $\theta_{2k+1} = 0$, for $k > 0$.

We follow the notational convention of relabelling the remaining $\theta_{2k} \in MSp_{8k-3}$ as φ_k, for $k = 1, 2, \ldots$. These elements were first described by Ray [42], who proved that they are multiplicatively indecomposable and generate part of the 2-torsion in MSp_*.

Later Ray utilised the bounded quaternionic flag manifolds to manufacture geometrical representatives for φ_n.

Proposition 7.1.12. *[47, Proposition 4.2] The torsion element $\varphi_n \in MSp_{8n-3}$ can be represented geometrically by $S^1 \times Q^{2n-1}$, with a suitable stably quaternionic structure.*

Proof. For notational convenience we will often denote $S^1 \times Q^k$ by W^{4k+1}.

Using the expansion (7.1.11), and the duality property (7.1.7) of the elements $q_k \in MSp_{4k}(\mathbb{H}P_+^\infty)$ we have

$$\varphi_n = \langle p_1(\eta \otimes_{\mathbb{R}} \xi_1), h_1 \otimes q_{2n} \rangle, \tag{7.1.13}$$

where $h_1 \in MSp_1(S^1)$ denotes the dual generator to g_1, and $\langle -, - \rangle$ is the Kronecker product for MSp.

We represent $h_1 \otimes q_{2n}$ by $f_{2n}\colon S^1 \times Q^{2n} \to \mathbb{H}P^{2n}$, where f_{2n} is the classifying map for the quaternionic bundle $\eta \otimes_{\mathbb{R}} \chi_{2n}$, and we obtain φ_n by making the map f_{2n} transverse to the inclusion $\mathbb{H}P^{2n-1} \to \mathbb{H}P^{2n}$.

The transversality construction yields the following diagram

$$\begin{array}{ccc} W^{8n-3} & \xrightarrow{j} & W^{8n+1} \\ \Big\downarrow{\scriptstyle f_{2n-1}} & & \Big\downarrow{\scriptstyle f_{2n}} \\ \mathbb{H}P^{2n-1} & \xrightarrow{i} & \mathbb{H}P^{2n}, \end{array} \tag{7.1.14}$$

where i is inclusion into the first $2n$ homogeneous coordinates of $\mathbb{H}P^{2n}$, and j embeds Q^{2n-1} in $Q^{2n} = \mathbb{H}P(\chi_{2n-1} \oplus \mathbb{H})$ as the subspace of quaternionic lines in χ_{2n-1}.

Now we describe the normal and tangent bundles of W^{8n-3}. Diagram (7.1.14) gives rise to an isomorphism

$$\nu^s(W^{8n-3}) \cong \nu(j) \oplus j^*\nu^s(W^{8n+1}). \tag{7.1.15}$$

The normal to the embedding i satisfies $\nu(i) \cong \xi_1$, where ξ_1 is the canonical quaternionic line bundle over $\mathbb{H}P^{2n-1}$, and pulling back ξ_1 along f_{2n-1}, we attain $\nu(j) \cong \eta \otimes_{\mathbb{R}} \chi_{2n-1}$.

Coupling the description of the stable normal bundle of Q^{2n} (7.1.2) with the isomorphism (7.1.15) yields

$$\nu^s(W^{8n-3}) \cong (\eta \otimes_{\mathbb{R}} \chi_{2n-1}) \oplus \left(\mathbb{R} \otimes_{\mathbb{R}} \left(\bigoplus_{k=1}^{2n-1} (2n-k)\chi_k^{\perp}\right)\right),$$

determining a stably quaternionic structure on W^{8n-3}.

In a similar manner, there is an isomorphism between the stable tangent bundle of W^{8n-3} and a direct sum of quaternionic bundles, given by

$$\tau^s(W^{8n-3}) \cong (\eta \otimes_{\mathbb{R}} \chi_{2n-1}) \oplus \left(\mathbb{R} \otimes_{\mathbb{R}} \left(\bigoplus_{k=0}^{2n-1} \chi_k\right)\right). \tag{7.1.16}$$

□

7.2 The manifolds Y^{4n+1}

We have established that $S^1 \times Q^{2n-1}$, equipped with a particular stably quaternionic structure, is a geometrical representative for $\varphi_n \in MSp_{8n-3}$. Now consider the following conjecture of Ray.

Conjecture 7.2.1. *[47] A simply-connected geometrical representative for φ_n may be constructed by choosing a suitable stably quaternionic structure on the S^5-bundle $S(\chi_{2n-2} \oplus \mathbb{R}^2)$ over Q^{2n-2}.*

We will denote the S^5-bundle $S(\chi_{n-1} \oplus \mathbb{R}^2)$ over Q^{n-1} by Y^{4n+1}, with projection $\pi_n \colon Y^{4n+1} \to Q^{n-1}$.

The remainder of our thesis is devoted to studying all the possible stably quaternionic structures on Y^{4n+1} by computing the image $\Psi(Y^{4n+1})$. Consequently, we will prove Conjecture 7.2.1. As a first step towards this goal, in this section we study the geometry of the manifolds Y^{4n+1}, which allows us to then determine their F-cohomology rings.

From the discussion that followed Lemma 6.1.8, it should be clear that a choice of nonzero vector in the $\mathbb{R}^2$ summand of $\chi_n \oplus \mathbb{R}^2$ gives a section $\omega_n \colon Q^{n-1} \to Y^{4n+1}$, while $\vartheta_n \colon Y^{4n+1} \to T(\chi_n \oplus \mathbb{R})$ is the usual quotient map. Then we have a commutative diagram

$$\begin{array}{ccccc}
Q^{n-1} & \xrightarrow{\omega_n} & Y^{4n+1} & \xrightarrow{\vartheta_n} & T(\chi_{n-1} \oplus \mathbb{R}) \\
\uparrow & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} \\
* & \longrightarrow & S^5 & \xrightarrow{1} & S^5,
\end{array} \tag{7.2.2}$$

in which ι is inclusion of fibres and 1 is the identity map.

Proposition 7.2.3. *There is an isomorphism*

$$\tau(Y^{4n+1}) \oplus \mathbb{R} \cong \mathbb{R}^2 \oplus \left(\bigoplus_{i=1}^{n-1} \chi_i \right), \tag{7.2.4}$$

which determines a bounding stably quaternionic structure on Y^{4n+1}.

Proof. There is a stably quaternionic structure on the base space Q^{n-1} of the S^5-bundle $Y^{4n+1} \to Q^{n-1}$, which arises from the isomorphism (7.1.1). Therefore we can apply Proposition 2.3.7 to get the required isomorphism on $\tau(Y^{4n+1}) \oplus \mathbb{R}$, which extends to the 6-disc bundle in the usual manner, to give a bounding stably quaternionic structure on Y^{4n+1}. □

We denote the resulting stably quaternionic structure by s, so that the cobordism class $[Y^{4n+1}, s]$ equals zero in MSp_{4n+1}. Similarly, we will take s' to be the bounding stably quaternionic structure on Q^n given by the isomorphism (7.1.1). As explained in [44, Lemma 4.1], a framing of the circle S^1, given by an element of $\pi_1(O) \cong \mathbb{Z}/2$, leads to a stably quaternionic structure on S^1. This is due to the fact that the 2-skeleton of the MSp spectrum is simply the sphere spectrum. Then the generator of $\pi_1(O)$ represents the nontrivial framing of S^1 that generates $\theta_1 \in MSp_1$, which appeared in the expansion (7.1.11). On the other hand, the trivial framing of S^1 gives rise to a trivial stably quaternionic structure on the circle, which we will denote by t.

Lemma 7.2.5. *There is a quaternionic bordism between $[Y^{4n+1}, s]$ and $[W^{4n+1}, t \times s']$ in MSp_*.*

Proof. The 6-disc bundle with total space $D(\chi_{n-1} \oplus \mathbb{R}^2)$ over Q^{n-1}, contains the subbundles whose total spaces are $Y^{4n+1} = S(\chi_{n-1} \oplus \mathbb{R}^2)$ and $Q^n = S(\chi_{n-1} \oplus \mathbb{R})$. It will be convenient to assume that the 6-disc and 5-sphere fibres of $D(\chi_{n-1} \oplus \mathbb{R}^2)$ and Y^{4n+1} respectively, have radius 2, and that the 4-sphere fibres of Q^n have radius 1.

The tubular neighbourhood of the embedding of Q^n in $D(\chi_{n-1} \oplus \mathbb{R}^2)$ is framed by the outward pointing normal vector of Q^n, and a choice of vector in the second $\mathbb{R}$ summand of $\chi_{n-1} \oplus \mathbb{R}^2$. Hence we have an embedding of $D^2 \times Q^n$ in $D(\chi_{n-1} \oplus \mathbb{R}^2)$.

Therefore we can define

$$X^{4n+2} := D(\chi_{n-1} \oplus \mathbb{R}^2) - (D^2 \times Q^n), \tag{7.2.6}$$

and so X^6 is simply $D^6 - (D^2 \times S^4)$. Then we have that X^{4n+2} is the total space of an X^6-bundle over Q^{n-1}. The boundary of X^{4n+2} is homeomorphic to a disjoint union

$$\partial X^{4n+2} \cong (S^1 \times Q^n) \amalg Y^{4n+1}, \tag{7.2.7}$$

and so X^{4n+2} provides a bordism between Y^{4n+1} and $S^1 \times Q^n = W^{4n+1}$.

Since X^{4n+2} is the total space of a disc bundle, an isomorphism of the form (2.3.9) leads to a stably quaternionic structure s'', which obviously restricts to s on Y^{4n+1}, since s also arose from an isomorphism of the form (2.3.9). It is also straightforward to check that the restriction of s'' to $S^1 \times Q^n$ simply gives $t \times s'$. It follows that X^{4n+2} provides a quaternionic bordism between $[Y^{4n+1}, s]$ and $[W^{4n+1}, t \times s']$. $\square$

We now construct a retraction map, which will prove useful throughout this chapter.

Lemma 7.2.8. *There exists a retraction $r_n \colon X^{4n+2} \to Y^{4n+1}$, whose restriction $r'_n = r_n|_{W^{4n+1}}$ is of degree 1.*

Proof. Consider the bundle $\pi_n \colon D(\chi_{n-1} \oplus \mathbb{R}^2) \to Q^{n-1}$, and its subbundles Y^{4n+1} and Q^n, with fibres as described in the proof of Lemma 7.2.5. Define a section $\omega_n \colon Q^{n-1} \to D(\chi_{n-1} \oplus \mathbb{R}^2)$ by taking $\omega_n(l)$ to be $+1$ in the first $\mathbb{R}$ summand of $\chi_{n-1} \oplus \mathbb{R}^2$, for each point l in Q^{n-1}. Denote this first $\mathbb{R}$ summand by $\mathbb{R}_1$. Then remove a 6-disc D^6_ε of small radius ε centred on $\omega_n(l)$ in each D^6-fibre $\pi_n^{-1}(l)$.

When we form X^{4n+2} as in (7.2.7), our 6-discs D^6_ε are removed from $D(\chi_{n-1} \oplus \mathbb{R}^2)$ as subspaces of $D^2 \times Q^n$. Then in each X^6-fibre of X^{4n+2} we can project radially from the centre of D^6_ε onto the 5-sphere fibres of Y^{4n+1}. This leads to a retraction $r_n \colon X^{4n+2} \to Y^{4n+1}$, as the radial projection is clearly the identity on the 5-sphere bundle Y^{4n+1} itself.

In each fibre of the X^6-bundle X^{4n+2}, the radial projection from the centre of D^6_ε in the direction of the vector $+1 \in \mathbb{R}_1$ passes through the point w of $S^1 \times S^4 \in X^6$ that has the highest value in the $\mathbb{R}_1$ coordinate; eventually it meets the end point p of the vector $+2 \in \mathbb{R}_1$, which lies on $S^5 \subset \partial X^6$. Therefore the restriction $r'_n = r_n|_{W^{4n+1}}$ has degree 1 since by choosing the point p in each fibre of Y^{4n+1}, the inverse image $(r'_n)^{-1}(p)$ consists of the single point w in each of the fibres of W^{4n+1}. Furthermore, the restriction r'_n preserves stably quaternionic structure (and thus orientation), since by Lemma 7.2.5, the radial projection line joining the points $w \in S^1 \times S^4$ and $p \in S^5$, is in fact a stably quaternionic bordism between w and p in each fibre. $\square$

It is straightforward (see e.g. [48, Lemma 10.3]) to show that the retraction r'_n

gives rise to a homotopy commutative diagram

$$\begin{array}{ccccc} S^1 \times S^4 & \xrightarrow{1\times\iota} & S^1 \times Q^n & \xrightarrow{1\times\vartheta_n} & S^1 \times T(\chi_{n-1}) \\ \downarrow r_1' & & \downarrow r_n' & & \downarrow c \\ S^5 & \xrightarrow{\iota} & Y^{4n+1} & \xrightarrow{\vartheta_n} & S^1 \wedge T(\chi_{n-1}), \end{array} \tag{7.2.9}$$

where c is the map which collapses onto the smash product, and ι is inclusion of fibres.

We now devote our attention to determining the F-cohomology of our manifolds Y^{4n+1}. To begin we restate Proposition 6.2.4 in the special case of a bounded quaternionic flag manifold.

Proposition 7.2.10. *For a bounded quaternionic flag manifold $Q^n = S(\chi_{n-1} \oplus \mathbb{R})$, the F_*-algebra $F^*(Q^n_+)$ is isomorphic to*

$$F_*[y_1^F, \dots, y_n^F]/\mathcal{L}_n \tag{7.2.11}$$

where $\mathcal{L}_n$ is the ideal

$$((y_i^F)(y_i^F - y_{i-1}^F)) : 1 \leq i \leq n). \tag{7.2.12}$$

By pulling back from the universal case of Example 6.1.10, we can view the generator y_n^F as $\vartheta_n^*(t_n^F)$, where t_n^F is a Thom class in $F^4(T(\chi_{n-1}))$.

Lemma 7.2.13. *There is a virtual bundle $\lambda_n \in KSp^0(T(\chi_{n-1}))$ such that $p_1^F(\lambda_n) = t_n^F$.*

Proof. From the discussion that followed Lemma 6.1.8 we have a split short exact sequence

$$KSp^0(Q^{n-1}) \xleftarrow{\omega_n^*} KSp^0(Q^n) \xleftarrow{\vartheta_n^*} KSp^0(T(\chi_{n-1})), \tag{7.2.14}$$

and since $\omega_n^*(\chi_n) = 0$, there must exist some $\lambda_n \in KSp^0(T(\chi_{n-1}))$ with $\vartheta_n^*(\lambda_n) = \chi_n$. Therefore $\vartheta_n^* p_1^F(\lambda_n) = p_1^F(\chi_n) = y_n^F$, and so $p_1^F(\lambda_n) = t_n^F$ as required. □

As a consequence of the remarks following Example 6.1.10, by considering the element $t_n^F y_{i_1}^F \dots y_{i_j}^F \in F^{4j+4}(T(\chi_{n-1}))$, we have the following equations

$$\vartheta_n^*(t_n^F y_{i_1}^F \dots y_{i_j}^F) = y_n^F \pi_n^*(y_{i_1}^F \dots y_{i_j}^F) = y_n^F y_{i_1}^F \dots y_{i_j}^F, \tag{7.2.15}$$

for any $\{i_1, \ldots, i_j\} \subseteq \{1, \ldots, n-1\}$.

Now we consider Y^{4n+1} over Q^{n-1}. Let σ denote the suspension isomorphism $F^*(T(\chi_{n-1})) \xrightarrow{\sigma} F^*(S^1 \wedge T(\chi_{n-1})) \cong F^*(S^1) \otimes F^*(T(\chi_{n-1}))$, where the latter isomorphism is given by the Künneth isomorphism for F-cohomology (see e.g. [56, Theorem 13.75]). Since the Thom space $T(\chi_{n-1} \oplus \mathbb{R})$ is homeomorphic to $S^1 \wedge T(\chi_{n-1})$, the isomorphism σ gives a Thom class $g_1^F \otimes t_n^F = \sigma t_n^F$ in $F^5(T(\chi_{n-1} \oplus \mathbb{R}))$, where g_1^F generates $F^1(S^1)$.

We will write $v_n^F \in F^5(Y^{4n+1})$ for the pullback $\vartheta_n^*(\sigma t_n^F)$ along $\vartheta_n \colon Y^{4n+1} \to T(\chi_{n-1} \oplus \mathbb{R})$.

Lemma 7.2.16. *The element v_n^F restricts to a generator of $F^5(S^5)$ in each fibre.*

Proof. By definition the Thom class $t_n^F \in F^4(T(\chi_{n-1}))$ restricts to a generator of $F^4(S^4)$ in a fibre. Therefore σt_n^F restricts in a fibre to a generator of $F^5(S^5)$, and the result then follows from the commutative diagram (7.2.2). □

This allows us to describe the F-cohomology of our manifolds Y^{4n+1}.

Proposition 7.2.17. *The F_*-algebra $F^*(Y_+^{4n+1})$ is a free module over $F^*(Q_+^{n-1})$, generated by 1 in dimension 0 and $v_n^F \in F^5(Y_+^{4n+1})$. The multiplicative structure is described by the single relation $(v_n^F)^2 = 0$.*

Proof. Given Lemma 7.2.16, the result follows from the Leray-Hirsch Theorem 2.5.1. The relation is due to the fact that v_n^F is a pullback of a suspension σt_n^F, and suspension kills all cup products. □

We conclude with a result that plays the role of [48, (11.13)] in our setting.

Lemma 7.2.18. *In $F^{4j+5}(Y^{4n+1})$ we have that $\vartheta_n^*(\sigma t_n^F y_{i_1}^F \ldots y_{i_j}^F) = v_n^F y_{i_1}^F \ldots y_{i_j}^F$.*

Proof. Referring to the commutative diagram (7.2.9), we have

$$r_n'^*(v_n^F) = r_n'^* \vartheta_n^*(\sigma t_n^F) = (1 \times \vartheta_n)^* c^*(\sigma t_n^F) = (1 \times \vartheta_n)^*(g_1^F \otimes t_n^F) \in F^5(S^1 \times Q^n).$$

By definition of $y_n^F \in F^4(Q^n)$, this implies $r_n'^*(v_n^F) = g_1^F \otimes y_n^F$.

Now consider the element $t_n^F y_{i_1}^F \dots y_{i_j}^F \in F^{4j+4}(T(\chi_{n-1}))$, a similar string of equalities leads to

$$r_n'^* \vartheta_n^*(\sigma t_n^F y_{i_1}^F \dots y_{i_j}^F) = g_1^F \otimes y_n^F y_{i_1}^F \dots y_{i_j}^F. \tag{7.2.19}$$

Now suppose that $\vartheta_n^*(\sigma t_n^F y_{i_1}^F \dots y_{i_j}^F) = l v_n^F y_{i_1}^F \dots y_{i_j}^F$, for some integer l. This would give

$$r_n'^* \vartheta_n^*(\sigma t_n^F y_{i_1}^F \dots y_{i_j}^F) = l(g_1^F \otimes y_n^F)(1 \otimes y_{i_1}^F \dots y_{i_j}^F) = l g_1^F \otimes y_n^F y_{i_1}^F \dots y_{i_j}^F, \tag{7.2.20}$$

and so, referring to equation (7.2.19), we have $l = 1$. □

Note that since σt_n^F is itself a Thom class, the result of Lemma 7.2.18 also follows by applying the usual properties of iterated bundle constructions, which are discussed at the close of Section 6.1. However, we have presented the above reasoning to illustrate some of the properties of the retraction r_n', which we will call upon in Section 7.5.

7.3 Tangential structures

Suppose we have a manifold M^n, which carries a G-structure g on the stable tangent bundle $\tau^s(M^n)$. If $G < H$, this also induces a H-structure on M^n. Recall that in Lemma 2.3.12, we described an element $\delta \in H/G^0(M^n)$ as a homotopy class of G-structure on the trivial H-bundle over M^n. Then we can change the G-structure on M^n by adding this trivial H-bundle, equipped with G-structure δ, to $\tau^s(M^n)$. We denoted the resulting G-structure on M^n by $g + \delta$. Note that since we only added a trivial H-bundle to $\tau^s(M^n)$, the H-structure on M^n is unchanged.

In this section we use our cohomology calculations to study $U/Sp^0(Y^{4n+1})$, which leads to a determination of $SO/Sp^0(Y^{4n+1})$. In other words, we obtain a complete description of all possible changes of stably quaternionic structure on Y^{4n+1}, which, respectively, leave the stably complex structure and orientation (that is, SO-structure) on Y^{4n+1} fixed. To aid our work with associated units in the next section, we then represent certain elements in $U/Sp^0(Y^{4n+1})$ by constructing appropriate virtual bundles in $KSp^0(Y^{4n+1})$.

For the lists of real K-theory generators that feature throughout this section, it will be helpful to recall the description (2.2.8) of the coefficient ring KO_*. As we remarked in the introduction, we will drop the superscript F from the notation for our cohomology generators when it is clear which cohomology theory $F^*(-)$ we are working with.

We commence with the computation of $U/Sp^0(-)$ for the bounded quaternionic flag manifold Q^n and the S^5-bundles Y^{4n+1}. From Lemma 2.3.13, this reduces to calculating $KO^{-3}(-)$ of our manifolds, so we can read off the following descriptions of $U/Sp^0(-)$ from Propositions 7.2.10 and 7.2.17.

Theorem 7.3.1. *The group of stably quaternionic structures on Q^n, considered as a fixed stably complex manifold, is given by an isomorphism*

$$U/Sp^0(Q^n) = 0. \tag{7.3.2}$$

The bounded quaternionic flag manifold Q^n has the bounding stably quaternionic structure s' as given by Proposition 6.3.3. Therefore Theorem 7.3.1 implies that it is impossible to change the stably quaternionic structure on Q^n, keeping the stably complex structure fixed, so that Q^n does not bound in MSp_*.

Let $R(j)$ denote a subset of length j from $[n-1] = \{1, \ldots, n-1\}$, so that if $R(j) = \{i_1, \ldots i_j\}$, then $y_{R(j)}$ denotes the element $y_{i_1} \ldots y_{i_j}$. We write $P[n-1]$ for the power set of $[n-1]$.

Theorem 7.3.3. *The group of stably quaternionic structures on Y^{4n+1}, considered as a fixed stably complex manifold, is given by an isomorphism*

$$U/Sp^0(Y^{4n+1}) \cong \bigoplus_{2^{n-1}} \mathbb{Z}, \tag{7.3.4}$$

on generators

$$\{\gamma v_n, \beta\gamma v_n y_{R(1)}, \gamma^2 v_n y_{R(2)}, \beta\gamma^2 v_n y_{R(3)}, \ldots, \beta\gamma^m v_n y_{R(n-1)} : R(j) \in P[n-1]\},$$

when $n = 2m$, and generators

$$\{\gamma v_n, \beta\gamma v_n y_{R(1)}, \gamma^2 v_n y_{R(2)}, \beta\gamma^2 v_n y_{R(3)}, \ldots, \gamma^{m+1} v_n y_{R(n-1)} : R(j) \in P[n-1]\},$$

when $n = 2m+1$.

Proof. The total number of $\mathbb{Z}$ summands in $U/Sp^0(Y^{4n+1})$ is equal to the cardinality of the power set $P[n-1]$, namely $\sum_{i=0}^{n-1}\binom{n-1}{i} = 2^{n-1}$. □

Remark 7.3.5. For simply connected manifolds $X = Q^n$ or Y^{4n+1}, we have that $SU/Sp^0(X_+) \cong U/Sp^0(X_+)$. This follows from the fact that there is a deformation retract r_U of U onto SU (to prove this we can use an argument analogous to the deformation retract of the group $GL^+(\mathbb{R})$ of invertible linear transformations of $\mathbb{R}^\infty$, with positive determinant, onto SO, as given by Hatcher [23, p. 26]). Any based map $f\colon X_+ \to U$ sends the basepoint of X_+ to $SU < U$, so by composing with r_U we have a homotopy of f to a map $X_+ \to SU$. This then induces the required isomorphism $SU/Sp^0(X_+) \cong U/Sp^0(X_+)$.

Now we wish to calculate $SO/Sp^0(-)$, which will describe the stably quaternionic structures on Q^n and Y^{4n+1} when they are considered as manifolds with a fixed orientation.

Consider the fibration $SO \to SO/Sp \to BSp$, which gives rise (see e.g. [15, Theorem 6.42]) to the exact sequence

$$\cdots \to Sp^0(-) \xrightarrow{h} SO^0(-) \to SO/Sp^0(-) \to KSp^0(-) \xrightarrow{h} KSO^0(-) \to \cdots, \quad (7.3.6)$$

in which the map $h\colon KO^{-4}(-) \to KO^0(-)$ can be taken to be multiplication by $\gamma^{-1}\beta$. Furthermore, $SO^0(X)$ is the group of homotopy classes $[X, SO] \cong [X, \Omega BSO] \cong [\Sigma X, BSO] = KSO^{-1}(X)$; the group $Sp^0(X) \cong KSp^{-1}(X) \cong KO^{-5}(X)$ is defined in a similar manner. Now, by Proposition 7.2.10, $Sp^0(Q^n)$ consists entirely of torsion generated by elements of the form $\alpha\gamma^{i+1}y_{R(2i+1)}$, while by Proposition 7.2.17, $Sp^0(Y^{4n+1})$ consists entirely of torsion generated by elements of the form $\alpha\gamma^{i+1}y_{R(2i+1)}$ and $\alpha^2\gamma^{i+1}v_n y_{R(2i)}$. Therefore when $X = Q^n$ or Y^{4n+1}, the image of $h\colon Sp^0(X) \to SO^0(X)$ is zero because $\alpha\beta = 0$ in KO_*. Similar reasoning confirms that the image of $h\colon TorsKSp^0(X) \to KSO^0(X)$ is also zero, so we have a short exact sequence

$$0 \longrightarrow SO^0(X) \longrightarrow SO/Sp^0(X) \longrightarrow TorsKSp^0(X) \longrightarrow 0. \quad (7.3.7)$$

Lemma 7.3.8. *[48, (12.6)] The short exact sequence (7.3.7) is split.*

Proof. The splitting is given by a map $TorsKSp^0(X) \to SO/Sp^0(X)$, which is the composite

$$TorsKSp^0(X) \xrightarrow{d} KO^{-3}(X) \cong U/Sp^0(X) \xrightarrow{j_U} SO/Sp^0(X), \tag{7.3.9}$$

where j_U is induced by the inclusion $U \to SO$, and d acts by $\alpha y \mapsto y$, on generators $\alpha y \in TorsKO^{-4}(X)$. □

We are now able to describe the groups $SO/Sp^0(-)$ for our manifolds Q^n and Y^{4n+1} in terms of the splitting of the sequence (7.3.7).

Theorem 7.3.10. *The group of stably quaternionic structures on Q^n, considered as a fixed SO-manifold, is given by an isomorphism*

$$SO/Sp^0(Q^n) \cong SO^0(Q^n) \cong \bigoplus_{2^{n-1}-1} \mathbb{Z}/2, \tag{7.3.11}$$

on generators

$$\{\alpha\gamma y_{R(2)}, \alpha\gamma^2 y_{R(4)}, \alpha\gamma^3 y_{R(6)}, \ldots, \alpha\gamma^m y_{R(n)} : R(j) \in P[n]\},$$

when $n = 2m$, and generators

$$\{\alpha\gamma y_{R(2)}, \alpha\gamma^2 y_{R(4)}, \alpha\gamma^3 y_{R(6)}, \ldots, \alpha\gamma^m y_{R(n-1)} : R(j) \in P[n]\},$$

when $n = 2m + 1$.

Proof. We can read off $SO^0(Q^n) \cong KO^{-1}(Q^n)$, and deduce its generators, from Proposition 7.2.10 as follows. The generators of $KO^*(Q^n)$ are of the form $y_{R(i)} \in KO^{4i}(Q^n)$, for $0 \leq i \leq n$. It is straightforward to check that we cannot multiply $y_{R(2i+1)}$ by any coefficient in KO_* so that the resulting product is an element of $KO^{-1}(Q^n)$. Therefore the generators of $KO^{-1}(Q^n)$ are of the form $\alpha\gamma^i y_{R(2i)}$ as listed above. We can use similar methods to verify that $TorsKSp^0(Q^n) = 0$, since we have $KSp^0(-) \cong KO^{-4}(-)$. The total number of $\mathbb{Z}/2$ summands is given by $\binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{2m}$, for both $n = 2m$ and $n = 2m + 1$. The sum of the even terms in the binomial expansion satisfies

$$\sum_{i=0}^{n} \binom{n}{2i} = 2^{n-1}, \tag{7.3.12}$$

so our number of summands is $2^{n-1} - 1$. □

Theorem 7.3.13. *The group of stably quaternionic structures on Y^{4n+1}, considered as a fixed SO-manifold, is described by an isomorphism*

$$SO/Sp^0(Y^{4n+1}) \cong SO^0(Y^{4n+1}) \oplus TorsKSp^0(Y^{4n+1}) \cong \bigoplus_{2^{n-1}-1} \mathbb{Z}/2 \bigoplus_{2^{n-2}} \mathbb{Z}/2,$$

on generators

$$\begin{aligned}&\{\alpha^2\gamma v_n y_{R(1)}, \alpha\gamma y_{R(2)}, \alpha^2\gamma^2 v_n y_{R(3)}, \alpha\gamma^2 y_{R(4)}, \ldots, \alpha^2\gamma^m v_n y_{R(n-1)} : R(j) \in P[n-1]\},\\ &\{\alpha\gamma v_n, \alpha\gamma^2 v_n y_{R(2)}, \ldots, \alpha\gamma^m v_n y_{R(n-2)} : R(j) \in P[n-1]\},\end{aligned} \tag{7.3.14}$$

when $n = 2m$, and generators

$$\begin{aligned}&\{\alpha^2\gamma v_n y_{R(1)}, \alpha\gamma y_{R(2)}, \alpha^2\gamma^2 v_n y_{R(3)}, \alpha\gamma^2 y_{R(4)}, \ldots, \alpha\gamma^m y_{R(n-1)} : R(j) \in P[n-1]\},\\ &\{\alpha\gamma v_n, \alpha\gamma^2 v_n y_{R(2)}, \ldots, \alpha\gamma^{m+1} v_n y_{R(n-1)} : R(j) \in P[n-1]\},\end{aligned} \tag{7.3.15}$$

when $n = 2m + 1$.

Proof. We can read off $SO^0(Y^{4n+1})$ and deduce its generating set, from Proposition 7.2.17 as follows. The generators of $KO^*(Y^{4n+1})$ are of the form $y_{R(i)} \in KO^{4i}(Y^{4n+1})$ and $v_n y_{R(i)} \in KO^{4i+5}(Y^{4n+1})$, for $0 \le i \le n-1$. It is straightforward to check that we cannot multiply elements of the form $y_{R(2i+1)}$ or $v_n y_{R(2i)}$ by coefficients in KO_* so that the resulting respective products lie in $KO^{-1}(Y^{4n+1})$. Hence $KO^{-1}(Y^{4n+1}) \cong SO^0(Y^{4n+1})$ is generated by elements of the form $\alpha\gamma^i y_{R(2i)}$ and $\alpha^2\gamma^{i+1} v_n y_{R(2i+1)}$ as listed above. The generators of $TorsKSp^0(Y^{4n+1})$ are determined in a similar manner. As usual, the enumeration of the $\mathbb{Z}/2$ summands follows from well-known identities on binomial coefficients. □

Arising from the fibration $U \to U/Sp \to BSp$ is a long exact sequence

$$\cdots \to Sp^0(-) \to U^0(-) \to U/Sp^0(-) \to KSp^0(-) \xrightarrow{h} KSO^0(-) \to \cdots, \tag{7.3.16}$$

that we can link with the sequence (7.3.6) via maps

$$i_U : U^0(-) \longrightarrow SO^0(-), \qquad j_U : U/Sp^0(-) \longrightarrow SO/Sp^0(-),$$

which are induced by the inclusion $U \to SO$.

Proposition 7.3.17. *For $X = Q^n, Y^{4n+1}$ we have an isomorphism*

$$Im(j_U) \cong Im(i_U) \oplus TorsKSp^0(X). \tag{7.3.18}$$

For $X = Q^n$ the image of i_U is zero, while for $X = Y^{4n+1}$ we have an isomorphism

$$Im(i_U) \cong \bigoplus_{2^{n-2}} \mathbb{Z}/2$$

on generators

$$\{\alpha^2\gamma v_n y_{R(1)}, \alpha^2\gamma^2 v_n y_{R(3)}, \alpha^2\gamma^3 v_n y_{R(5)}, \ldots, \alpha^2\gamma^m v_n y_{R(n-1)} : R(j) \in P[n-1]\},$$

when $n = 2m$, and on generators

$$\{\alpha^2\gamma v_n y_{R(1)}, \alpha^2\gamma^2 v_n y_{R(3)}, \alpha^2\gamma^3 v_n y_{R(5)}, \ldots, \alpha^2\gamma^m v_n y_{R(n-1)} : R(j) \in P[n-2]\},$$

when $n = 2m + 1$.

Proof. By applying exactly the same reasoning as in [48, Theorem 12.7], studying the interaction between the long exact sequences (7.3.6) and (7.3.16) will lead to the isomorphism (7.3.18). The descriptions of $\mathrm{Im}(i_U)$ for Q^n and Y^{4n+1} follow from the $SO^0(-)$ groups determined in Theorems 7.3.10 and 7.3.13; therein, $TorsKSp^0(-)$ is also described for each of our manifolds Q^n and Y^{4n+1}. □

In light of Remark 7.3.5 we have that $\mathrm{Im}(i_{SU}) = \mathrm{Im}(i_U)$, and so $\mathrm{Im}(j_{SU}) = \mathrm{Im}(j_U)$.

To conclude this section, we will study the elements of $SU/Sp^0(Y^{4n+1})$ in detail, which will allow us to describe associated units in Section 7.4. First we present the necessary background from [48, Chapter 12].

Applying the long exact sequence (7.3.16) to the suspension $S^1 \wedge BSp$ we have,

$$\cdots \longrightarrow U/Sp^0(S^1 \wedge BSp) \xrightarrow{a} KSp^0(S^1 \wedge BSp) \longrightarrow K^0(S^1 \wedge BSp) \longrightarrow \cdots,$$

where the map a can be taken to be multiplication by $\alpha \in KO_*$. The virtual quaternionic bundle $(\eta - \mathbb{R}) \otimes_{\mathbb{R}} \xi$ in $KSp^0(S^1 \wedge BSp)$ vanishes when mapped to $K^0(S^1 \wedge BSp) = 0$, and so the exactness of the above sequence implies that $(\eta - \mathbb{R}) \otimes_{\mathbb{R}} \xi \in \mathrm{Im}(a)$. In [42, Lemma 3.2] it is shown that the first real K-theory quaternionic Pontryagin class $p_1((\eta - \mathbb{R}) \otimes_{\mathbb{R}} \xi) = g_1 \otimes \alpha p_1$ in $KO^4(S^1 \wedge BSp)$.

Now consider the element $\Omega \in SU/Sp^0(S^1 \wedge BSp)$, represented by a map $S^1 \wedge BSp \to SU/Sp$, which arises by adjointing the equivalence $BSp \simeq \Omega(SU/Sp)$ (see e.g. [29, Theorem 5.22]).

Lemma 7.3.19. *[48, Lemma 12.9] The map a sends Ω to $(\eta - \mathbb{R}) \otimes_{\mathbb{R}} \xi \in KSp^0(S^1 \wedge BSp)$, and Ω corresponds to $g_1 \otimes \gamma p_1 \in KO^{-3}(S^1 \wedge BSp)$.*

We will regard Ω as a particular SU trivialisation of the virtual quaternionic bundle $(\eta - \mathbb{R}) \otimes_{\mathbb{R}} \xi$. Moreover, under the map $j_{SU} \colon SU/Sp^0(-) \to SO/Sp^0(-)$ we have that $j_{SU}(\Omega) := \Delta$ is a particular SO trivialisation of $(\eta - \mathbb{R}) \otimes_{\mathbb{R}} \xi$.

Given a map $f \colon X \to S^1 \wedge BSp$, we can pull back these notions, as illustrated by the commutative diagram

$$\begin{array}{ccc} SU/Sp^0(X) & \xrightarrow{j_{SU}} & SO/Sp^0(X) \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\ SU/Sp^0(S^1 \wedge BSp) & \xrightarrow{j_{SU}} & SO/Sp^0(S^1 \wedge BSp), \end{array} \tag{7.3.20}$$

so that we have elements $\Omega(f) := f^*(\Omega)$ and $\Delta(f) := f^*(\Delta)$. This will allow us to give an explicit description of some of the elements in $SU/Sp^0(Y^{4n+1})$ and $SO/Sp^0(Y^{4n+1})$ as specific SU and SO trivialisations of virtual quaternionic bundles over Y^{4n+1}, which we will construct in due course. Ultimately this facilitates our calculations with associated units in the next section.

First we require some preliminary facts on virtual bundles.

Suppose we have virtual bundles θ_i over spaces X_i, for $i = 1, 2, 3$. The usual properties of commutativity and associativity apply to tensor products of virtual bundles to give respective isomorphisms

$$\theta_1 \otimes_{\mathbb{K}} \theta_2 \cong \theta_2 \otimes_{\mathbb{K}} \theta_1, \tag{7.3.21}$$

of virtual bundles over $X_1 \wedge X_2$, and

$$(\theta_1 \otimes_{\mathbb{K}} \theta_2) \otimes_{\mathbb{K}} \theta_3 \cong \theta_1 \otimes_{\mathbb{K}} (\theta_2 \otimes_{\mathbb{K}} \theta_3), \tag{7.3.22}$$

over $X_1 \wedge X_2 \wedge X_3$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ when the θ_i are real, complex or quaternionic, respectively. In the quaternionic case, when we consider the tensor product $\theta_1 \otimes_{\mathbb{H}} \theta_2$

it will be understood to mean that the quaternions act on the right of θ_1 and on the left of θ_2.

Assume now that θ_1 and θ_2 are virtual quaternionic bundles, and that θ_3 is a virtual real bundle. We have an isomorphism of virtual real bundles

$$(\theta_1 \otimes_{\mathbb{H}} \theta_2) \otimes_{\mathbb{R}} \theta_3 \cong \theta_1 \otimes_{\mathbb{H}} (\theta_2 \otimes_{\mathbb{R}} \theta_3), \tag{7.3.23}$$

in $KO^0(X_1 \wedge X_2 \wedge X_3)$ (see e.g. [32, Proposition 5.15]). It follows that we have a second isomorphism of virtual real bundles

$$(\theta_1 \otimes_{\mathbb{H}} \theta_2) \otimes_{\mathbb{R}} (\theta_1 \otimes_{\mathbb{H}} \theta_2) \cong (\theta_1 \otimes_{\mathbb{H}} \theta_1) \otimes_{\mathbb{R}} (\theta_2 \otimes_{\mathbb{H}} \theta_2), \tag{7.3.24}$$

in $KO^0(X_1 \wedge X_1 \wedge X_2 \wedge X_2)$, since the tensor product of two quaternionic bundles is a real bundle, as discussed in Chapter 6. These isomorphisms will prove crucial in our calculations below.

Now let θ be a virtual quaternionic bundle over X. Let $\theta^{\langle 2k\rangle}$ denote the virtual real bundle

$$(\theta \otimes_{\mathbb{H}} \theta) \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} (\theta \otimes_{\mathbb{H}} \theta), \tag{7.3.25}$$

over the smash product $X \wedge \cdots \wedge X$ of $2k$ copies of X. Furthermore, let $\theta^{\langle 2k+1\rangle}$ denote the virtual quaternionic bundle

$$(\theta \otimes_{\mathbb{H}} \theta) \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} (\theta \otimes_{\mathbb{H}} \theta) \otimes_{\mathbb{R}} \theta = \theta^{\langle 2k\rangle} \otimes_{\mathbb{R}} \theta, \tag{7.3.26}$$

over the smash product of $2k+1$ copies of X. Of course, $\theta^{\langle 1\rangle}$ should be interpreted as θ itself, and virtual bundles of the form $\theta^{\langle i\rangle}$ are subject to the usual properties of commutativity and associativity of tensor products, as discussed above. By pulling back along diagonal maps of the form $\Delta : X \to X \wedge \cdots \wedge X$, we can also define

$$\begin{aligned} \theta^{2k} &:= \Delta^*\theta^{\langle 2k\rangle} && \text{in } KO^0(X), \\ \theta^{2k+1} &:= \Delta^*\theta^{\langle 2k+1\rangle} && \text{in } KSp^0(X). \end{aligned} \tag{7.3.27}$$

With these notions in place, we have the following result.

Lemma 7.3.28. *In $KO^4(\mathbb{H}P^\infty)$ the first quaternionic Pontryagin class of the virtual quaternionic line bundle $(\xi_1 - \mathbb{H})^{2i+1}$ satisfies*

$$p_1((\xi_1 - \mathbb{H})^{2i+1}) = \gamma^i p_1^{2i+1}. \tag{7.3.29}$$

Proof. Our proof is based on the reasoning given in [48, page 41]. First recall that in Section 2.2 we saw that the generator $\gamma \in KO_*$ is represented as a virtual bundle by $(\xi_1 - \mathbb{H}) \otimes_{\mathbb{H}} (\xi_1 - \mathbb{H})$. Then we can use the associativity and commutativity of the tensor product, the virtual real bundle isomorphisms (7.3.24) and (7.3.23), and well-known properties of the real K-theory spectrum (see e.g. [56, page 302]) to attain the equations

$$\begin{aligned} \gamma^i \wedge \gamma p_1(\xi^{\langle 2i+1\rangle}) &= (\xi_1 - \mathbb{H})^{\langle 2i\rangle} \otimes_{\mathbb{R}} ((\xi_1 - \mathbb{H}) \otimes_{\mathbb{H}} \xi^{\langle 2i+1\rangle}) \\ &= ((\xi_1 - \mathbb{H}) \otimes_{\mathbb{H}} \xi)^{\langle 2i+1\rangle} \\ &= \gamma p_1 \wedge \cdots \wedge \gamma p_1 \in KO^0(S^{8i+4} \wedge BSp \wedge \cdots \wedge BSp). \end{aligned}$$

Since we only work with quaternionic line bundles, we will restrict from BSp to $BSp(1) \cong \mathbb{H}P^\infty$ throughout to obtain

$$\gamma^i \wedge \gamma p_1((\xi_1 - \mathbb{H})^{\langle 2i+1\rangle}) = \gamma p_1 \wedge \cdots \wedge \gamma p_1 \in KO^0(S^{8i+4} \wedge \mathbb{H}P^\infty \wedge \cdots \wedge \mathbb{H}P^\infty),$$

and pulling back along the diagonal map $\mathbb{H}P^\infty \to \mathbb{H}P^\infty \wedge \cdots \wedge \mathbb{H}P^\infty$ gives

$$\begin{aligned} \gamma^{i+1} p_1((\xi_1 - \mathbb{H})^{2i+1}) &= \gamma^{2i+1} p_1^{2i+1} \\ \Rightarrow \quad p_1((\xi_1 - \mathbb{H})^{2i+1}) &= \gamma^i p_1^{2i+1} \in KO^4(\mathbb{H}P^\infty). \end{aligned}$$

□

This result will prove crucial for our constructions in $SU/Sp^0(Y^{4n+1})$, which are now given in the following theorem.

Theorem 7.3.30. *Given any subset $\{l_1, \ldots, l_{2j}\} \subseteq \{1, \ldots, n-1\}$ of even cardinality, there is a quaternionic bundle $\mu_{n,l_1,\ldots,l_{2j}}$ over Y^{4n+1}, which is trivial when viewed as an SU-bundle, and is such that the element $\Omega(\mu_{n,l_1,\ldots,l_{2j}}) \in SU/Sp^0(Y^{4n+1})$ corresponds to $\gamma^{j+1} v_n y_{l_1} \ldots y_{l_{2j}}$ in $KO^{-3}(Y^{4n+1})$.*

Proof. We adapt the proof of Theorem 12.11 in [48].

Recalling the definition of λ_n from Lemma 7.2.13, write $\theta_{n,l_1,\ldots,l_{2j}}$ for the virtual quaternionic bundle $\lambda_n \otimes_{\mathbb{R}} (\chi_{l_1} - \mathbb{H}) \otimes_{\mathbb{H}} \cdots \otimes_{\mathbb{H}} (\chi_{l_{2j}} - \mathbb{H}) \in KSp^0(T(\chi_{n-1}))$. It follows from equation (7.2.15) that $\vartheta_n^*(\theta_{n,l_1,\ldots,l_{2j}}) \in KSp^0(Q^n)$ is equal to

$$(\chi_n - \mathbb{H}) \otimes_{\mathbb{R}} ((\chi_{l_1} - \mathbb{H}) \otimes_{\mathbb{H}} (\chi_{l_2} - \mathbb{H})) \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} ((\chi_{l_{2j-1}} - \mathbb{H}) \otimes_{\mathbb{H}} (\chi_{l_{2j}} - \mathbb{H})).$$

Now define a virtual quaternionic bundle $\mu_{n,l_1,\ldots,l_{2j}}$ to be the pullback

$$\vartheta_n^*((\eta-\mathbb{R})\otimes_{\mathbb{R}}\theta_{n,l_1,\ldots,l_{2j}})\in KSp^0(Y^{4n+1}). \tag{7.3.31}$$

A map $Y^{4n+1}\to S^1\wedge BSp$ is given by the composite

$$Y^{4n+1}\xrightarrow{\vartheta_n}S^1\wedge T(\chi_{n-1})\xrightarrow{f}S^1\wedge BSp, \tag{7.3.32}$$

where f is the suspension of the classifying map for $\theta_{n,l_1,\ldots,l_{2j}}$, and so $\Omega(\mu_i)$ is defined. This gives a lift of the classifying map of $\mu_{n,l_1,\ldots,l_{2j}}$ to SU/Sp, and thus an SU-trivialisation of $\mu_{n,l_1,\ldots,l_{2j}}$. It follows from Lemma 7.3.19 that $\Omega(\mu_{n,l_1,\ldots,l_{2j}})$ corresponds to

$$\gamma\vartheta_n^*(\sigma p_1(\theta_{n,l_1,\ldots,l_{2j}}))\in KO^{-3}(Y^{4n+1}). \tag{7.3.33}$$

As in the universal example of Lemma 7.3.28 we can pullback $(\xi_1-\mathbb{H})^{\langle 2j+1\rangle}$ over $\mathbb{H}P^\infty\wedge\cdots\wedge\mathbb{H}P^\infty$ to $Q^n\wedge\cdots\wedge Q^n$ to obtain

$$p_1(\vartheta_n^*(\widehat{\theta}_{n,l_1,\ldots,l_{2j}}))=\gamma^j y_n\wedge y_{l_1}\wedge\cdots\wedge y_{l_{2j}}\in KO^4(Q^n\wedge\cdots\wedge Q^n),$$

where $\widehat{\theta}_{n,l_1,\ldots,l_{2j}}$ denotes the external tensor product $\lambda_n\otimes_{\mathbb{R}}((\chi_{l_1}-\mathbb{H})\otimes_{\mathbb{H}}(\chi_{l_2}-\mathbb{H}))\otimes_{\mathbb{R}}\cdots\otimes_{\mathbb{R}}((\chi_{l_{2j-1}}-\mathbb{H})\otimes_{\mathbb{H}}(\chi_{l_{2j}}-\mathbb{H}))$ over $Q^n\wedge\cdots\wedge Q^n$. Pulling back once more to Q^n via the diagonal map $Q^n\to Q^n\wedge\cdots\wedge Q^n$ we have

$$p_1(\vartheta_n^*(\theta_{n,l_1,\ldots,l_{2j}}))=\gamma^j y_n y_{l_1}\ldots y_{l_{2j}}\in KO^4(Q^n),$$

and applying Lemma 7.2.18 yields

$$\begin{aligned}\vartheta_n^*(\sigma p_1(\theta_{n,l_1,\ldots,l_{2j}})) &= \gamma^j\vartheta_n^*(\sigma t_n y_{l_1}\ldots y_{l_{2j}})\\ &= \gamma^j v_n y_{l_1}\ldots y_{l_{2j}}\in KO^5(Y^{4n+1}).\end{aligned}$$

So by (7.3.33) we have that our SU trivialisation $\Omega(\mu_{n,l_1,\ldots,l_{2j}})$ corresponds to $\gamma^{j+1}v_n y_{l_1}\ldots y_{l_{2j}}$ in $KO^{-3}(Y^{4n+1})$. □

Corollary 7.3.34. *The* $TorsKSp^0(Y^{4n+1})$ *summand that appears in the splitting of* $SO/Sp^0(Y^{4n+1})$ *in Theorem 7.3.13 is generated by* $\Delta(\mu_{n,l_1,\ldots,l_{2j}})$*, as* $\{l_1,\ldots,l_{2j}\}$ *ranges over all subsets of* $\{1,\ldots,n-1\}$ *that have even cardinality.*

Proof. We defined $\Delta(-)$ to be $j_{SU}\Omega(-)$, so the result follows by combining Proposition 7.3.17 and Theorem 7.3.30. □

7.4 Associated units

In Section 2.3.4, we defined the associated unit of $\delta \in H/G^0(M^n_+)$ to be the image of δ under the J-homomorphism $J\colon H/G^0(M^n_+) \to MG^0(M^n_+)$. Let $As(M^n)$ denote the group $\{J(\delta) : \delta \in H/G^0(M^n_+)\}$. In this section we will determine $As(Y^{4n+1})$ in the case when $H/G = SO/Sp$.

The elements of $As(Y^{4n+1})$ may be divided into two disjoint types: *type A* are elements arising from $SU^0(Y^{4n+1})$ or $SO^0(Y^{4n+1})$, forming a subgroup $\mathcal{A}(Y^{4n+1}) < As(Y^{4n+1})$; *type B* are elements arising from $TorsKSp^0(Y^{4n+1})$, forming a subgroup $\mathcal{B}(Y^{4n+1}) < As(Y^{4n+1})$. This dichotomy follows from our calculations of the groups $U/Sp^0(Y^{4n+1})$ and $SO/Sp^0(Y^{4n+1})$, and Proposition 7.3.17.

To deal with the elements of type A, we will rely heavily on the theory expounded in [48, Sections 13, 14]. Rather than regurgitate all of this background detail, we will give a rapid summary of the specific results that we need to reach a description of $\mathcal{A}(Y^{4n+1})$; any omitted proofs can of course be found in [48]. We conclude by determining $\mathcal{B}(Y^{4n+1})$, using the quaternionic bundles $\mu_{n,l_1,\ldots,l_{2j}}$, which we defined at the end of the previous section.

Suppose we have a finite CW complex X and a ring spectrum E. Let $ET(X) < KO^0(X)$ denote the subgroup of virtual bundles over X, which are E *orientable*; that is, bundles for which an appropriate Thom class in $E^*(X)$ exists.

Definition 7.4.1. The *exotic J group* of X associated with the spectrum E is the quotient $KO^0(X)/ET(X)$, which we denote $EJ(X)$.

When E is the sphere spectrum, $EJ(X)$ reduces to the group $J(X)$, introduced by Atiyah in [3]; the notation is due to the fact that when X is a sphere S^n, $J(X)$ is the image of the classical stable J-homomorphism $J\colon \pi_i(O(n)) \to \pi_{n+i}(S^n)$ (see e.g. [15, Chapter 8]).

We will focus on the case when the spectrum E is taken to be the MSp spectrum of Example 2.2.9, which is associated with quaternionic cobordism theory. In the case of quaternionic projective space, we have the following result.

Lemma 7.4.2. *[48, (14.6), (14.7)] For all $n \in \mathbb{Z}$, the exotic J groups $MSpJ(\mathbb{H}P^n)$ and $MSpJ(S^1 \wedge \mathbb{H}P^n)$ are zero.*

The notion of the exotic J group is intimately linked to the type $\mathcal{A}$ associated units of X.

Lemma 7.4.3. *[48, Lemma 15.1] The subgroup $\mathcal{A}(X)$ of associated units of type A is isomorphic to $MSpJ(S^1 \wedge X)$.*

So our aim is to compute $MSpJ(S^1 \wedge Y^{4n+1})$. To begin, we consider the case of the bounded quaternionic flag manifold Q^n.

Theorem 7.4.4. *The exotic J groups $MSpJ(Q^n)$ and $MSpJ(S^1 \wedge Q^n)$ are zero, for all n.*

Proof. The proof will be by induction on n. The base case $MSpJ(S^4) = 0$ is confirmed by Lemma 7.4.2.

The classifying map of the bundle λ_n, as defined in Lemma 7.2.13, induces an epimorphism

$$KO^*(\mathbb{H}P^n) \longrightarrow KO^*(T(\chi_{n-1})), \tag{7.4.5}$$

as $\lambda_n^*(t^{KO}) = t_n^{KO} \in KO^4(T(\chi_{n-1}))$, where t^{KO} is the universal Thom class of Example 6.1.10. It follows that $MSpJ(T(\chi_{n-1})) = 0$.

Now assume that $MSpJ(Q^i) = 0$ for $i \leq n-1$. We have that (6.1.9) in the case of Q^n induces a short exact sequence

$$MSpJ(Q^{n-1}) \xleftarrow{\omega_n^*} MSpJ(Q^n) \xleftarrow{\vartheta_n^*} MSpJ(T(\chi_{n-1})). \tag{7.4.6}$$

Since $MSpJ(T(\chi_{n-1})) = 0$, the inductive hypothesis implies that $MSpJ(Q^n) = 0$, and so by induction the result holds for all n.

Similar reasoning shows that $MSpJ(S^1 \wedge Q^n) = 0$, for all n. □

This allows us to determine a similar result for Y^{4n+1}.

Theorem 7.4.7. *The exotic J groups $MSpJ(Y^{4n+1})$ and $MSpJ(S^1 \wedge Y^{4n+1})$ are zero, for all n.*

Proof. Since we have a short exact sequence

$$MSpJ(Q^{n-1}) \overset{\omega_n^*}{\longleftarrow} MSpJ(Y^{4n+1}) \overset{\vartheta_n^*}{\longleftarrow} MSpJ(T(\chi_{n-1} \oplus \mathbb{R})), \tag{7.4.8}$$

similar reasoning to Theorem 7.4.4 will suffice to prove the result by induction; the base case $MSpJ(S^5) = MSpJ(S^1 \wedge S^4)$ is given by Lemma 7.4.2. □

So we have established that $\mathcal{A}(Q^n)$ and $\mathcal{A}(Y^{4n+1})$ are both zero.

Now we turn our attention to the associated units of type B, which arise from $TorsKSp^0(-)$. Following Theorem 7.3.30, the elements in $\mathcal{B}(Y^{4n+1})$ are of the form $J\Delta(\mu_{n,l_1,\ldots,l_{2j}})$. We will describe our type B associated units by pulling back from the universal case of $S^1 \wedge BSp$. We have a composition

$$S^1 \wedge BSp \xrightarrow{\Delta} SO/Sp \xrightarrow{j} \Omega^\infty MSp, \tag{7.4.9}$$

using the map j, which was introduced in Section 2.3.4. The unit represented by the composite $j(\Delta) \in MSp^0(S^1 \wedge BSp_+)$ is called the *universal unit*, denoted by $\mathcal{U}$. Using the torsion elements $\varphi_n \in MSp_{8n-3}$, and the generators $P_i \in MSp^{4i}(BSp)$ of primitive elements in $MSp^*(BSp)$, we can describe $\mathcal{U}$ as follows.

Theorem 7.4.10. *[48, Theorem 16.2] The universal unit is given by*

$$\mathcal{U} = 1 + g_1 \otimes \sum_{i\geq 1} \varphi_i P_{2i-1} \text{ in } MSp^0(S^1 \wedge BSp_+).$$

This allows us to describe $J\Delta(\mu_{n,l_1,\ldots,l_{2j}}) \in MSp^0(Y^{4n+1})$, which is represented by the composite map

$$Y^{4n+1} \xrightarrow{\vartheta_n} T(\chi_{n-1} \oplus \mathbb{R}) \cong S^1 \wedge T(\chi_{n-1}) \xrightarrow{f} S^1 \wedge BSp \xrightarrow{\Delta} SO/Sp \xrightarrow{j} \Omega^\infty MSp(\infty),$$

where f is the smash product of the identity and the classifying map for the virtual bundle $\theta_{n,l_1,\ldots,l_{2j}}$. By Theorem 7.4.10 we have

$$J\Delta(\mu_{n,l_1,\ldots,l_{2j}}) = 1 + \vartheta_n^* \left(g_1 \otimes \sum_{i\geq 1} \varphi_i P_{2i-1}(\theta_{n,l_1,\ldots,l_{2j}}) \right). \tag{7.4.11}$$

From the short exact sequence (7.4.6) we have $\vartheta_n^* \colon MSp^*(T(\chi_{n-1})) \to MSp^*(Q^n)$ is a monomorphism, so we may work with $P_{2i-1}(\vartheta_n^* \theta_{n,l_1,\ldots,l_{2j}}) \in MSp^{8i-4}(Q^n)$, which is equal to

$$P_{2i-1}((\chi_n - \mathbb{H}) \otimes_\mathbb{R} ((\chi_{l_1} - \mathbb{H}) \otimes_\mathbb{H} (\chi_{l_2} - \mathbb{H})) \otimes_\mathbb{R} \cdots \otimes_\mathbb{R} ((\chi_{l_{2j-1}} - \mathbb{H}) \otimes_\mathbb{H} (\chi_{l_{2j}} - \mathbb{H}))),$$

in place of $P_{2i-1}(\theta_{n,l_1,\ldots,l_{2j}})$.

First we consider the universal case of $P_{2i-1}((\xi_1-\mathbb{H})^{\langle 2j+1\rangle})$ in $MSp^{8i-4}(\mathbb{H}P^\infty\wedge\cdots\wedge\mathbb{H}P^\infty)$, which may be expanded as

$$P_{2i-1}((\xi_1-\mathbb{H})^{\langle 2j+1\rangle})=\sum_{k_1,\ldots,k_{2j+1}\geq 1} c^i_{k_1,\ldots,k_{2j+1}} p_1^{k_1}\otimes\cdots\otimes p_1^{k_{2j+1}}, \tag{7.4.12}$$

for some coefficients $c^i_{k_1,\ldots,k_{2j+1}}\in MSp_{4(k_1+\cdots+k_{2j+1}-2i+1)}$, and with the summation taken over all possible combinations of $k_1\geq 1,\ldots,k_{2j+1}\geq 1$. We will return to study the cobordism classes $c^i_{k_1,\ldots,k_{2j+1}}$ in detail later.

By way of contrast, we note that in [48], the universal case of $P_{2i-1}((\xi_1-\mathbb{H})^{2j+1})$ in $MSp^{8i-4}(\mathbb{H}P^\infty)$ is considered, in which case the expansion is

$$P_{2i-1}((\xi_1-\mathbb{H})^{2j+1})=\sum_{k\geq 0}\varepsilon_{i,j,k}p_1^{2j+1+k}, \tag{7.4.13}$$

for some $\varepsilon_{i,j,k}\in MSp_{4(2j+k+2-2i)}$. We will also look at the relationship between $\varepsilon_{i,j,k}$ and $c^i_{k_1,\ldots,k_{2j+1}}$ in due course.

Now, pulling back (7.4.12) first to $Q^n\wedge\cdots\wedge Q^n$, and then to a single copy of Q^n via the diagonal map, we have

$$P_{2i-1}(\vartheta_n^*(\theta_{n,l_1,\ldots,l_{2j}}))=\sum_{k_1,\ldots,k_{2j+1}\geq 1} c^i_{k_1,\ldots,k_{2j+1}} y_n^{k_1}y_{l_1}^{k_2}\ldots y_{l_{2j}}^{k_{2j+1}}. \tag{7.4.14}$$

Therefore, using the relation $y_n^{k_1}=y_ny_{n-1}^{k_1-1}$ in the quaternionic cobordism ring of Q^n (6.2.4), and recalling that $y_n\pi_n^*(x)$ may be written as $\vartheta_n^*(t_nx)$, for any $x\in MSp^*(Q^{n-1})$, we can express (7.4.11) as

$$J\Delta(\mu_{n,l_1,\ldots,l_{2j}})=1+\sum_{i=1}^{\lfloor\frac{n+1}{2}\rfloor}\varphi_i\sum_{k_1,\ldots,k_{2j+1}\geq 1} c^i_{k_1,\ldots,k_{2j+1}}\vartheta_n^*(\sigma t_ny_{n-1}^{k_1-1}y_{l_1}^{k_2}\ldots y_{l_{2j}}^{k_{2j+1}}),$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. The first summation runs only between $i=1$ and $\lfloor\frac{n+1}{2}\rfloor$, because $P_{2i-1}(\vartheta_n^*(\theta_{n,l_1,\ldots,l_{2j}}))\in MSp^{8i-4}(Q^n)$, as in equation (7.4.14), vanishes once $8i-4>4n$. As in the proof of Theorem 7.3.30, we are able to rewrite this expression to describe our type B associated units as follows.

Theorem 7.4.15. *The associated unit $J\Delta(\mu_{n,l_1,\ldots,l_{2j}})\in\mathcal{B}(Y^{4n+1})$ is given by*

$$J\Delta(\mu_{n,l_1,\ldots,l_{2j}})=1+\sum_{i=1}^{\lfloor\frac{n+1}{2}\rfloor}\varphi_i\sum_{k_1,\ldots,k_{2j+1}\geq 1} c^i_{k_1,\ldots,k_{2j+1}} v_ny_{n-1}^{k_1-1}y_{l_1}^{k_2}\ldots y_{l_{2j}}^{k_{2j+1}}, \tag{7.4.16}$$

for $2j \le n-1$, *and* $k_1 + \cdots + k_{2j+1} \le n$.

Proof. We insist that $k_1 + \cdots + k_{2j+1} \le n$, because the relations in the cohomology ring (7.2.17) of Y^{4n+1} give $v_n y_{n-1}^{k_1-1} y_{l_1}^{k_2} \ldots y_{l_{2j}}^{k_{2j+1}} = 0$, if $k_1 + \cdots + k_{2j+1} > n$. The constraint that $2j \le n-1$ is a necessary consequence of Theorem 7.3.30. □

Remark 7.4.17. In light of Proposition 7.3.17, since $\mathcal{A}(Y^{4n+1}) = 0$, the units in $MSp^0(Y^{4n+1})$ associated with $SU/Sp^0(Y^{4n+1})$ are the same as those associated with $SO/Sp^0(Y^{4n+1})$, namely the type $\mathcal{B}$ associated units arising from $TorsKSp^0(Y^{4n+1})$.

Let us now investigate the elements $c^i_{k_1,\ldots,k_{2j+1}} \in MSp_{4(k_1+\cdots+k_{2j+1}-2i+1)}$. Though we have no systematic procedure for determining these cobordism classes in terms of known generators of MSp_*, we can combine a variety of approaches to study some low dimensional examples.

The case when $i = 1$ appeared in a paper of Ray [44], who denoted the sum of all $c^1_{k_1,\ldots,k_{2j+1}}$ with $k_1 + \cdots + k_{2j+1} = p$ by $m(j+1, \Delta_p)$. This helped determine examples of the classes $\varepsilon_{i,j,k}$, which appeared in the expansion (7.4.13), in [48, Lemma 16.3]. Our cobordism classes are related to $\varepsilon_{i,j,k}$ by the equation

$$\varepsilon_{i,j,k} = \sum_{k_1,\ldots,k_{2j+1} \ge 1} c^i_{k_1,\ldots,k_{2j+1}}, \text{ such that } k_1 + \cdots + k_{2j+1} = 2j+1+k,$$

where as usual, the summation is over all possible combinations of $k_1 \ge 1, \ldots, k_{2j} \ge 1$. For example, we have

$$\begin{aligned}
\varepsilon_{1,1,2} &= c^1_{1,2,2} + c^1_{2,1,2} + c^1_{2,2,1} + c^1_{3,1,1} + c^1_{1,3,1} + c^1_{1,1,3} \\
\varepsilon_{1,2,0} &= c^1_{1,1,1,1,1} \\
\varepsilon_{2,1,1} &= c^2_{2,1,1} + c^2_{1,2,1} + c^2_{1,1,2} \\
\varepsilon_{2,1,0} &= c^2_{1,1,1}
\end{aligned}$$

The case $i = 1$ was also studied in detail by Imaoka [28], whose results were utilised in [4]. By a combination of all of the above methods, and some slight modifications for our particular situation, we can determine some examples of the classes $c^i_{k_1,\ldots,k_{2j+1}}$ in low dimensions by studying their relationship with $\varepsilon_{i,j,k}$. In common with other authors, we will express our answers in terms of certain generators $2x_i \in MSp_{4i}$, using the notation established in [45].

We can determine

$$\begin{aligned} c^i_{k_1} &= 0, \text{ if } k_1 \neq 2i-1, \\ c^i_{2i-1} &= 1, \\ c^1_{1,1,1} &= 36x_2 + x_1^2, \\ c^1_{2,1,1} &= 60x_3 - 15x_1x_2, \\ c^2_{1,1,1} &= 36, \\ c^2_{2,1,1} &= 110x_1, \\ c^1_{1,1,1,1,1} &= 302400x_4 + 10800x_1x_3 + 12096x_2^2 + 972x_1^2x_2 + x_1^4. \end{aligned} \tag{7.4.18}$$

Clearly no obvious pattern arises from this list to allow us to describe more examples of $c^i_{k_1,\dots,k_{2j+1}}$ in terms of known generators of MSp_*.

Remark 7.4.19. Due to the associative (7.3.22) and commutative (7.3.21) nature of the tensor product of virtual bundles, we were able to obtain virtual real bundle isomorphisms of the form (7.3.24) and (7.3.23). Various combinations of these isomorphisms are enough to show that $c^i_{k_1,k_2,k_3} = c^i_{k'_1,k'_2,k'_3}$ for any permutation k'_1, k'_2, k'_3 of k_1, k_2, k_3. Of course, any other relation between tensor products of virtual bundles will yield similar relations on the elements $c^i_{k_1,\dots,k_{2j+1}}$.

Now suppose that we can represent the element $c^1_{k_1,\dots,k_{2j+1}} \in MSp_{4(k_1+\cdots+k_{2j+1}-2i+1)}$ geometrically by a $4(k_1+\cdots+k_{2j+1}-2i+1)$-dimensional stably quaternionic manifold $M_{k_1,\dots,k_{2j+1}}$. Then the following fact could prove helpful in describing $c^1_{k_1,\dots,k_{2j+1}}$.

Lemma 7.4.20. *The manifold $M_{k_1,\dots,k_{2j+1}}$ may be embedded as a codimension-$(8i-4)$ submanifold of $Q^{k_1+\cdots+k_{2j+1}}$.*

Proof. With $q_k \in MSp_{4k}(\mathbb{H}P^\infty_+)$ as in Proposition 7.1.6, and with reference to the expansion (7.4.12), it is straightforward to check that the Kronecker product

$$\langle p_1((\xi_1 - \mathbb{H})^{2j+1}), q_{k_1+\cdots+k_{2j+1}} \rangle, \tag{7.4.21}$$

yields $c^1_{k_1,\dots,k_{2j+1}}$ in $MSp_{4(k_1+\cdots+k_{2j+1}-1)}$.

In Proposition 7.1.8, we proved that the generator $q_{k_1+\cdots+k_{2j+1}}$ is represented by the map $Q^K \to \mathbb{H}P^K$ classifying the quaternionic line bundle χ_K, where $K = k_1 + \cdots + k_{2j+1}$. The Kronecker product (7.4.21) with the first Pontryagin class is equivalent to making this classifying map transverse to the inclusion $\mathbb{H}P^{K-1} \subset \mathbb{H}P^K$. Therefore, in a similar manner to the transversality construction illustrated by diagram (7.1.14), we have an embedding of the manifold $M_{k_1,\ldots,k_{2j+1}}$, representing $c^1_{k_1,\ldots,k_{2j+1}}$, as a codimension-$(8i-4)$ submanifold of Q^K. □

While we do not have a complete understanding of the elements $c^i_{k_1,\ldots,k_{2j+1}}$ in terms of generators of MSp_*, we have gleaned enough information to conclude this section with a list of the type B associated units for Y^{4n+1} in low dimensions.

Corollary 7.4.22. *For $n \leq 4$ the associated units $\mathcal{B}(Y^{4n+1})$ in $MSp^0(Y^{4n+1}_+)$ are as follows.*

For $n = 1$, $J\Delta(\mu_1) = 1 + \varphi_1 v_1.$

For $n = 2$, $J\Delta(\mu_2) = 1 + \varphi_1 v_2.$

For $n = 3$,

$$\begin{aligned} J\Delta(\mu_3) &= 1 + \varphi_1 v_3 + \varphi_2 v_3 y_2 y_1, \\ J\Delta(\mu_{3,2,1}) &= 1 + \varphi_1 x_1^2 v_3 y_2 y_1. \end{aligned}$$

For $n = 4$,

$$\begin{aligned} J\Delta(\mu_4) &= 1 + \varphi_1 v_4 + \varphi_2 v_4 y_3 y_2, \\ J\Delta(\mu_{4,3,2}) &= 1 + \varphi_1 (x_1^2 v_4 y_3 y_2 + x_1 x_2 v_4 y_3 y_2 y_1), \\ J\Delta(\mu_{4,3,1}) &= 1 + \varphi_1 x_1^2 v_4 y_3 y_1, \\ J\Delta(\mu_{4,2,1}) &= 1 + \varphi_1 (x_1^2 v_4 y_2 y_1 + x_1 x_2 v_4 y_3 y_2 y_1). \end{aligned}$$

7.5 Fundamental classes

Having determined our associated units, we must now obtain the fundamental class $[Y^{4n+1}]_s \in MSp_{4n+1}(Y^{4n+1})$, so that we can calculate $\Psi(Y^{4n+1})$.

To get started, since $MSp^*(Q^n)$ and $MSp^*(Y^{4n+1})$ are free over MSp_* by Propositions 6.2.4 and 7.2.17 we can dualise to deduce the following results.

Corollary 7.5.1. *A basis for the quaternionic bordism module of Q^n is given by*

$$MSp_*(Q^n) = MSp_*\{z_{R(i)} : R(i) \in P[n]\}, \tag{7.5.2}$$

where the monomial $z_{R(i)} \in MSp_{4i}(Q^n)$ is the dual homology element to the monomial $y_{R(i)} \in MSp^{4i}(Q^n)$, for $1 \leq i \leq n$.

A basis for the quaternionic bordism module of Y^{4n+1} is given by

$$MSp_*(Y^{4n+1}) = MSp_*\{w_n, z_{R(i)}, w_n z_{R(i)} : R(i) \in P[n-1]\}, \tag{7.5.3}$$

where $w_n \in MSp_5(Y^{4n+1})$, $z_{R(i)} \in MSp_{4i}(Y^{4n+1})$ and $w_n z_{R(i)} \in MSp_{4i+5}(Y^{4n+1})$ are the dual homology elements to $v_n \in MSp^5(Y^{4n+1})$, $y_{R(i)} \in MSp^{4i}(Y^{4n+1})$ and $v_n y_{R(i)} \in MSp^{4n+5}(Y^{4n+1})$ respectively, for $1 \leq i \leq n-1$.

Given the bounding stably quaternionic structure s' on Q^n from Proposition 6.3.3, we will determine the fundamental class $[Q^n]_{s'}$ in terms of the above basis of $MSp_*(Q^n)$.

Proposition 7.5.4. *The fundamental class of Q^n with stably quaternionic structure s', is given by $[Q^n]_{s'} = z_n z_{n-1} \ldots z_1 \in MSp_{4n}(Q^n_+)$.*

Proof. Referring to Corollary 7.5.1, the fundamental class is of the form

$$[Q^n]_{s'} = \sum_{R(i) \in P[n-1]} \zeta_{\{R(i)\}} z_{R(i)} + z_n z_{n-1} \ldots z_1, \tag{7.5.5}$$

where $\zeta_{\{R(i)\}} \in MSp_{4(n-i)}$. Using the projection $\pi_n \colon Q^n \to Q^{n-1}$ we can determine these coefficients $\zeta_{\{R(i)\}}$ to be

$$\begin{aligned} \zeta_{\{R(i)\}} &= \langle y_{R(i)}, [Q^n]_{s'} \rangle \\ &= \langle \pi_n^*(y_{R(i)}), [Q^n]_{s'} \rangle \\ &= \langle y_{R(i)}, (\pi_n)_*[Q^n]_{s'}, \rangle \end{aligned}$$

but $(\pi_n)_*[Q^n]_{s'} = 0 \in MSp_{4n}(Q^{n-1})$, since π_n maps the bounding Q^n into Q^{n-1}, representing zero in $MSp_{4n}(Q^{n-1})$. It follows that $\zeta_{\{R(i)\}} = 0$, for any subset $R(i) \in P[n-1]$, and so the expression (7.5.5) reduces to $[Q^n]_{s'} = z_n z_{n-1} \ldots z_1$. □

Using similar methods, the fundamental class of $[S^1 \times Q^n, t \times s']$, where t is the trivial stably quaternionic structure on S^1, is given by $[S^1 \times Q^n]_{t\times s'} = h_1 \otimes z_n z_{n-1} \dots z_1$, where $h_1 \in MSp_1(S^1)$ is the dual generator to $g_1 \in MSp^1(S^1)$.

In Lemma 7.2.8 we defined a degree 1 retraction r'_n, which arose from the quaternionic bordism between $[S^1 \times Q^n, t \times s']$ and $[Y^{4n+1}, s]$. We can easily dualise the action of $(r'_n)^*$, as described in Lemma 7.2.18, to determine $(r'_n)_*([S^1 \times Q^n]_{t\times s'}) = [Y^{4n+1}]_s$.

Proposition 7.5.6. *The fundamental class of Y^{4n+1} with stably quaternionic structure s, is given by $[Y^{4n+1}]_s = w_n z_{n-1} z_{n-2} \dots z_1$ in $MSp_{4n+1}(Y^{4n+1}_+)$.*

7.6 Determination of $\Psi(Y^{4n+1})$

We now have all the information we need to compute the image $\Psi(Y^{4n+1})$ of the map $\Psi_s \colon SO/Sp^0(Y^{4n+1}_+) \to MSp_{4n+1}$. After comparing our results with those in [48], we prove Conjecture 7.2.1.

According to the instructions that concluded Section 2.3.4, to compute $\Psi(Y^{4n+1})$ we must form the associated dual $\Gamma_s(\Delta(\mu_{n,l_1,\dots,l_{2j}})) = J\Delta(\mu_{n,l_1,\dots,l_{2j}}) \frown [Y^{4n+1}]_s$ and take the quaternionic bordism class of a manifold that represents $\Gamma_s(\Delta(\mu_{n,l_1,\dots,l_{2j}}))$.

Theorem 7.6.1. *The image $\Psi(Y^{4n+1})$, which is a subgroup of MSp_{4n+1}, is generated by*

$$\left\{ \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \varphi_i \sum_{k_1,\dots,k_{2j+1}\geq 1} c^i_{k_1,\dots,k_{2j+1}} \right\}$$

where $2j \leq n-1$, and the integers $k_1, \dots, k_{2j+1}$ are such that

$$y_{n-1}^{k_1-1} y_{l_1}^{k_2} \dots y_{l_{2j}}^{k_{2j+1}} = y_{n-1} y_{n-2} \dots y_1 \in MSp^{4K}(Y^{4n+1}), \tag{7.6.2}$$

where $K = k_1 - 1 + k_2 + \cdots + k_{2j+1}$.

Proof. Compare the expressions for the associated units (7.4.16) and the fundamental class (7.5.6). Clearly the associated dual is nonzero only when $k_1 \geq 1, \dots, k_{2j+1} \geq 1$ are such that the monomial $y_{n-1}^{k_1-1} y_{l_1}^{k_2} \dots y_{l_{2j}}^{k_{2j+1}}$ reduces to $y_{n-1} y_{n-2} \dots y_1$ under the cohomology relations in $MSp^{4K}(Y^{4n+1})$. Hence we have the generators of $\Psi(Y^{4n+1})$ in MSp_* as described above.

To see that $\Psi(Y^{4n+1})$ is a subgroup in MSp_{4n+1}, we utilise the reduced map $\widetilde{\Psi}$. Given $a_1, a_2 \in SO/Sp^0(Y^{4n+1})$, in equation (2.3.19) the reduced J homomorphism $\widetilde{J}$ was shown to have the property that $\widetilde{J}(a_1 + a_2) = \widetilde{J}(a_1) + \widetilde{J}(a_2) + \widetilde{J}(a_1)\widetilde{J}(a_2)$. Then we have

$$\begin{aligned}\widetilde{\Psi}_s(a_1 + a_2) &= \langle \widetilde{J}(a_1) + \widetilde{J}(a_2) + \widetilde{J}(a_1)\widetilde{J}(a_2), [Y^{4n+1}]_s\rangle \\ &= \widetilde{\Psi}_s(a_1) + \widetilde{\Psi}_s(a_2) + \langle \widetilde{J}(a_1)\widetilde{J}(a_2), [Y^{4n+1}]_s\rangle.\end{aligned}$$

By Theorems 7.4.7, 7.4.15, any $\widetilde{J}(a) \in As(Y^{4n+1})$ is in the ideal generated by v_n, but $v_n^2 = 0$ in $MSp^{10}(Y^{4n+1})$, so it follows that $\widetilde{J}(a_1)\widetilde{J}(a_2) = 0$. Therefore $\widetilde{\Psi}$ is a homomorphism.

The stably quaternionic structure s bounds, so by the formula (2.3.20) we have $\Psi(Y^{4n+1}) = \widetilde{\Psi}(Y^{4n+1})$, which completes the proof. □

Note that every element in the subgroup $\Psi(Y^{4n+1})$ has order 2, since the generators all have factors the 2-torsion elements φ_i.

Following Remark 7.4.17, the units associated to $SO/Sp^0(Y^{4n+1})$ coincide with those associated to $SU/Sp^0(Y^{4n+1})$, coming from $TorsKSp^0(Y^{4n+1})$. Therefore the changes of stably quaternionic structure on Y^{4n+1} that give rise to the elements in $\Psi(Y^{4n+1})$ are such that they leave the SU-structure on the manifold unchanged. Since we started with a bounding stably quaternionic structure s on Y^{4n+1} (and hence by forgetting, a bounding SU-structure), then $[Y^{4n+1}, s + \delta]$ also bounds in MSU_*, for any $\delta \in SO/Sp^0(Y^{4n+1})$.

As an illustration of our results, in low dimensions the subgroups $\Psi(Y^{4n+1}) < MSp_{4n+1}$, and their generators are

$$\begin{aligned}\Psi(Y^5) &= \mathbb{Z}/2, && \{\varphi_1\}, \\ \Psi(Y^9) &= 0, && \\ \Psi(Y^{13}) &= \mathbb{Z}/2 \oplus \mathbb{Z}/2, && \{\varphi_2, x_1^2\varphi_1\}, \\ \Psi(Y^{17}) &= \mathbb{Z}/2, && \{x_1x_2\varphi_1\},\end{aligned}$$

In the case when $n = 5$, there is a subgroup $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 < \Psi(Y^{21})$ generated by $\{\varphi_3, x_1^4\varphi_1, x_2^2\varphi_1\}$. Unfortunately, obtaining a complete description of $\Psi(Y^{21})$, and indeed $\Psi(Y^{4n+1})$, for $n > 5$, is hampered by difficulties in computing the elements $c^i_{k_1,\ldots,k_{2j+1}} \in MSp_{4(k_1+\cdots+k_{2j+1}-2i+1)}$. Without further work, we can only describe the remaining generators of $\Psi(Y^{21})$ as $g_{2,2,1} = x_2^2\varphi_1 + \varphi_2 c^2_{2,2,1} + \varphi_3 c^3_{2,2,1}$ and $g_{3,1,1} = (x_1x_3 + ex_2^2)\varphi_1 + \varphi_2 c^2_{3,1,1} + \varphi_3 c^3_{3,1,1}$, for some $e \in \mathbb{Z}/2$.

Remark 7.6.3. By comparing our results with the description of MSp_* in low dimensions given in [45], we have that Y^5 and Y^{13} are SO-universal stably quaternionic manifolds, in the sense of Definition 2.3.21. While we do not have a complete description of $\Psi(Y^{21})$, we do not rule out the possibility that Y^{21} is also an SO-universal stably quaternionic manifold; to prove this we would need to represent $x_1^2\varphi_2 \in MSp_{21}$ via a stably quaternionic structure on Y^{21}. Again, further investigation into the elements $c^i_{k_1,\ldots,k_{2j+1}}$ is needed before we can say anything more about the possibility of Y^{4n+1} being SO-universal stably quaternionic manifolds, for $n \geq 5$.

Let us briefly compare our results with those of [48], where $\Psi(-)$ was calculated on the $(4n+1)$-dimensional manifolds A^{4n+1} defined by Alexander in [2] (note that these calculations of $\Psi(A^{4n+1})$ are in MSp_*/V_*, where V_* is a certain ideal in MSp_*).

The manifolds A^{4n+1} are defined to be $S(\xi_1 \otimes_{\mathbb{H}} \overline{\xi}_1 \oplus \mathbb{R}^2)$, the total space of a 5-sphere bundle over $\mathbb{H}P^{n-1}$. Each A^{4n+1} carries a bounding stably quaternionic structure via the usual application of Proposition 2.3.7. Alexander's manifolds rely on a technique used by Landweber [31], to build stably quaternionic manifolds over quaternionic projective space, which is itself not stably quaternionic. In contrast, the manifolds Y^{4n+1} are built over a quaternionic tower $(Q^k : k \leq n-1)$ of bounded quaternionic flag manifolds, each of which is stably quaternionic. So in terms of quaternionic cobordism theory, it is perhaps more natural to work with Y^{4n+1}.

Certainly we are able to represent more elements in MSp_* using the various stably quaternionic structures on Y^{4n+1}: in dimensions when n is even, $\Psi(Y^{4n+1})$ is not necessarily zero, while $\Psi(A^{4n+1}) = 0$ for all $n = 2m$. In the case when n is odd, we have the following.

Corollary 7.6.4. *The image $\Psi(A^{8m-3})$ is a subgroup of $\Psi(Y^{8m-3})$, which is generated* *by the quaternionic bordism classes of the associated duals*

$$\Gamma_s(\Delta(\mu_n)), \Gamma_s(\Delta(\mu_{n,n-1,n-2})), \Gamma_s(\Delta(\mu_{n,n-1,n-2,n-3,n-4})), \ldots, \Gamma_s(\Delta(\mu_{n,n-1,\ldots,1})).$$

Proof. It is straightforward, though long-winded, to show this, so we offer only a sketch here.

The proof relies on the fact that for virtual bundles $\mu_{n,n-1,\ldots,n-2j}$, the associated units will arise from an expansion of the form

$$P_{2i-1}((\chi_n - \mathbb{H})^{2j+1}) = \sum_{k=0}^{n-2j-1} \varepsilon_{i,j,k} y_n y_{n-1}^{2l+1+k},$$

which plays the role of (7.4.14) in this case. Following the reasoning as in Section 7.4, we obtain the expression for the associated units to be

$$J\Delta(\mu_{n,n-1,\ldots,n-2j}) = 1 + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \varphi_i \sum_{k=0}^{n-2j-1} \varepsilon_{i,j,k} v_n y_{n-1}^{2j+k}, \tag{7.6.5}$$

for $2j \leq n-1$. The result follows by comparing (7.6.5) with the corresponding expression for the associated units in $\mathcal{B}(A^{8m-3})$ given by [48, Theorem 16.6]. □

Returning to the manifolds we have been working with, we can easily relate our results to bounded quaternionic flag manifolds.

Proposition 7.6.6. *There is an isomorphism* $\Psi(S^1 \times Q^n) \cong \Psi(Y^{4n+1})$.

Proof. The isomorphism is induced by the degree 1 map $r_n' \colon W^{4n+1} = S^1 \times Q^n \to Y^{4n+1}$ of Lemma 7.2.8. First we will prove that there is an injection $f \colon \Psi(Y^{4n+1}) \to \Psi(W^{4n+1})$, and then show that f is in fact an epimorphism.

From the proof of Lemma 7.2.18, we know that ${r_n'}^*(v_n) = g_1 \otimes y_n \in MSp^5(W^{4n+1})$. So using r_n' and Theorem 7.3.30, we can pull back $\Omega(\mu_{n,l_1,\ldots,l_{2j}})$, which corresponds to $\gamma^{j+1} v_n y_{l_1} \ldots y_{l_{2j}}$ in $KO^{-3}(Y^{4n+1})$, to the SU-trivialisation $\Omega((\eta - \mathbb{R}) \otimes_{\mathbb{R}} \theta_{n,l_1,\ldots,l_{2j}})$ corresponding to $\gamma^{j+1} g_1 \otimes y_n y_{l_1} \ldots y_{l_{2j}}$ in $KO^{-3}(W^{4n+1})$.

Applying the techniques of Section 7.4 to W^{4n+1}, it is straightforward to determine the formula

$$J\Delta((\eta - \mathbb{R}) \otimes_{\mathbb{R}} \theta_{n,l_1,\ldots,l_{2j}}) = 1 + g_1 \otimes \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \varphi_i \sum_{k_1,\ldots,k_{2j+1} \geq 1} c^i_{k_1,\ldots,k_{2j+1}} y_n^{k_1} y_{l_1}^{k_2} \ldots y_{l_{2j}}^{k_{2j+1}},$$

for $2j \leq n-1$, and $k_1 + \cdots + k_{2j+1} \leq n$.

Recall from Section 7.5, that the retraction map induces an equality between fundamental classes $(r'_n)_*([S^1 \times Q^n]_{t\times s'}) = [Y^{4n+1}]_s$, so that

$$(r'_n)_*(h_1 \otimes z_n z_{n-1} \ldots z_1) = w_n z_{n-1} \ldots z_1. \tag{7.6.7}$$

Now, at the beginning of this section we computed $\Psi(Y^{4n+1})$ by forming the associated dual $\Gamma_s(\Delta(\mu_{n,l_1,\ldots,l_{2j}})) = J\Delta(\mu_{n,l_1,\ldots,l_{2j}}) \frown [Y^{4n+1}]_s$, but using (7.6.7), we have

$$\begin{aligned}
\Gamma_s(\Delta(\mu_{n,l_1,\ldots,l_{2j}})) &= J\Delta(\mu_{n,l_1,\ldots,l_{2j}}) \frown (r'_n)_*(h_1 \otimes z_n z_{n-1} \ldots z_1) \\
&= J(r'_n)^*(\Delta(\mu_{n,l_1,\ldots,l_{2j}})) \frown (h_1 \otimes z_n z_{n-1} \ldots z_1) \\
&= J\Delta((\eta - \mathbb{R}) \otimes_{\mathbb{R}} \theta_{n,l_1,\ldots,l_{2j}}) \frown (h_1 \otimes z_n z_{n-1} \ldots z_1) \\
&= \Gamma_{t\times s'}(\Delta((\eta - \mathbb{R}) \otimes_{\mathbb{R}} \theta_{n,l_1,\ldots,l_{2j}})) \in MSp_{4n+1}(W^{4n+1}),
\end{aligned}$$

and then we clearly have an injection $f\colon \Psi(Y^{4n+1}) \to \Psi(W^{4n+1})$, given by

$$f(\Gamma_s(\Delta(\mu_{n,l_1,\ldots,l_{2j}}))) = (\Gamma_{t\times s'}(\Delta((\eta - \mathbb{R}) \otimes_{\mathbb{R}} \theta_{n,l_1,\ldots,l_{2j}}))). \tag{7.6.8}$$

We must show that f is in fact an epimorphism. It is straightforward to check that r'_n induces an injection $(r'_n)^*\colon KO^{-3}(Y^{4n+1}) \to KO^{-3}(W^{4n+1})$, but not an isomorphism, as there are elements in $KO^{-3}(W^{4n+1})$ of the form $\beta\gamma^j g_1 \otimes y_{R(2j)}$, for $R(2j) \in P[n]$, and of the form $\gamma^{j+1} g_1 \otimes y_{R(2j+1)}$, for $R(2j+1) \in P[n-1]$, neither of which are in the image of $(r'_n)^*$. No associated units arise from the former elements, while it can be shown that for $R(2j+1) = \{l_1, \ldots, l_{2j+1}\}$, the latter give rise to associated units in $MSp^0(W^{4n+1})$ of the form

$$J\Delta(\kappa_{l_1,\ldots,l_{2j+1}}) = 1 + g_1 \otimes \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \varphi_i \sum_{k_1,\ldots,k_{2j+1} \geq 1} c^i_{k_1,\ldots,k_{2j+1}} y_{l_1}^{k_1} \ldots y_{l_{2j+1}}^{k_{2j+1}}, \tag{7.6.9}$$

for some virtual quaternionic bundle $\kappa_{l_1,\ldots,l_{2j+1}}$. However, the corresponding associated dual $\Gamma_{t\times s'}(\Delta(\kappa_{l_1,\ldots,l_{2j+1}}))$ in $MSp_{4n+1}(W^{4n+1})$ is always zero, since the cap product of (7.6.9) and the fundamental class $[S^1 \times Q^n]_{t\times s'}$ is zero. This can be proved by using the fact that $n \notin R(2j+1)$ and applying similar reasoning to that of Lemma 5.1.33. Hence we have no new elements in $\Psi(W^{4n+1})$ arising from the elements $\beta\gamma^i g_1 \otimes y_{R(2i)}$ and $\gamma^{i+1} g_1 \otimes y_{R(2i+1)}$ in $KO^{-3}(W^{4n+1})$. It is also easy to use the tools of Section 7.4 to show that, as for Q^n and Y^{4n+1}, there are no type A associated units for W^{4n+1} that would contribute to the image $\Psi(W^{4n+1})$. This is enough to show that $f\colon \Psi(Y^{4n+1}) \to \Psi(W^{4n+1})$ is indeed an epimorphism, and so we have the required isomorphism. □

Remark 7.6.10. With suitable modifications, it should be possible to use our results to gain an insight into the images $\Psi(S^1 \times Y^{4n+1})$ and $\Psi(Y^{4n+2})$, where the simply-connected manifold Y^{4n+2} is the total space of the 6-sphere bundle $S(\chi_{n-1} \oplus \mathbb{R}^3)$ over Q^{n-1}.

The spur for our investigation of $\Psi(Y^{4n+1})$ was Ray's Conjecture 7.2.1, which we are now able to verify.

Proposition 7.6.11. *The element $\varphi_m \in MSp_{8m-3}$ may be represented geometrically by the simply-connected manifold Y^{8m-3}, equipped with stably quaternionic structure $s + \Delta(\mu_{2m-1})$.*

Proof. By putting $n = 2m-1$ in the basis described by Theorem 7.6.1, we have that the associated dual $\Gamma_s(\Delta(\mu_{2m-1})) = J\Delta(\mu_{2m-1}) \frown [Y^{8m-3}]_s$ in $MSp_{8m-3}(Y^{4n+1}_+)$ satisfies

$$\Gamma_s(\Delta(\mu_{2m-1})) = \varphi_1 c^1_{2m-1} + \varphi_2 c^2_{2m-1} + \cdots + \varphi_m c^m_{2m-1}.$$

This expression reduces to φ_m since from the list (7.4.18) we see that $c^i_{k_1} = 0$, if $k_1 \neq 2i-1$, and $c^i_{2i-1} = 1$. □

Recall that the indecomposable torsion elements $\varphi_n \in MSp_{8n-3}$ are actually the coefficients θ_{2n} in the expansion (7.1.11). On the other hand, the elements $\theta_{2n+1} \in MSp_{8n+1}$ are zero for $n > 0$. However, there is nothing to stop us rewriting the expression of Theorem 7.4.10 to give the universal unit purely in terms of the elements θ_n, for all n. Checking through the original calculation in [48, Chapter 16] it is clear that the universal unit then takes the form

$$\mathcal{U} = 1 + g_1 \otimes \sum_{i\geq 2} \theta_i P_{i-1} \text{ in } MSp^0(S^1 \wedge BSp_+). \tag{7.6.12}$$

Subsequently, the associated unit $J\Delta(\mu_{n,l_1,\ldots,l_{2j}})$ given in Theorem 7.4.15 can be rewritten as

$$\begin{aligned} J\Delta(\mu_{n,l_1,\ldots,l_{2j}}) = 1 + &\sum_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} \theta_{2i} \sum_{k_1,\ldots,k_{2j+1}\geq 1} c^i_{k_1,\ldots,k_{2j+1}} v_n y_{n-1}^{k_1-1} y_{l_1}^{k_2} \ldots y_{l_{2j}}^{k_{2j+1}} \\ &+ \sum_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} \theta_{2i+1} \sum_{k_1,\ldots,k_{2j+1}\geq 1} d^i_{k_1,\ldots,k_{2j+1}} v_n y_{n-1}^{k_1-1} y_{l_1}^{k_2} \ldots y_{l_{2j}}^{k_{2j+1}} \end{aligned} \tag{7.6.13}$$

for $2j \leq n-1$, and $k_1 + \cdots + k_{2j+1} \leq n$, and some coefficients $d^i_{k_1,\ldots,k_{2j+1}} \in MSp_{4(k_1+\cdots+k_{2j+1}-2i)}$. In a similar manner to the elements $c^i_{k_1,\ldots,k_{2j+1}}$ of (7.4.12), this set of coefficients has its true origins in an expansion of the form

$$P_{2i}((\xi_1 - \mathbb{H})^{2j+1}) = \sum_{k_1,\ldots,k_{2j+1}\geq 1} d^i_{k_1,\ldots,k_{2j+1}} p_1^{k_1} \ldots p_1^{k_{2j+1}} \in MSp^{8i}(\mathbb{H}P^\infty). \tag{7.6.14}$$

Now, the restriction of the primitive element $P_i \in MSp^{4i}(BSp)$ to $\mathbb{H}P^\infty$ is simply p_1^i (see e.g. [42, page 263]). By restricting (7.6.14) to the case when $j = 0$, this fact is enough to show that

$$\begin{aligned} d^i_{k_1} &= 0, \text{ if } k_1 \neq 2i, \\ d^i_{2i} &= 1. \end{aligned} \tag{7.6.15}$$

This leads us to our final result.

Proposition 7.6.16. *The element $\theta_{2m+1} \in MSp_{8m+1}$ may be represented geometrically by Y^{8m+1} equipped with stably quaternionic structure $s + \Delta(\mu_{2m})$.*

Proof. The proof follows the same reasoning as that of Proposition 7.6.11. The associated dual $\Gamma_s(\Delta(\mu_{2m})) = J\Delta(\mu_{2m}) \frown [Y^{8m+1}]_s$ satisfies

$$\Gamma_s(\Delta(\mu_{2m})) = \theta_3 d^1_{2m} + \theta_5 d^2_{2m} + \cdots + \theta_{2m+1} d^m_{2m},$$

which reduces to θ_{2m+1} via (7.6.15). □

Consequently, we have the intriguing conclusion that when Y^{4n+1} is equipped with the stably quaternionic structure associated to $\Delta(\mu_n) \in SO/Sp^0(Y^{4n+1})$, our manifolds bound only when n is even; moreover when n is odd, since φ_n is of order two, we must have that the disjoint union of two copies of Y^{8n-3} is a boundary. It would be very interesting to understand the geometry underpinning these phenomena.

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