

ON FAMILIES OF NESTOHEDRA

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Contents

Abstract	4
Declaration	5
Copyright Statement	6
Acknowledgements	7
1 Introduction	8
2 Motivation	10
2.1 Toric Topology	11
2.2 Algebraic Geometry	20
2.3 Symplectic Geometry	23
2.4 Combinatorics	25
3 Polytope Theory	31
3.1 Polytopes	31
3.2 Vectors	37
3.3 Rings	42
3.4 Polynomials	44
4 Nestohedra	49
4.1 Definition	49
4.2 Examples	51
4.3 Background	56

4.4	Boundaries	64
4.5	Nestohedra and Toric Topology	69
4.6	Bigraded Betti Numbers	74
5	Families of Polytopes	77
5.1	Families	77
5.2	Family Subrings	81
5.3	Generating Functions	85
6	Calculating Combinatorial Invariants	92
6.1	Families of Nestohedra	93
6.2	Cubes	95
6.3	Simplices	97
6.4	Permutohedra	98
6.5	Stellohedra	100
6.6	Associahedra	102
6.7	Cyclohedra	107
6.8	Multi-Parameter Families	110
6.9	Helianthahedra	112
6.10	Bipartohedra	117
	Bibliography	126

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In toric topology it is important to have a way of constructing Delzant polytopes, which have canonical combinatorial data. Furthermore we have examples of quasi-toric manifolds with representatives in all even dimensions. These examples give rise to sequences of polytopes with related combinatorial invariants.

In this thesis we intend to formalise the concept of a family of polytopes, which will behave in a similar way to the quotient spaces of quasi-toric manifolds. We will then compute certain combinatorial invariants in this context.

Recently, polytope theory has developed to include \mathcal{P} with homogeneous polynomial invariants and an important operator d , which takes a polytope to the disjoint union of its facets. We will examine our families against this background and extend the polynomial invariants and d to entire families. In particular we will introduce a method to calculate polynomial invariants of families by the use of partial differential equations.

We will also look at some polytopes called nestohedra, which arise from building sets. These nestohedra give us a construction of Delzant polytopes. We will show that it is possible to calculate d for any given nestohedra directly from its building set. We will also show that the canonical characteristic function of a nestohedron, F , which is a facet of a nestohedron, P , agrees with the characteristic function of F as a facet of P .

We will see that nestohedra naturally form families. We will end this work by combining the work on nestohedra with the work on families and calculating the combinatorial invariants of some families of nestohedra.

Declaration

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Chapter 1

Introduction

Toric topology is a discipline which has a heavy reliance on polytope theory. In this thesis we will be looking at some problems in polytope theory arising from toric topology.

In chapter 2 we will overview toric topology itself. We begin by defining the basic concepts and demonstrate the importance of Delzant polytopes. We will also introduce some examples of quasi-toric manifolds with representatives in all even dimensions. These examples give rise to sequences of polytopes with related combinatorial invariants. We will end the chapter by stating four questions which we will find answers in later chapters. These questions will concern: finding examples of Delzant polytopes, calculating their combinatorial invariants, finding families of polytopes that behave like those from the quasi-toric manifolds, and developing ways to study these families.

In chapter 3 we will look at polytope theory. We will look at the historical methods of studying polytopes as well as highlighting the recent work based on ring of polytopes \mathcal{P} . The work on \mathcal{P} includes homogeneous polynomial invariants and an important operator d , which takes a polytope to the disjoint union of its facets.

In chapter 4 we will look at some polytopes called nestohedra, which arise from building sets. These nestohedra give us a construction of Delzant polytopes and answer one of our questions. We will look at what is already known about nestohedra and then show that it is possible to calculate d for any given nestohedron directly

from its building set. We will also show that the canonical characteristic function of a nestohedron, F , which is a facet of a nestohedron, P , agrees with the characteristic function of F as a facet of P .

In chapter 5 we will formalise the concept of a family of polytopes, which will behave in a similar way to the quotient spaces of a family of quasi-toric manifolds. We will examine our families against the background of \mathcal{P} and extend the polynomial invariants and d to entire families. In particular we will introduce some methods to calculate polynomial invariants of families by the use of partial differential equations.

In chapter 6 we will see that nestohedra naturally form families. We will then combine the work on nestohedra with the work on families by calculating the combinatorial invariants of several families of nestohedra.

Chapter 2

Motivation

Toric Topology is the study of quasi-toric manifolds, which are manifolds with well-behaved torus actions. It lies at the intersection of several major areas of geometrical mathematics. In their paper “*An Invitation to Toric Topology: Vertex Four of a Remarkable Tetrahedron*” [11] Buchstaber and Ray use the image of a tetrahedron as the convex hull of four points, or “vertex disciplines”, to demonstrate this interconnectedness.

In this chapter I am going to provide an overview of this subject, highlighting the natural and important role played by polytopes. I will make use of the image of the tetrahedron here because it is both a good image and the tetrahedron itself is a polytope!

The four “vertex disciplines” of the toric tetrahedron in [11] are algebraic geometry, combinatorics, symplectic geometry and toric topology. They are given the labels A , C , S and T respectively for ease of reference. I will start my exposition with the last of these, Toric Topology, as it is the one on which I shall mostly focus in this chapter.

While the nature of toric topology is going to motivate our work on polytopes, we will need some concepts of polytope theory to talk about toric topology. While formal definitions of these concepts will appear in chapter 3, it is important to note at this point that: A *polytope*, P , is the intersection of a finite number of half spaces in \mathbb{R}^n . The *dimension* of a polytope is the dimension of its affine hull. A *supporting*

hyperplane is a hyperplane which defines one of the half-spaces. A *face* of a polytope is the intersection of the polytope with some collection of supporting hyperplanes and is a polytope in its own right. A dimension 0 face is called a *vertex*, a dimension 1 face is called an *edge* and a codimension 1 face is called a *facet*. A polytope is *simplicial* if all its faces (apart from itself) are simplices and *simple* if exactly n facets meet at each vertex. Two polytopes are *combinatorially equivalent* if their faces form isomorphic posets (the *face poset*) under inclusion. The *dual* of P is a polytope with face poset given by the faces of P under reverse inclusion; every polytope has a dual, often referred to as its *polar* P^* .

2.1 Toric Topology

We will begin our look at Toric Topology by setting up some standard notation that will allow us to fully understand what is meant by the statement:

Toric Topology is the study of quasi-toric manifolds, which are smooth manifolds with well-behaved torus actions.

The n -dimensional torus, $(S^1)^n$, shall be denoted by T^n . We will also denote the set of real numbers greater than or equal to 0 by \mathbb{R}_{\geq} . We call the n -fold direct product of copies of \mathbb{R}_{\geq} the n -dimensional positive orthant, denoted by \mathbb{R}_{\geq}^n .

In order to understand the statement we shall begin by asking “What is a torus action?”

Definition 2.1.1. Let G be a Lie group. A G action on a smooth manifold, M , is an action of G on the set of points in M such that each element of G defines a diffeomorphism $g : M \rightarrow M$ and each point in M defines a smooth map $x : G \rightarrow M$.

A torus action is a group action where the group is the n -torus. The n -torus is a group since S^1 is a Lie group with multiplication $e^{i\pi\varphi} \times e^{i\pi\psi} = e^{i\pi(\varphi+\psi)}$ and T^n is the direct product of n copies of this.

While we are describing group actions it would be useful to define two more group theoretic concepts that we will be using later.

Definition 2.1.2. The *orbit* of a point $x \in M$ under a G -action is

$$Gx := \{gx \in M : g \in G\} \subset M.$$

The orbit space of the G -action on M is the space of all orbits and is denoted M/G .

Definition 2.1.3. The *isotropy subgroup* of a point $x \in M$ under a G -action is the closed subgroup

$$G_x := \{g \in G : gx = x\} < G.$$

In order to fully understand our original statement we must now answer the question “What do we mean by a well-behaved torus action?”

The first part of the answer is that we would like it to be *locally standard*. That is, we would like it to look locally like the standard action of T^n on \mathbb{C}^n which is given coordinate-wise

$$(\theta_1, \dots, \theta_n)(z_1, \dots, z_n) = (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n).$$

The orbit of the standard action of T^1 on \mathbb{C}^1 , considering polar coordinates on \mathbb{C}^1 , identifies any two points in \mathbb{C}^1 that have the same modulus and so the quotient space can then clearly be seen to be \mathbb{R}_{\geq}^1 . Since in the standard action of T^n on \mathbb{C}^n each coordinate of the torus acts independently, the quotient space \mathbb{C}^n/T^n can be seen to be \mathbb{R}_{\geq}^n .

What then can we say about the orbit spaces of general quasi-toric manifolds under these well behaved torus-actions? In [4] Bredon shows that the orbit space of M^{2n} under T^n is a smooth n -manifold with n -corners.

An *n -manifold with m -corners* is defined analogously to an n -dimensional manifold, M^n , where instead of the target of the charts being \mathbb{R}^n , it is $\mathbb{R}^{n-m} \times \mathbb{R}_{\geq}^m$. In particular a manifold with boundary is a manifold with 1-corner. The properties of smoothness and differentiability for manifolds with corners are defined analogously to those for normal manifolds.

Unfortunately, not all locally standard actions are sufficiently well-behaved for our purposes. Consequently, to properly answer the question of what a well-behaved torus action is, we shall place a further restriction on the orbit space in our definition of quasi-toric manifold.

Definition 2.1.4. A quasi-toric manifold is a smooth manifold, M^{2n} , with a locally standard action of T^n such that the quotient space M^{2n}/T^n is homeomorphic to an n -dimensional compact simple polytope, P^n .

This is not as large a restriction as it first appears, since if an n -manifold with n -corners is homeomorphic to a polytope, P , then P must be simple. This is because the any chart on the manifold containing the preimage a vertex of P must map that vertex to 0 and the surrounding facets into distinct coordinate planes. Thus there are exactly n facets meeting at each vertex and P is simple.

The condition of an n -manifold with n -corners being homeomorphic to a polytope can also be expressed as insisting that the manifold have the following properties of a polytope; contractibility, compactness and the intersection of any two distinct edges is at most one point.

We will now look at a couple of examples of torus actions on manifolds.

Example 2.1.5. The natural, coordinate wise action of T^n on \mathbb{C}^n as discussed above, is not a quasi-toric manifold, since its orbit space, \mathbb{R}_{\geq}^n , is not a compact polytope. It is however the local case of all quasi-toric manifolds, so will be known as the motivating example.

Example 2.1.6. The n -torus also acts coordinate wise on the first n homogeneous coordinates of $\mathbb{C}P^n$ by the action

$$(t_1, \dots, t_n) \times [x_1 : \dots : x_n : x_{n+1}] \mapsto [t_1 x_1 : \dots : t_n x_n : x_{n+1}].$$

In this case the quotient space is the n simplex.

Let us now look at more closely at the orbit space of a quasi-toric manifold under a well behaved torus-action, in particular which points are mapped to the various faces of the polytope. Examining our motivating example, we see that the preimage of $(0, \dots, 0)$ in \mathbb{R}_{\geq}^n is the single point $(0, \dots, 0)$ which is the only point to have the whole group as its isotropy subgroup. Furthermore if we examine the isotropy subgroups of two distinct points in the interiors of a particular face then we notice

that they are identical. This will hold for any quasi-toric manifold so hence forth we will refer to the isotropy subgroup of a face and mean the isotropy subgroup of any point within the interior of that face; we denote it by $T(F)$ for a face F .

We notice three important properties of the isotropy subgroups of faces.

Property 2.1. A face of codimension k has isotropy subgroup $T^k < T^n$ and so, in particular, a facet will have an isotropy subgroup isomorphic to T^1 .

Property 2.2. Since we are dealing with simple polytopes, any face F of codimension d can be written as the intersection of d facets F_{i_k} . Clearly $T(F_{i_k}) \subset T(F)$ for any i_k , but we also have that

$$T(F) \cong \times_{k=1}^d T(F_{i_k})$$

because the $T(F_{i_k})$ are transversal in T^n .

Property 2.3. Any copy of T^1 considered as a subgroup of T^n is defined by a line of rational slope, which can be written as a n -tuple of integers. An n -tuple, (x_1, \dots, x_n) , defines a map which embeds T^1 into T^n by

$$e^{i\theta} \mapsto (e^{ix_1\theta}, \dots, e^{ix_n\theta}).$$

What do these properties tell us? We see from 2.2 that the isotropy subgroup of any face is specified if we know the isotropy subgroup of the facets. Then from 2.1 we have that the isotropy subgroups of the facets are circles and by 2.3 we know that these can each be specified by an n -tuple of integers.

We have one n -tuple for each facet, so given a numbering of the facets we can combine these vectors as the columns of an $n \times m$ matrix which we call Λ . This matrix defines a function $\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^n$ which we call the *characteristic function*. The i -th column of this matrix specifies which subtorus is the isotropy subgroup of the i -th facet. Furthermore, we can say that the square matrix Λ_v given by Λ restricted to the facets which meet at any particular vertex, v , must have determinant ± 1 , since the isotropy subgroup of any vertex is all of T^n . Alternatively this condition means the vectors associated to the facets which meet at any vertex must form a basis of \mathbb{Z}^n . Thus a quasi-toric manifold defines a pair (P, Λ) , called the *combinatorial data*

of the manifold, consisting of a combinatorial polytope with ordered facets and a characteristic function.

Let us now look at example of the combinatorial data from a particular quasi-toric manifold.

Example 2.1.7. From example 2.1.6 we know that $\mathbb{C}P^2$ is a quasi-toric manifold and that the orbit space is the 2 simplex, Δ . There are three facets of Δ which we will denote F_1 , F_2 and F_3

The action of T^2 on $\mathbb{C}P^2$ is given by

$$(t_1, t_2) \times [x_1 : x_2 : x_3] \mapsto [t_1x_1 : t_2x_2 : x_3].$$

This action fixes three points, $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$, these points are mapped under the orbit map to the vertices of Δ . There are three 1 dimensional subtori which fix the points of $\mathbb{C}P^2$:

1. The subtorus given by $(t, 1) \in T^2$ for $t \in S^1$ fixes the points $[0 : x_2 : x_3] \in \mathbb{C}P^2$.
2. The subtorus given by $(1, t) \in T^2$ for $t \in S^1$ fixes the points $[x_1 : 0 : x_3] \in \mathbb{C}P^2$.
3. The subtorus given by $(t, t) \in T^2$ for $t \in S^1$ fixes the points $[x_1 : x_2 : 0] \in \mathbb{C}P^2$.

Without loss of generality we can say that under the orbit map points of the form $[0 : x_2 : x_3]$ are mapped to F_1 , points of the form $[x_1 : 0 : x_3]$ are mapped to F_2 and points of the form $[x_1 : x_2 : 0]$ are mapped to F_3 .

The subtori that fix the points mapped to each facet are determined by a pair of integers. The pair associated to F_1 is $(1, 0)$, because the subtorus that fixes it is given by pairs (t^1, t^0) . Similarly the pair associated to F_2 is $(0, 1)$ and the pair associated to F_3 is $(-1, -1)$.

This means the characteristic function of the quasi-toric manifold $\mathbb{C}P^2$ is given by the matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

and the combinatorial data for $\mathbb{C}P^2$ is (Δ, Λ) .

So far we have seen how we can find the combinatorial data from a quasi-toric manifold. We will now look at how we can construct a quasi-toric manifold from a set of combinatorial data. There are three ways of doing this. One way to construct a manifold from a pair is as a quotient space of $T^n \times P$ as done by Davis and Januszkiewicz in their paper [13]. Alternatively, we can follow the method of Buchstaber, Panov and Ray in [10] and use a construction as a pullback. This construction has the advantage that it is easier to see the result is a smooth manifold. The final option is to use the more explicit construction given by Buchstaber and Panov in [9]. In order to demonstrate the construction of a quasi-toric manifold from combinatorial data, we will take the final option and overview the explicit construction of the manifolds known as moment angle manifolds.

We will begin with a set of combinatorial data, (P, Λ) . We will first look at the boundary of the dual polytope; $K = \partial P^*$. Since P is a simple polytope we have that K is a simplicial sphere, that is a simplicial complex which is homeomorphic to a sphere. We know that K has m vertices, v_1, \dots, v_m , where m is the number of facets of P . For any simplex $\omega \in K$ we define a subset of the m -fold product of the unit disc $D^2 \subset \mathbb{C}$, $B_\sigma = \{(z_1, \dots, z_m) \in (D^2)^m : |z_i| = 1 \ \forall v_i \notin \sigma\}$, using this we can define the *moment angle complex* of K to be

$$Z_K = \bigcup_{\sigma \in K} B_\sigma \subset (D^2)^m.$$

There is a natural action of T^m on $Z_K \subset (D^2)^m$ that is induced from the coordinate wise action of T^m on $(D^2)^m \subset \mathbb{C}^m$. Furthermore we have that the orbit space Z_K/T^m is homeomorphic to P . This construction can be made for any simplicial complex, but when K is a simplicial sphere the moment angle complex is a smooth manifold of dimension $m + n$. In this case we call the moment angle complex a *moment angle manifold*. When K is the boundary of the dual of a simple polytope P we will refer to this manifold as Z_P rather than $Z_{\partial P^*}$ for notational convenience.

So from a set of combinatorial data, (P, Λ) , we have constructed Z_P , an $(m + n)$ -dimensional manifold with an action of T^m . We have only used P so far and ignored the characteristic function. The characteristic function is a map $\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^n$ with

kernel isomorphic to \mathbb{R}^{m-n} . We can combine this with the quotient map of \mathbb{R}^i by its integer lattice, $L_i : \mathbb{R}^i \rightarrow T^i$, to get:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R}^{m-n} & \hookrightarrow & \mathbb{R}^m & \xrightarrow{\lambda} & \mathbb{R}^n & \longrightarrow & 0 \\ & & \downarrow L_{m-n} & & \downarrow L_m & & \downarrow L_n & & \\ 0 & \longrightarrow & T^{m-n} & \hookrightarrow & T^m & \xrightarrow{\lambda_T} & T^n & \longrightarrow & 0 \end{array}$$

We will pay particular attention to the $m - n$ -dimensional torus that is the kernel of λ_T , we shall call this torus S . We have from section 3 of [10] that S acts freely on Z_P .

We can now look at the orbit space of Z_P under the action of S , which we shall call $M_{(P,\Lambda)}$. Since Z_P is a $(m + n)$ -dimensional manifold and S is a $(m - n)$ -dimensional Lie group, we have that $M_{(P,\Lambda)}$ is a $2n$ -dimensional manifold. Furthermore, $M_{(P,\Lambda)}$ has an induced action of T^n as the quotient of T^m by S . Since S acts freely on Z_P the orbit space $M_{(P,\Lambda)}/T^n$ is homeomorphic to the orbit space $Z_P/T^m \cong P$. Since P is a simple polytope, the action of T^n on $M_{(P,\Lambda)}$ is locally standard and we have constructed a quasi-toric manifold with combinatorial data (P, Λ) .

So now we have one construction that takes a quasi-toric manifold and produces some combinatorial data, and another which takes a set of combinatorial data, constructs a moment angle manifold and then turns the moment angle manifold into to a quasi-toric manifold. This begs the question of whether if we apply these one after the other we get the same quasi-toric manifold? In [9] it is proved that the original and final quasi-toric manifolds are the same up to the equivalence relation given in [13]:

Definition 2.1.8. Two quasi-toric manifolds are said to be equivalent if they are weakly equivariantly homeomorphic.

In [10] it is proved that the combinatorial data completely classifies quasi-toric manifolds up to an equivalence relation on the combinatorial data also given in [13]:

Definition 2.1.9. Two sets of combinatorial data, (P, Λ) and (P', Λ') are said to be equivalent if $P = P'$ (as combinatorial polytopes) and the maps $\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^n$

and $\lambda' : \mathbb{R}^m \rightarrow \mathbb{R}^n$ have isomorphic kernels after factoring out by the integral lattice \mathbb{Z}^{m-n} .

We will now have a look at some specific cases that are covered by these equivalence relations and some of the reasoning behind the definitions.

The construction of the moment angle complex depends on the combinatorial type of the polytope, so combinatorial data can only be equivalent if the polytopes are combinatorially equivalent.

The quasi-toric manifold is obtained from the moment angle complex by factoring out the action of a freely acting subtorus which is found as the kernel of the map given by Λ . In particular two matrices Λ and Λ' should be equivalent if they differ only by a change of basis of \mathbb{R}^n .

Two maps $\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $\lambda' : \mathbb{R}^m \rightarrow \mathbb{R}^n$ that differ only by the order of the columns have isomorphic kernels after factoring out by the integral lattice \mathbb{Z}^{m-n} . This means that the choice of numbering of the facets of the polytope does not affect the equivalence class of the combinatorial data. In particular, we can renumber the facets should we need to.

The weak equivariant homeomorphism type of a quasi-toric manifold only depends on the image of T^1 that is the isotropy subgroup of each facet. Let $A \in \mathbb{Z}^n$ be the n -tuple representing the isotropy subgroup of some facet, F , then we can see that $-A$ represents the same isotropy subgroup. In fact A and $-A$ are the only n -tuples which can represent this subtorus as a characteristic function, since Λ_v the minor relating to a vertex v must have determinant ± 1 . We will say that one of A or $-A$ is *inward pointing* and the other *outward pointing*. This means that we can replace any column of Λ by its negative without changing the equivalence class of the combinatorial data.

However, whether the n -tuples that make up a characteristic function are inward or outward pointing does give more structure on the quasi-toric manifold than just its weak equivariant homeomorphism type. We refer to this extra information as an *omniorientation*, which is an orientation of P and a choice of matrix Λ . This choice of Λ specifies whether its columns are inward or outward pointing. An orientation

of a combinatorial polytope is an orientation of a specified representative of the equivalence class.

Omniorientations are important because we know from [10, 4.5] that any omniorientated quasi-toric manifold admits a canonical stably complex structure which is invariant under the torus action. Since our combinatorial data is the pair of the polytope, P , and matrix, Λ , it specifies an omnioriented quasi-toric manifold.

An omniorientation is specified by an orientation of the polytope and a choice of inwards or outwards for each column of Λ . These can be used to define the sign of a vertex, which will be useful later in the chapter.

Definition 2.1.10. Let (P, Λ) be the combinatorial data of a quasi-toric manifold and let A be the $n \times m$ matrix with rows given by the a_i which define P as

$$P = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle + 1 \geq 0, \ i = 1, \dots, m\}$$

with the a_i numbered in the same way as the facets they represent. A vertex v of P can be expressed as the intersection of n facets. Let A_v and Λ_v be the minors of A and Λ corresponding to these facets. Then the *sign* of v is

$$\text{Sign}(v) = \det A_v \cdot \det \Lambda_v.$$

One final property of quasi-toric manifolds that we shall look at in this section is the existence of *facial submanifolds*. We take M^{2n} to be a quasi-toric manifold with an action of T^n , combinatorial data (P, Λ) and quotient map $q : M \rightarrow P$. We then take F to be a facet of P with isotropy subgroup $S \cong T(F)$. We then have from [13] that $M_F = q^{-1}(F)$ is a $2(n-1)$ dimensional manifold with an action of $T^{n-1} \cong T^n/S$ and orbit space F , so M_F is a quasi-toric manifold whose characteristic function can be calculated from Λ .

When two facial submanifolds, M_F and M_G , meet, their normal bundles in M are transversal and so their intersection is a manifold of dimension $2(n-2)$. Furthermore we know from [13] that this manifold is the preimage of the face $F \cap G$ under the quotient map q . This property means we can define the *face submanifold* of any face of P by the same method as we defined the facial submanifolds for the facets of P .

So, vertex T on the toric tetrahedron concerns quasi-toric manifolds; what of the other three vertices? Two of them, A and S , are areas of mathematics which feature constructions that are sometimes quasi-toric manifolds. The remaining vertex, C , is involved because quasi-toric manifolds are defined by their combinatorial data. We will leave that vertex until last.

2.2 Algebraic Geometry

In this section we will look at some algebraic geometry. This is an area of mathematics which provides many important examples of quasi-toric manifolds. The type of examples we will see here are toric manifolds, which are a special case of a toric variety. As we shall see, they are only quasi-toric manifolds in certain circumstances.

Toric varieties appear as the compactification of the action of $(\mathbb{C}^*)^n$ on itself along a general orbit. More formally, we define a toric variety as follows:

Definition 2.2.1. [9] and [18]. A *toric variety* is a normal variety, M , which contains an algebraic torus, $\mathbb{T}^n \cong (\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$, as a Zariski open set in such a way that M has an action of \mathbb{T}^n which extends the natural action of \mathbb{T}^n on itself .

Note that the algebraic torus, \mathbb{T}^n , is different from the torus that has appeared before, which was $T^n = (S^1)^n$. It is for this reason that we call it the algebraic torus.

Before we can look at when these toric varieties are also quasi-toric manifolds we will need some preliminaries.

Definition 2.2.2. [9]. A *polyhedral cone* spanned by a finite set of vectors $l_1, \dots, l_s \in \mathbb{R}^n$ is the set

$$\sigma = \{r_1 l_1 + \dots + r_s l_s \in \mathbb{R}^n \mid r_i \geq 0\}.$$

A polyhedral cone is *rational* if the generating vectors can be taken as vectors in \mathbb{Z}^n and is *strongly convex* if it does not contain a line through the origin. Strongly convex, rational polyhedral cones will be referred to as cones. Cones have faces and facets since they are convex polyhedra as they are the intersection of a finite number

of half spaces bounded by supporting hyperplanes, but they are not compact. The faces are the intersection of the cone with some of these hyperplanes.

A cone is *simplicial* if it is generated by part of a basis for \mathbb{R}^n and is *non-singular* if in addition it is generated by part of a basis for \mathbb{Z}^n .

Definition 2.2.3. [9] A *fan* is a set of cones, Σ , of cones in \mathbb{R}^n such that

- every face of a cone in Σ is in Σ ,
- the intersection of any two cones in Σ is a face of both.

A fan is *complete* if the union of all the cones in it is \mathbb{R}^n . It is *simplicial* or *non-singular* if all the cones in it are respectively simplicial or non-singular.

Fans are useful objects to study in conjunction with toric varieties because of the *fundamental varietal correspondence* between toric varieties and fans. We will not go into details of this correspondence here, other than to note its existence. Details can be found in many places, notably [18].

We can use the fundamental varietal correspondence to look at the properties of the varieties that correspond to those we have already seen for fans. If a fan is complete then the corresponding variety is compact. Simplicial fans correspond to orbifolds. Finally a variety corresponding to a non-singular fan is a smooth manifold; in this case the toric variety is known as a *toric manifold*.

We would like to know when toric manifolds are also quasi-toric manifolds. Toric manifolds are generated by fans and quasi-toric manifolds are generated by simple combinatorial polytopes. However we can move between the underlying combinatorics of these two settings. If we start with a simple combinatorial polytope, P^n , we can choose a representative for P^* which has vertices in \mathbb{Z}^n . We can then define a vector, l_i , as the inward pointing normal vectors of the facet, F_i , for $i = 1, \dots, m$. It is known from [9] that these vectors can be chosen as integer and primitive. We can then define $\Sigma(P)$ the *normal fan* of P to be the fan defined on cones given by those sets of vectors which correspond to sets of facets with non empty intersections. We call a fan *strongly polytopal* if it is realisable as the normal fan of a simple polytope.

We can construct a simple polytope from a fan as the dual of the underlying simplicial complex provided that this is a polytopal sphere, i.e. combinatorially equivalent to the boundary of a simplicial polytope. We will call such a fan *weakly polytopal*. The *underlying simplicial complex* of a fan is the simplicial complex with vertex set given by the end points of all the vectors which define cones in the fan, and simplices for a given set of vertices if and only if there is a cone defined by those vectors.

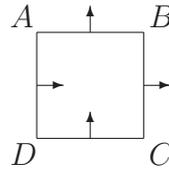
Let Σ be a fan and M_Σ the corresponding variety. Since M_Σ is a toric variety it has an action of \mathbb{T}^n and thus has an action of $T^n \subset \mathbb{T}^n$. Furthermore, if Σ is non-singular then M_Σ is a manifold. So when is M_Σ a quasi-toric manifold as well as a toric manifold? It turns out that when Σ is weakly polytopal, with P being the simple polytope dual to the underlying simplicial complex, then M_Σ is a quasi-toric manifold and has combinatorial data (P, Λ) where the columns of Λ are given by the vectors l_i which define Σ . When in addition Σ is strongly polytopal then M_Σ is also projective, i.e. a variety that can be embedded as a complex submanifold of $\mathbb{C}P^N$ for some N . Also we have from [9] that any projective toric manifold is a quasi-toric manifold.

It is important to note that not all quasi-toric manifolds are toric manifolds. The action of \mathbb{T}^n restricts naturally to an action of T^n , but an action of T^n may not extend to an action of \mathbb{T}^n . This is demonstrated well by the following observation.

An important fact about toric manifolds is that the sign of each vertex will be $+1$, because the calculation of sign uses the matrices A and Λ and we have $A^T = \Lambda$, so $\text{Sign}(v) = (\det \Lambda_v)^2$. Thus any quasi-toric manifold which does not have the sign of each vertex being $+1$ cannot be a toric manifold. For example the pair (P_4^2, Λ) where P_4^2 is the 2 dimensional polytope with four facets (the square) and

$$\Lambda = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

which gives the labeling of the facets as



which is not a toric manifold as the sign of vertices A , B and D are -1 , while the sign of vertex C is 1 .

2.3 Symplectic Geometry

In this section we will look at symplectic geometry. Symplectic geometry provided examples in Hamiltonian T^n -manifolds which inspired toric topology. The study of these manifolds gives rise to a type of polytopes that will be important later, Delzant polytopes. Symplectic geometry concerns the geometry of symplectic manifolds.

Definition 2.3.1. [1] A symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a closed non-degenerate 2-form.

What then is a closed non-degenerate 2-form? A 2-form can be written as, for some choice of coordinates x_i ,

$$\omega(x_1, x_2) = \sum_{i,j=1}^2 f_{i,j} dx_i \wedge dx_j$$

with $dx_i \wedge dx_j = -dx_j \wedge dx_i$ and some functions $f_{i,j}$. It is closed when its exterior derivative is identically 0 and non-degenerate if it is not 0 itself.

What this means in practice is that a symplectic manifold is a manifold together with a function that gives each tangent space the structure of a symplectic vector space in such a way that the structure at a point x varies smoothly with x . A *symplectic vector space* is a vector space with ω a non-degenerate skew-symmetric bilinear form. A simple result in [1] shows us that a symplectic vector space must be even dimensional.

The existence of such a 2-form on M implies that M must be of even dimension, because its tangent space at any point is even dimensional. An automorphism

of a symplectic manifold which preserves the 2-form will be known as a *symplectomorphism*. A group action is *symplectic* if each element of the group defines a symplectomorphism.

A natural question to ask is when symplectic manifolds are also quasi-toric manifolds. To be a quasi-toric manifold a manifold must have a torus of appropriate dimension acting on it. Naturally on a symplectic manifold that action should preserve the symplectic structure, i.e., be one where every element of the group defines a symplectomorphism.

An important class of symplectic manifold are manifolds with Hamiltonian actions of some lie group G . Since we are interested in symplectic manifolds that are also quasi-toric manifolds, we will only look at symplectic manifolds with Hamiltonian torus actions. Hamiltonian actions preserve the symplectic structure of the manifold.

Definition 2.3.2. [6] A *Hamiltonian T^n -manifold* is a triple (M^{2n}, ω, μ) consisting of a symplectic manifold (M, ω) and a map $\mu : M \rightarrow (\mathbb{R}^n)^*$ called *the momentum map*.

The momentum map is from the manifold to the dual of the Lie algebra of T^n and defines the Hamiltonian action. It is constant on the orbits of T^n , is locally modelled on the standard action of T^n on \mathbb{C}^n and identifies M^{2n}/T^n with the polytope P^n [13]. It should also be noted that the toric manifolds of section 2.2 are symplectic manifolds with Hamiltonian torus action, with symplectic structure given by the Kähler form; for details see [1].

The natural question to ask at this point is when are quasi-toric manifolds also symplectic manifolds with Hamiltonian actions? The answer to this question actually predates the definition of a quasi-toric manifold and was provided by Delzant in [14].

Definition 2.3.3. [14] A polytope $P^n \subset \mathbb{R}^n$ is called *Delzant* if and only if at each vertex the normal vectors through each facet which meets the vertex form a \mathbb{Z} -basis for \mathbb{Z}^n

It is important to note that this condition applies to individual polytopes and not to the combinatorial polytopes used in the construction of quasi-toric manifolds. This

condition is expressed in the following equivalent way in [12]: “A Delzant polytope in \mathbb{R}^n is a convex polytope which is simple, rational and smooth. A rational polytope is on where the edges meeting at a vertex, v , are of the form $v + tu_{v,i}$ for $t \geq 0$ and some $u_{v,i} \in \mathbb{Z}^n$. Such a polytope is smooth if for each vertex, v , we have that $u_{v,1}, \dots, u_{v,n}$ can be chosen to be a \mathbb{Z} -basis of \mathbb{Z}^n .”

Theorem 2.3.4. [14] There is a bijective correspondence between Delzant polytopes and Hamiltonian toric manifolds.

While we will not include the proof of this here, we shall note that a Delzant polytope has canonically specified characteristic function given by the outward pointing normal vectors of each facet. Furthermore, we know from [13] that the construction used in the proof of this gives a manifold which is equivalent to the quasi-toric manifold constructed in the normal way from the polytope with this combinatorial data. Since the Hamiltonian toric manifold given by the correspondence is a symplectic manifold this gives us a canonical symplectic structure on any quasi-toric manifold with combinatorial data (P, Λ) , such that P has a canonical realisation, P_D that is Delzant and (P, Λ) is equivalent to the canonical combinatorial data for P_D .

2.4 Combinatorics

We have seen three of our “vertex disciplines” so far. Toric topology gave us quasi-toric manifolds and algebraic geometry and symplectic geometry showed us some examples of these manifolds. The remaining “vertex discipline” is combinatorics, which is slightly different. Combinatorics is part of the “toric tetrahedron” because we have seen that quasi-toric manifolds are defined by their combinatorial data, which consists of a polytope and a matrix.

In this section we will look at how combinatorics links with with toric topology, as well as motivating and posing some questions that will form the basis of the work on polytopes we will undertake in the rest of this thesis.

We saw in section 2.1 that the combinatorial data required for a quasi-toric manifold is not just the combinatorial polytope, but also a matrix defining a characteristic

function. This matrix is required to fulfil the condition that the columns representing each facet must form a basis of \mathbb{Z}^n when they meet at a vertex. Such a characteristic function may not exist for certain polytopes, as in this example.

Example 2.4.1. [9, 1.17 and 5.26] Let $x : \mathbb{R} \rightarrow \mathbb{R}^n$ be a curve in \mathbb{R}^n given by

$$t \mapsto x(t) = (t, t^2, \dots, t^n).$$

We call this curve the *moment curve* and we define the *cyclic polytope* $C^n(t_1, \dots, t_m)$ to be the convex hull of the points $x(t_1), \dots, x(t_m)$. Since no more than n points in this list can be coplanar, by the Vandemonde determinant, this is a simplicial polytope.

Being simplicial, its dual is a simple polytope. For $n \geq 4$ and $m \geq 2^n$ this simple polytope cannot admit any characteristic function. This is because any two facets of this polytope have non empty intersection, so every pair of columns in Λ must be extendable to a basis of \mathbb{Z}^n . However, since $m \geq 2^n$ and no columns are 0, there must be a pair of columns which coincide modulo 2. These columns are not extendable to a basis of \mathbb{Z}^n .

We do know however that a characteristic function must exist for any three dimensional polytope as this was proved by Davis and Januszkiewicz in [13] using the four colour theorem.

When a characteristic function, and so a quasi-toric manifold, does exist for a certain polytope we are able to discover information about the manifold directly from the polytope. For a quasi-toric manifold M with data (P, Λ) , we have that

$$\dim H^{2i}(M^{2n}, \mathbb{Z}) = h_i$$

where h_i is the i -th coefficient of the h -polynomial of the polytope (see definition 3.4.3). We also know, from theorem 3.2.11, that the invariants g_i , which are defined as

$$g_i = h_{i+1} - h_i,$$

are positive multiples of another set of invariants γ_i . If we could calculate these γ_i for a given polytope, or at least show that they are positive as suggested in conjecture 3.2.14, then we would know that in this case $\dim H^{2i}(M^{2n}, \mathbb{Z})$ is an increasing sequence which gives us strong information about the topology of M^{2n} .

For these reasons it is desirable to be able to calculate the combinatorial invariants of polytopes exactly, if this is possible.

As we have seen we can use our knowledge of polytopes to learn about the quasi-toric manifolds, similarly we will now see that we can use our knowledge of manifolds to learn about polytopes. The g -theorem, 3.2.13, which gives conditions on a vector with integer values for it to be the f -vector of simple polytope, was proved using arguments on toric varieties the algebraic geometry counterpart of quasi-toric manifolds in [26]. This result was one of the first major break throughs in toric topology. Later advances include, the Buchstaber invariant of a polytope, which is the maximal dimension of freely acting subtorus on the moment angle complex, has emerged from toric topology. It is a measure of how far a polytope is from being the quotient space of a quasi-toric manifold. The Betti numbers of a polytope, which are an algebraic invariant of the Stanley-Reisner ring can be identified with invariants of the cohomology ring of the moment angle complex. This has lead to the defining of a refined version of the invariant; bigraded Betti numbers.

As we have seen so far, that it is desirable to study polytopes which admit characteristic functions. One of the major problems with doing this is finding such polytopes.

A combinatorial polytope is defined by any of its representatives, which in turn are defined by finite collections of points or half spaces. Given a particular collection of points or half-spaces it is not obvious to tell if such a polytope is simple, simplicial or neither and these properties are some of the easier ones to determine. For a more complicated property, such as Delzant polytopes, finding examples becomes distinctly non-trivial. As such, polytopes are normally constructed indirectly and we will be looking at a particular construction in later chapters.

Next we will highlight a difference in the approaches of toric topology and polytope theory, one that we will look at later in this thesis. In all our discussions of polytopes above we have referred to individual polytopes, their individual invariants and characteristic functions. By way of contrast we will look at three sets of examples of quasi-toric manifolds.

Example 2.4.2 (Motivating Example). Our first example here is the motivating example we have seen before. The natural, coordinate wise action of T^n on \mathbb{C}^n over \mathbb{R}_{\geq}^n , is the local case of all quasi-toric manifolds.

We can consider the face submanifolds of the motivating example and see that for a face $\mathbb{R}_{\geq}^k \subset \mathbb{R}_{\geq}^n$ has facial submanifold which is $\mathbb{C}^k \subset \mathbb{C}^n$.

Example 2.4.3 (Complex Projective Space). The second example we will consider here is complex projective n -space. The action on this space is coordinate wise on the homogeneous coordinates of $\mathbb{C}P^n$. The quotient space is the n -simplex.

Again considering the face submanifolds we see that any face of Δ^n is a k -simplex and that the action restricts to the face submanifold in such a way that the face submanifold is $\mathbb{C}P^k \subset \mathbb{C}P^n$.

Example 2.4.4 (Bott Towers). The third example we will look at here are called Bott towers, which can be found in [21]. A Bott tower is a family of compact complex manifolds M_1, M_2, \dots, M_n , in which $M_1 = \mathbb{C}P^1$ and each successive stage M_i is obtainable as the projectivisation of the sum of the trivial complex line bundle and an arbitrary complex line bundle L_i over the previous stage M_{i-1} , together with iterated fibre maps and distinguished sections. Given a complex manifold M_n with appropriate fibration and section structure, we can deduce the intermediate manifolds M_2, \dots, M_{n-1} that make up a Bott tower. Thus, we shall refer to M_n as an *n -step Bott tower*

For example,

$$\underbrace{\mathbb{C}P^1 \times \dots \times \mathbb{C}P^1}_{n \text{ times}}$$

the n -fold product of $\mathbb{C}P^1$ is an n -stage Bott tower.

A construction of a general Bott tower is given in [21, §2.2]. Following this rather technical construction, we get that an n -step Bott tower is given by $n(n-1)/2$ integers, $\{c_{ij}\}_{1 \leq i < j \leq n}$, and

$$M_n = (\mathbb{C}^2 \setminus 0)^n / G$$

with $G = (\mathbb{C}^\times)^n$ and action of its i -th factor defined by

$$[z_1, w_1, \dots, z_n, w_n]a_i = [z_1, w_1, \dots, z_i a_i, w_i a_i, \dots, z_j, a_i^{c_{ij}} w_j, \dots].$$

Bott towers are quasi-toric manifolds. An n -step Bott tower has a standard action of T^n ([21, §2.4]) given by

$$(\lambda_1, \dots, \lambda_n)[z_1, w_1, \dots, z_n, w_n] = [z_1, \lambda_1 w_1, \dots, z_n, \lambda_n w_n].$$

The quotient space of any n -step Bott tower is the n cube, I^n . The integers c_{ij} give rise to the different characteristic functions for different Bott towers.

In [22] it is proved that any quasi-toric manifold over a cube is a Bott tower. Since the facets of an n cube are $(n-1)$ cubes the face manifolds of a Bott tower are also Bott towers.

These examples all occur in families, spanning multiple dimensions but all the manifolds have very similar properties and definitions. It would be desirable if we could find a way to study polytopes in a similar fashion.

We have seen what toric topology is and also some of the problems involved. Many of these problems involve finding suitable polytopes to look at quasi-toric manifolds over. We now state these problems as four questions which we will attempt to find solutions for in the remainder of this thesis.

Question 1. Can we find families of polytopes that can underlie quasi-toric manifolds with similar properties?

Question 2. What machinery can we develop to study the combinatorics of such families?

Question 3. Can we construct polytopes which have canonical combinatorial data?

Question 4. Can we calculate the combinatorial invariants of these polytopes?

As we have seen in this chapter, Question 3 can be answered if we can find a construction of Delzant polytopes. If this construction forms families which answer question 1 we could then use the solution of question 2 to answer question 4. This leads to a final question:

Question 5. Can we answer all these questions with one type of polytope?

The answer is yes, and we shall see how and why in the following chapters.

Chapter 3

Polytope Theory

We have posed in the last chapter some questions motivated by toric topology which require answers in the language of polytopes. In this chapter we will introduce the main concepts and methods of polytope theory that will allow us to provide solutions to these questions.

3.1 Polytopes

The study of polytopes, also called polyhedra, is one of the oldest areas of mathematics. It is one that is easily traced back to the ancient Greeks, in particular Plato. In this section we will overview the basics of polytope theory.

Let us begin with the basic definitions, for which we shall follow the logical structure of [9]. There are two ways of defining a polytope.

Definition 3.1.1. A *convex polytope* is the convex hull of a finite number of points in some \mathbb{R}^n .

Definition 3.1.2. A *convex polytope* is an intersection of a finite number of half-spaces in some \mathbb{R}^n .

Provided that the polytope given by the second definition is bounded, these definitions are equivalent as is proved in this theorem from Ziegler's book. We will therefore insist that our polytopes are bounded from now onwards.

Theorem 3.1.3. [30, pp. 29 – 39] A bounded intersection of a finite number of half spaces in \mathbb{R}^n is the convex hull of a finite number of points in the same \mathbb{R}^n . Conversely, the convex hull of a finite number of points in \mathbb{R}^n is the intersection of a finite number of half spaces and is bounded.

Definition 3.1.4. The *dimension* of a polytope is the dimension of its affine hull.

We will normally assume that any n -dimensional polytope is embedded in \mathbb{R}^n , particularly for the next set of definitions. In the definitions given here, a polytope P^n will be of dimension n .

Definition 3.1.5. The *boundary* of a polytope P^n is the boundary of the polytope as a subspace of \mathbb{R}^n .

One of the nice properties of polytopes is that the boundary of any polytope is the union of lower dimensional polytopes. To prove this we will have to define what these polytopes are. We begin with the definition of a polytope as the intersection of a finite number of half-planes. A *supporting hyperplane* is a hyperplane which defines one of the half-spaces.

Definition 3.1.6. A *face* of a polytope is the intersection of the polytope and some collection of supporting hyperplanes.

The polytope itself meets this definition, as does the empty set. We call all the other faces *proper faces*. It is easy to see that the boundary of a polytope is the union of all its proper faces.

We will have that the boundary of any polytope is the union of lower dimensional polytopes if any face is also a polytope. This is true and we again refer the reader to Ziegler's book.

Lemma 3.1.7. [30, 2.3] A face of a polytope is a polytope in its own right. Furthermore if F is a face of P then the faces of F are the faces of P that are contained in F .

We will give some of these faces particular names as they are the ones we will refer to often. A dimension 0 face is called a *vertex*, a dimension 1 face is called an *edge* and a codimension 1 face is called a *facet*.

The faces of a polytope can be ordered by inclusion to give a partially ordered set, which we will call the *face poset*. The face poset of a polytope defines an equivalence relation on the set of all polytopes, *combinatorial equivalence*, and we will only study polytopes up to this relation.

Definition 3.1.8. Two polytopes are *combinatorially equivalent* if there is a bijection between their face posets.

A equivalence class of polytopes under this relation is called a *combinatorial polytope*. We can now distinguish certain classes of combinatorial polytopes which have some nice properties.

Definition 3.1.9. A *simplicial polytope* is a polytope in which every proper face is a simplex of appropriate dimension.

Definition 3.1.10. A *simple polytope* is a polytope, P^n , where exactly n facets meet at each vertex.

It is worth noting at this stage that faces of simple polytopes are also simple. These are key definitions so I shall illustrate them with regards to the classical examples; the Platonic solids in \mathbb{R}^3 .

Example 3.1.11 (Tetrahedron). The tetrahedron, or 3-dimensional simplex, is the convex hull of four points in general position. It has four triangular faces, with three meeting at each vertex. It is both a simplicial polytope and a simple polytope.

Example 3.1.12 (Cube). The cube, I^3 , is the direct product of three unit intervals. It has six square faces with three meeting at each vertex. It is a simple polytope only.

Example 3.1.13 (Octahedron). The octahedron, sometimes known as the 3 dimensional cross polytope, has eight triangular faces, with four meeting at each vertex. Thus it is a simplicial polytope only.

Example 3.1.14 (Dodecahedron). The dodecahedron has twelve pentagonal faces, with three meeting at each vertex, making it a simple polytope but not a simplicial polytope.

Example 3.1.15 (Icosahedron). The icosahedron has twenty triangular faces with five meeting at each vertex, which means it is a simplicial polytope but not a simple one.

The simplex is the only polytope which is both simplicial and simple. There is an intrinsic relationship between these two properties; one is the dual of the other.

Definition 3.1.16. [9, 1.10] The *dual* of a polytope, P^n , is

$$P^* = \{x' \in \mathbb{R}^n \mid \langle x', x \rangle \geq -1 \quad \forall x \in P\}.$$

Lemma 3.1.17. For a polytope P , the dual P^* is a polytope called the *dual* or *polar* polytope.

Proof. Certainly the dual of a polytope is defined as the intersection of infinitely many half spaces in \mathbb{R}^n . If we can prove that only finitely many of these are necessary then we will have shown that the dual of a polytope is itself a polytope, using definition 3.1.2 of a polytope.

Using the definition of a polytope as the convex hull of a finite set of points, we have that any point, x , in P can be expressed as

$$x = t_1.v_1 + \dots + t_m.v_m$$

where the v_i are the vertices of P and the t_i are positive real numbers with $\sum_{i=1}^m t_i = 1$.

Then we have that, for any x' ,

$$\langle x', x \rangle = \sum_{i=1}^m \langle x', t_i.v_i \rangle = \sum_{i=1}^m t_i \langle x', v_i \rangle. \quad (3.1)$$

Let us compare P^* with a new polytope

$$P^\Delta = \{x' \in (\mathbb{R}^n)^* \mid \langle x', v_i \rangle \geq -1 \quad \forall i\}.$$

Clearly we have that $P^* \subset P^\Delta$ since if $\langle x', x \rangle \geq -1 \forall x \in P$ then $\langle x', v_i \rangle \geq -1 \forall i$ as $v_i \in P$. It remains to show that $P^\Delta \subset P^*$. Let $x^\Delta \in P^\Delta$ and we have that $\langle x^\Delta, v_i \rangle \geq -1$ for all i . We now examine $\langle x^\Delta, x \rangle$ for $x \in P$. We have by equation 3.1 that

$$\begin{aligned} \langle x^\Delta, x \rangle &= \sum_{i=1}^m t_i \langle x^\Delta, v_i \rangle \\ &\geq \sum_{i=1}^m t_i (-1) \\ &= - \sum_{i=1}^m t_i \end{aligned}$$

since each $\langle x^\Delta, v_i \rangle \geq -1$. We know that $\sum_{i=1}^m t_i = 1$ and so $x^\Delta \in P^*$.

We have that $P^\Delta \subset P^*$ and $P^* \subset P^\Delta$, so they are equal. Since P^Δ is a polytope, the dual of a polytope is a polytope. \square

This duality gives rise to a third type of polytope that we will distinguish by name, flag polytopes, along with simple polytopes and simplicial polytopes. Flag polytopes are inspired by noticing that the boundary of a simplicial polytope is a simplicial complex.

Definition 3.1.18. [9] A simplicial complex, K , is *flag* if for any set of vertices $\{v_1, \dots, v_l\}$ such that the edge between v_i and v_j is in K for all $1 \leq i < j \leq l$, we have that the simplex with vertices v_1, \dots, v_l is in K .

We can use this property of simplicial complexes to define a property of polytopes in the following ways.

Definition 3.1.19. [9] A simplicial polytope is *flag* if its boundary is flag.

Definition 3.1.20. [9] A simple polytope is *flag simple* if its dual is a flag polytope.

Finally in this section we should look at the direct product of two combinatorial polytopes.

Definition 3.1.21. The *direct product* of two combinatorial polytopes, P^n and Q^m , is the combinatorial equivalence class of direct product of a representative of each

polytope as subsets of \mathbb{R}^n and \mathbb{R}^m , which is a subset of \mathbb{R}^{n+m} . We will write the direct product of P and Q as $P \times Q$.

Proposition 3.1.22. The direct product of two combinatorial polytopes is well defined. Furthermore for any two combinatorial polytopes P^n and Q^m , we have that $P \times Q$ is a $n + m$ -dimensional polytope and that the facets of $P \times Q$ are precisely the products $F_P \times Q$ for F_P a facet of P and the products $P \times F_Q$ for F_Q a facet of Q .

Proof. Let P_1^n and P_2^n be representatives of the combinatorial equivalence class P^n and let Q_1^m and Q_2^m be representatives of the combinatorial equivalence class Q^m . We have that $P_1^n \times Q_1^m$ and $P_2^n \times Q_2^m$ are possible representatives of $P \times Q$, which will be well defined if they are combinatorially equivalent.

We know from [30, pg 9] that for any two non-combinatorial polytopes, P^n and Q^m , we know that the face poset of $P \times Q$ consists of the products of a non-empty face of P and a non empty face of Q , with ordering $P_1 \times Q_1 \leq P_2 \times Q_2$ if and only if $P_1 \leq P_2$ and $Q_1 \leq Q_2$.

Thus we know that the face poset of $P_1^n \times Q_1^m$ consists of the products of a non-empty face of P_1^n and a non empty face of Q_1^m and that the face poset of $P_2^n \times Q_2^m$ consists of the products of a non-empty face of P_2^n and a non empty face of Q_2^m . Because P_1^n and P_2^n are combinatorially equivalent, as are Q_1^m and Q_2^m , there is a bijection between the face posets of each pair. These bijections give a bijection between the face posets of $P_1^n \times Q_1^m$ and $P_2^n \times Q_2^m$ and the direct product is well defined.

That $P \times Q$ is $n + m$ dimensional follows directly from the definition of dimension. Since $P \times Q$ is of dimension $n + m$, facets are faces of dimension $n + m - 1$. Since they are faces, they are products of a non-empty face of P and a non-empty face of Q . Thus they must be the product of either an $n - 1$ dimensional face of P and a m dimensional face of Q or an $n - 1$ dimensional face of P and a m dimensional face of Q . Which means a facet of $P \times Q$ is the product of either a facet of F and Q or F and a facet of Q . Moreover all products of this type will be faces of dimension $n + m - 1$ and so facets. \square

3.2 Vectors

Having seen in the previous section what a polytope is, in this section we will look at how polytopes have been studied and the main strategies that have been used to investigate these objects.

Combinatorial polytopes are defined as abstract entities by their face posets. The study of polytopes has historically centred around this fact, with the basic tool in the armoury being the face vector.

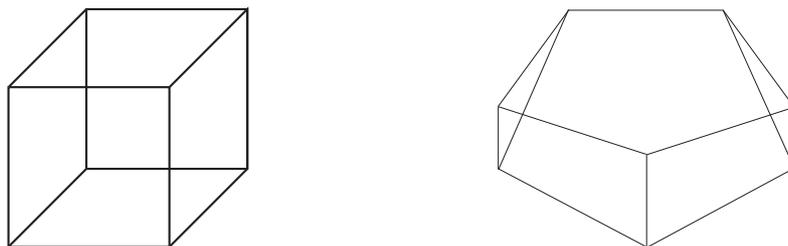
Definition 3.2.1. The f -vector, also known as the *face vector*, of a polytope P^n is the integral vector

$$\mathbf{f}(P) = (f_0, \dots, f_{n-1})$$

where f_i is the number of i -dimensional faces of P^n .

It is clear that the f -vector of a polytope is a combinatorial invariant since it only depends on the face poset of the polytope. On the other hand it is also easy to construct an example which shows that the f -vector is not unique to a particular polytope.

Example 3.2.2. Let us examine the two 3 dimensional polytopes given in the following diagrams.



The figure on the left is the standard cube I^3 , with six 4-gon facets and f -vector $(8, 12, 6)$. The figure on the right has facets consisting of two 5-gons, two 4-gons and two 3-gons. It also has f -vector $(8, 12, 6)$.

Another aspect of convex polytopes which distinguishes them is their 1-skeleton, the graph of their vertices linked by their edges. Another way of studying polytopes

starts off by looking at the 1-skeletons of simple polytopes from the viewpoint of Morse theory.

Definition 3.2.3. [9] Examine the 1-skeleton of a simple polytope, P^n , which is embedded in some \mathbb{R}^n . We can choose a linear function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ which distinguishes all the vertices of P . This function shall be known as the height function. Such a function is $\phi(x) = \langle d, x \rangle$ for some vector, d , in \mathbb{R}^n which is not parallel to any edge of P .

We then define the index of a vertex, v , to be the number of vertices, w , connected to v such that $\phi(v) \geq \phi(w)$. We then define h_i to be the number of vertices with index i . Because P is simple we know that the maximum index of any vertex is n .

The h -vector, also known as the *height vector*, of P^n is the vector

$$\mathbf{h}_\phi(P) = (h_0, \dots, h_n)$$

This vector is a combinatorial invariant of a polytope as proved below.

Theorem 3.2.4. The h -vector is a combinatorial invariant and is independent of the choice of ϕ .

Proof. (Adapted from [9, 1.20]) We will show that the h -vector is a combinatorial invariant by proving that the h -vector can be written in terms of the f -vector, which we already know to be combinatorially invariant.

Let P^n be a simple polytope with f -vector (f_0, \dots, f_{n-1}) and h -vector (h_0, \dots, h_n) , given by some height function ϕ . It is our intention to either express h_i in terms of the f_j for all i or f_j in terms of the h_i for all j .

To do this we examine the faces of P^n . Looking at a face F^k , a k -dimensional face of P^n , we see that ϕ gives us unique top and bottom vertices. We shall denote the top vertex of F by v_F . Since P is simple we know that exactly k edges of F meet at v_F and so the index of $v_F \geq k$. By the same reasoning we can see that each vertex of index q is the top vertex of exactly $\binom{q}{k}$ faces of dimension k . Since each face has exactly one top vertex and h_q is the number of vertices of index q , we get the relation

$$f_k = \sum_{q \geq k} \binom{q}{k} h_q$$

as required. \square

This proof shows the equivalence of the approaches of using the faces and the 1-skeleton. It also shows that the h -vector is independent of the choice of height function ϕ . We shall now use this equivalence to extend the definition of the h -vector from simple polytopes to all polytopes.

Definition 3.2.5. For a polytope, P^n , we define h_i , for i from 0 to n , to be the solutions of the equation,

$$h_0 t^n + \dots + h_{n-1} t + h_n = (t-1)^n + f_0(t-1)^{n-1} + \dots + f_{n-1}.$$

Then we define the h -vector of P^n to be the vector

$$\mathbf{h}(P) = (h_0, \dots, h_n).$$

In particular we have the relations

$$f_k = \sum_{q \geq k} \binom{q}{k} h_q$$

and

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{n-i}{n-k} f_{i-1}$$

which match the relation given in theorem 3.2.4.

Theorem 3.2.6 (Dehn-Sommerville relations [9]). The h -vector of a simple n -dimensional polytope is symmetric, i.e.

$$h_i = h_{n-i}, i = 0, 1, \dots, n.$$

Proof. By theorem 3.2.4 we have that the original definition of the h -vector is independent of the choice of $\phi(x)$. In particular we can look at what happens when we use $\psi(x) = -\phi(x)$ instead. For ease of reference we will refer to the index of a vertex, v , in each case as $\text{ind}_\phi(v)$ or $\text{ind}_\psi(v)$ and the h -vector developed as $\mathbf{h}_\phi(P) = (h_0^\phi, h_1^\phi, \dots, h_n^\phi)$ and $\mathbf{h}_\psi(P) = (h_0^\psi, h_1^\psi, \dots, h_n^\psi)$.

Because of the similarities between the functions, we have that $\text{ind}_\phi(v)$ is equal to the total number of edges at v minus $\text{ind}_\psi(v)$. Since the polytope is simple, the total number of edges at v is n , so we have $\text{ind}_\phi(v) = n - \text{ind}_\psi(v)$. This means that

$$h_i^\phi = h_{n-i}^\psi$$

for all $i = 0, 1, \dots, n$. However 3.2.4 tells us that $\mathbf{h}_\phi = \mathbf{h}_\psi$ and so we have

$$h_i = h_{i-n}, \quad i = 0, 1, \dots, n$$

as required. □

By a dual argument we have that.

Corollary 3.2.7. The Dehn-Sommerville relations also hold for simplicial polytopes.

Furthermore it is possible to show that.

Theorem 3.2.8. [9, 1.25] The Dehn-Sommerville relations are the most general linear equations satisfied by the f -vectors, or equivalently h -vectors, of all simple (or simplicial) polytopes.

However, we will not prove this explicitly here. This theorem tells us though that if we want to know more about the vectors of polytopes we will have to restrict the polytopes we study in some way. One way we can do this was introduced in the last section, eventually we will look at just flag simple polytopes.

Since we are looking at a subset of simple (simplicial) polytopes we know that the Dehn-Sommerville relations must hold. We have that the h -vector is symmetric and so we know that $\lfloor \frac{n}{2} \rfloor$ of the entries are repeated. We also know from the definition of the h -vector that $h_0 = h_n = 1$. We can therefore look at two ways of reducing the h vector of length $n + 1$ to a vector of length $\lfloor \frac{n}{2} \rfloor + 1$ which contains the same information. The g -vector does this by recording the successive differences between the entries of the h -vector. Alternatively, the γ -vector does this by rewriting the equation which defines h -vector in terms of elementary symmetric polynomials in t and 1. These two vectors arise from different way of considering the invariants.

The g -vector comes from thinking about the invariants as vectors, while the γ -vector arises from considering the invariants as polynomials.

Definition 3.2.9. The g -vector of a polytope, P^n , is a vector $\mathbf{g}(P) = (g_0, g_1, \dots, g_m)$ where $g_0 = 1$, $g_i = h_i - h_{i-1}$ and $m = \lfloor \frac{n}{2} \rfloor$.

Definition 3.2.10. The γ -vector of a simple polytope, P^n , is a vector $\gamma(P) = (\gamma_0, \gamma_1, \dots, \gamma_m)$ where the γ_i are defined to be the solutions of the equation,

$$h_0 t^n + \dots + h_{n-1} t + h_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i (1+t)^{n-2i} t^i.$$

From the definitions we can derive relations between the g -vector and γ -vector.

Theorem 3.2.11. [5] We have that, for a simple polytope, P^n ;

$$\gamma_i(P^n) = (-1)^i \sum_{j=0}^i (-1)^j \binom{n-i-j}{i-j} g_j(P^n)$$

for $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$. We conversely have that

$$g_i(P^n) = (n-2i+1) \sum_{j=0}^i \frac{1}{n-i-j+1} \binom{n-2j}{i-j} \gamma_j(P^n)$$

Proof. These relations come from simply rearranging the definitions of the g -vector and γ -vector, both of which are defined in terms of the h -vector. \square

We can now look for relations on these vectors which tell us more than the Dehn-Sommerville relations.

The g -vector gives rise to the g -theorem which gives a necessary and sufficient condition for a vector to be the f -vector of a simple polytope.

Definition 3.2.12. [9, 1.27] For any two positive integers a and i there exists a unique *binomial i -expansion* of the form

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \dots + \binom{a_j}{j}$$

where $a_i > a_{i-1} > \dots > a_j \geq j \geq 1$. Define

$$a^{(i)} = \binom{a_i+1}{i+1} + \binom{a_{i-1}+1}{i} + \dots + \binom{a_j+1}{j+1}$$

and $0^{(i)} = 0$

Theorem 3.2.13 (*g*-theorem). [9, 1.29] An integer vector $(f_0, f_1, \dots, f_{n-1})$ is the *f*-vector of a simple polytope if and only if the corresponding *h*-vector, (h_0, \dots, h_n) satisfies

1. $h_i = h_{n-i} \forall i$, the Dehn-Sommerville relations;
2. $h_0 \leq h_1 \leq \dots \leq h_{\lfloor \frac{n}{2} \rfloor}$;
3. $h_0 = 1$ and $h_{i+1} - H_i \leq (h_i - h_{i-1})^{(i)}$ for $i = 1, \dots, \lfloor \frac{n}{2} \rfloor - 1$.

This theorem was one of the major advances in polytope theory and one of the first to use toric topology. The necessity of these conditions were proved by Stanley in [26] and that the conditions are sufficient was proved by Billera and Lee in [2].

These conditions on the *g*-vector have lead to the conjecture of stricter conditions on the γ -vector of flag simple polytopes.

Conjecture 3.2.14 (Gal's Conjecture [19]). For a flag simple polytope we have

$$\gamma_i \geq 0 \quad \forall i.$$

Finally, we will introduce the concept of the Stanley-Reisner ring. It is a combinatorial invariant of simple polytopes which expresses the face poset as a ring.

Definition 3.2.15. [9] The *face ring* or *Stanley-Reisner ring* of a simple polytope P with m facets, F_1, \dots, F_m , is the quotient ring

$$k(P) = k[v_1, \dots, v_m] / \mathcal{I}_P$$

where k is a commutative ring with unit and \mathcal{I}_P is the ideal generated by all square free monomials v_{i_1}, \dots, v_{i_s} such that $F_{i_1} \cap \dots \cap F_{i_s} = \varphi$ in P and $i_1 < \dots < i_s$.

3.3 Rings

In the previous two sections we have seen how polytopes have been studied in the past. In this section we will look at a new approach that was introduced in [5]. We will be considering a differential graded ring generated by combinatorial polytopes.

For a ring structure we will require two operations, an addition and a multiplication. For a differential graded ring structure will also require a grading and a differential operator. We will start with the grading, which we shall take to be twice the dimension of the polytope. We will then define addition formally on polytopes of the same dimension.

Definition 3.3.1. [5] We denote by P_{2n} the abelian group generated by combinatorial n -dimensional polytopes, with the zero given by the empty n -dimensional polytope.

When considering this addition we can think about taking the disjoint union of two polytopes, although this operation does not have inverses. If we now define a multiplication on elements of these groups we get a graded ring. We will base this multiplication on the direct product of combinatorial polytopes.

Definition 3.3.2. [5] We denote by $\mathcal{P} = \sum_{n \geq 0} P_{2n}$ the graded commutative associative ring in which multiplication is given on the generators of the groups by the direct product of combinatorial polytopes, and extended distributively to all elements. The unit then corresponds to the 0 dimensional polytope, $P^0 = (\text{point})$. We call \mathcal{P} the *ring of polytopes*.

This multiplication is commutative because the operation of taking the direct product of two non-combinatorial polytopes is commutative. It is for this reason that we take the grading of P^n to be $2n$ rather than n , because this gives \mathcal{P} both commutative and anti-commutative structure since it contains no elements of odd grading.

We can finally define a differential operator to make \mathcal{P} into a differential graded ring.

Definition 3.3.3. Define $d : \mathcal{P} \rightarrow \mathcal{P}$ to be the map which takes a polytope to the sum of its facets.

This is a differential operator because $d(P_{2i}) \subset P_{2(i-1)}$ and from proposition 3.1.22 we have

$$d(P_1^{n_1} \times P_2^{n_2}) = d(P_1^{n_1}) \times P_2^{n_2} + P_1^{n_1} \times d(P_2^{n_2}). \quad (3.2)$$

We can mimic the construction of this ring structure to give subrings for that correspond to certain types of polytopes. For example we know that the product of two simple polytopes is simple we can create a ring of simple polytopes, \mathcal{P}_S .

Proposition 3.3.4. The ring of simple polytopes, \mathcal{P}_S , is a differential graded subring of the ring of polytopes \mathcal{P} .

Proof. We know that the grading of a polytope in \mathcal{P}_S is the same as in \mathcal{P} , since in both cases the grading is the dimension of the polytope. All that remains to show is that \mathcal{P}_S is closed under d . This follows as the facets of a simple polytope are simple. \square

We can do the same with simplicial polytopes to give a ring of simplicial polytopes, \mathcal{P}_Δ , which by a similar argument is also a differential graded subring of \mathcal{P} .

3.4 Polynomials

Along with the new algebraic approach to polytopes given in [5], a new approach to the classical vector-based tools for studying polytopes is presented. This new approach is to replace the vectors with a set of homogeneous polynomials which encode the same data about the polytopes. We will replace the f -vector with the f -polynomial.

Definition 3.4.1. The f -polynomial, also known as the *face polynomial*, of a polytope P^n is the homogeneous polynomial

$$f(P)(\alpha, t) = \sum_{i=0}^n f_i \alpha^i t^{n-i}$$

where f_i is the number of i -dimensional faces of P^n .

Like the f -vector, the f -polynomial is a combinatorial invariant and, like the f -vector, it is unable to distinguish all polytopes. However unlike the f -vector we have the very powerful property

Theorem 3.4.2. For a polytope $P \in \mathcal{P}_S$, we have that

$$f(dP)(\alpha, t) = \frac{\partial}{\partial t} f(P)(\alpha, t).$$

Proof. Let the facets of P^n be F_i^{n-1} for $i = 1, \dots, m$. Then

$$f(dP) = \sum_{i=1}^m f(F_i^{n-1})(\alpha, t).$$

On the other hand

$$\frac{\partial}{\partial t} f(P)(\alpha, t) = \sum_{i=0}^n (n-i) f_i \alpha^i t^{n-i-1}.$$

Thus the theorem is proved if each face, F_l^j , of P is contained in exactly $n-j$ facets of P .

Let us examine a tower of faces $F_l^0 \subset F_l^1 \subset \dots \subset F_l^n$. We have that each F_l^j is contained in strictly more facets than F_l^{j+1} , that F_l^n is a facet and so is contained in precisely 1 facet (itself) and that as P is simple, F_l^0 is the intersection of precisely n facets. Thus any face F_l^j is contained in exactly $n-j$ facets as required. \square

This theorem will prove to be quite important later. We can also convert the h -vector into a homogeneous polynomial.

Definition 3.4.3. [9] Examine the 1-skeleton of a simple polytope, P^n , which is embedded in some \mathbb{R}^n . We can choose a linear function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ which distinguishes all the vertices of P . Such a function is $\phi(x) = \langle d, x \rangle$ for some vector, d , in \mathbb{R}^n which is not parallel to any edge of P . This function gives a direction to every edge of P .

Let the set of vertices of P be V and the set of edges be E . We can define a function $w : V \times E \rightarrow \{\alpha, 1, t\}$ such that $w(v, e)$ is α if e exits v , is t if e enters v and 1 otherwise.

The h -polynomial, also known as the *height polynomial*, of P^n is the polynomial

$$h(P)(\alpha, t) = \sum_{v \in V} \prod_{e \in E} w(v, e).$$

This will be a homogenous polynomial whenever every vertex of P has the same number of edges joining at it. Because P is a simple polytope of dimension n , we have that $h(P)(\alpha, t)$ is homogeneous of degree n .

Proposition 3.4.4. For $P \in \mathcal{P}_S$ we have that the h -polynomial as defined in 3.4.3 can be written as

$$h(P)(\alpha, t) = \sum_{i=0}^n h_i \alpha^i t^{n-i}$$

where h_i is the i -th entry of the h -vector.

Proof. Let P^n be a simple polytope. Now let us examine $\prod_{e \in E} w(v, e)$ for a particular vertex v . An edge, e contributes α if it exits v , t if it enters v and 1 if and only if the intersection of e and v is empty. If we let i_v be the number of incoming edges and j_v the number of outgoing edges we have that

$$\prod_{e \in E} w(v, e) = \alpha^{i_v} t^{j_v}.$$

However since P is a simple polytope we have that $j_v = n - i_v$, furthermore we have that i_v is the index of v in the definition of the h -vector. Thus we have that the coefficient of $\alpha^i t^{n-i}$ in the h -polynomial is $\sum_{v | \text{ind}(v)=i} 1$ which is the number of vertices with index i , which is h_i as required. \square

This proposition extends the definition of the h -polynomial from simple polytopes to all polytopes. As was the case with the f - and h -vectors, the f - and h -polynomials can be related by a change of coordinates.

Proposition 3.4.5. For a polytope, P^n , we have

$$f(P)(\alpha - t, t) = h(P)(\alpha, t).$$

We can also re-express the Dehn-Sommerville relations in the new language.

Theorem 3.4.6 (Dehn-Sommerville relations [5]). The h -polynomial of a simple n -dimensional polytope is symmetric in α and t , i.e.

$$h(P)(\alpha, t) = h(P)(t, \alpha).$$

Proof. By proposition 3.4.4 we have that $h(P)(\alpha, t) = \sum_{i=0}^n h_i \alpha^i t^{n-i}$. So we have that

$$h(P)(t, \alpha) = \sum_{i=0}^n h_i t^i \alpha^{n-i} = \sum_{j=0}^n h_{n-j} \alpha^j t^{n-j}$$

after the substitution $j = n - i$. By comparing coefficients we see that $h(P)(\alpha, t) = h(P)(t, \alpha)$ precisely when $h_i = h_{n-i}$ for all $i = 0, \dots, n$. This is the Dehn-Sommerville relation for vectors. \square

Since the Dehn-Sommerville relations show that for a simple polytope the h -polynomial is symmetric in α and t . By the fundamental theorem of symmetric polynomials, we can rewrite the h -polynomial in terms of the elementary symmetric polynomials.

Definition 3.4.7. The γ -polynomial of a simple polytope, P^n , is the polynomial

$$\gamma(P)(\tau) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i \tau^i$$

where $a = (\alpha + t)$, $b = \alpha t$, $\tau = \frac{b}{a^2}$ and the γ_i are defined to be the solutions of the equation

$$h(P^n)(\alpha, t) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i a^{n-2i} b^i.$$

This definition gives us that $h(P^n)(\alpha, t) = a^n \gamma(P)(\tau)$. We also notice that when $\alpha = 1$ the equation defining the γ_i is the same as the one used for the γ -vector, meaning that the γ_i are the same in both cases, as the notation suggests. The conjectured conditions on the γ -vector can then be restated for the γ -polynomial.

Conjecture 3.4.8 (Gal's Conjecture [19]). For a flag simple polytope the γ -polynomial has all non-negative coefficients.

We will also introduce one new type of polynomial at this point. An advantage that the ring of polytopes gives us is that instead of looking at equations in $\mathbb{Z}[t]$, or $\mathbb{Z}[\alpha, t]$, we can instead look at $\mathcal{P}[t]$ which can give us a more nuanced invariant polynomial for our polytopes. We shall now take a brief look at a generalization of the approach that gave us the f -polynomial to this new setting.

Definition 3.4.9. [7] The *generalised face polynomial* of a polytope $P^n \in \mathcal{P}$ is a map $F : \mathcal{P} \rightarrow \mathcal{P}[t]$ given by

$$F(P)(t) = \sum_{k=0}^n d_k(P) t^k$$

where d_k is the map which takes P to the sum of the codimension k faces of P .

We will fix $0^0 = 1$ so that $F(P)(0) = P$ for all P . It is shown in [7] that in general the d_k do not commute with each other, but that they do commute when P is a simple polytope. Indeed we have that the F -polynomial coincides with

$$F(P)(t) = \sum_{k=0}^n d^k(P) \frac{t^k}{k!}$$

when $P \in \mathcal{P}_S$. In this case the F -polynomial obeys the same differential relations as the f -polynomial does in theorem 3.4.2.

Theorem 3.4.10. [7] For a polytope $P \in \mathcal{P}_S$, we have

$$F(dP)(t) = \frac{d}{dt} F(P)(t).$$

Proof. Since the dP is a collection of $n - 1$ dimensional polytopes,

$$F(dP) = \sum_{k=0}^{n-1} d^k(dP) \frac{t^k}{k!} = \sum_{k=0}^{n-1} d^{k+1}(P) \frac{t^k}{k!}.$$

We also have that

$$\frac{d}{dt} F(P)(t) = \sum_{l=0}^n l d^l(P) \frac{t^{l-1}}{l!} = \sum_{l=1}^n d^l(P) \frac{t^{l-1}}{(l-1)!} = \sum_{k=0}^{n-1} d^{k+1}(P) \frac{t^k}{k!}$$

as required. □

Armed with these techniques, let us now go in search of some polytopes which will answer the questions posed at the end of the last chapter.

Chapter 4

Nestohedra

We ended the Motivation chapter by posing four questions to which we would like to find answers. In this chapter we will examine an answer to the third question:

Question 3. Can we construct polytopes which have canonical combinatorial data?

In order to do this we will look at a recent method of constructing polytopes called nestohedra. In this chapter we will define these polytopes, examine some examples and look at what can be said about them. We will then look at their links to Toric Topology, show that they are Delzant and demonstrate their canonical combinatorial data.

4.1 Definition

In this section we will overview the construction of nestohedra from [24]. We will be constructing these polytopes using a Minkowski sum, so we will start by defining this.

Definition 4.1.1. The *Minkowski sum* of two subsets, A and B , of some \mathbb{R}^n is the set of points

$$A + B := \{c = (c_1, \dots, c_n) \mid c_i = a_i + b_i \forall a = (a_1, \dots, a_n) \in A, b = (b_1, \dots, b_n) \in B\}.$$

The polytopes are then built out of the appropriately named building sets.

Definition 4.1.2. A *building set*, B , is a set of subsets of $[n + 1]$, the set consisting of the first $n + 1$ integers, such that

1. If $S_1, S_2 \in B$ such that $S_1 \cap S_2 \neq \Phi$ then $S_1 \cup S_2 \in B$.
2. The set $\{i\} \in B$ for all $i \in [n + 1]$.

A building set is called *connected* if $[n + 1] \in B$.

For a graph, Γ , on $n + 1$ nodes any numbering of the nodes produces a building set $B(\Gamma)$. This building set consists of all non-empty subsets, $I \subset [n + 1]$, such that the induced graph $\Gamma|_I$ is connected. A building set constructed from a graph in this way will be called a *graphical building set*. A connected graph will produce a connected building set.

Taking this set we then construct the nestohedron by converting the building set to a set of simplices and then adding them together using the Minkowski sum.

Definition 4.1.3. For a building set B , the *nestohedron*, P_B , is the Minkowski sum

$$P_B = \sum_{S \in B} \Delta_S$$

where $\Delta_S := \text{ConvexHull}\{e_i | i \in S\}$ and e_i is the tip of the standard unit basis vector. A nestohedron with a connected building set will be called connected.

A useful property of Minkowski sums is that the Minkowski sum of two orthogonal subsets of \mathbb{R}^n is the direct product of those subsets. Conversely, a direct product can be regarded as the Minkowski sum of its factors provided that those factors are embedded orthogonally in some \mathbb{R}^n . This gives us a useful property of nestohedra.

Theorem 4.1.4. The direct product of two nestohedra is a nestohedron.

Proof. Let us consider two nestohedra P_A and P_B with building sets A on $[n]$ and B on $[m]$ respectively. We will let X be the set $\{\{x + n | x \in S\} | S \in B\}$, which is B under a renumbering of elements. We now consider the nestohedron P_C with building set C on $[n + m]$ defined by $C := A \cup X$. By the property just discussed, we

see that the nestohedron P_C is the product of P_A and P_X . It is also clear from the construction that $P_B = P_X$ and so we have

$$P_C = P_A \times P_B.$$

□

With this result we know that we can mimic the construction of \mathcal{P}_S with nestohedra to give a ring \mathcal{P}_N .

Corollary 4.1.5. The ring of nestohedra, \mathcal{P}_N , is a graded subring of \mathcal{P}_S .

Proof. We need to check multiplicative closure and the presence of the multiplicative identity in \mathcal{P}_N . Multiplicative closure is given by 4.1.4. The multiplicative identity is P^0 , which is a nestohedron with building set $\{\{1\}\}$. □

We would also like \mathcal{P}_N to be a differential graded subring and we will see that it is later.

4.2 Examples

In this section we will look at some examples of polytopes which can be constructed as nestohedra. These polytopes are sometimes famous polytopes that appear in other parts of the literature, so we will have a brief look at the history of the polytope in these cases.

Because this construction is not intuitively obvious we will start by looking at illustrated examples of all possible two dimensional connected nestohedra, since there are sufficiently few of them to give a complete illustrated list. There are 30 possible connected building sets on four elements up to isomorphism, which is why we will not also illustrate the possible three dimensional nestohedra.

The two dimensional nestohedra are those with building sets on three elements,

they are listed here up to isomorphism.

$$\{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\} \quad (4.1)$$

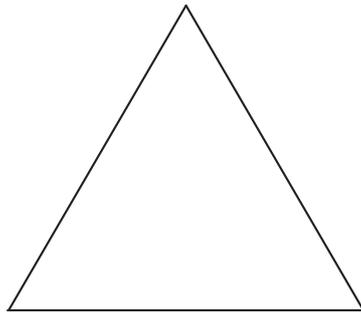
$$\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\} \quad (4.2)$$

$$\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\} \quad (4.3)$$

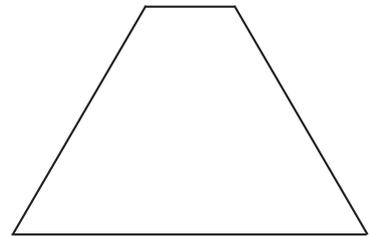
$$\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \quad (4.4)$$

These nestohedra are shown in the following diagrams.

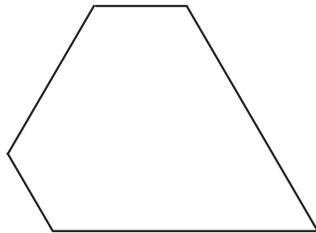
4.1



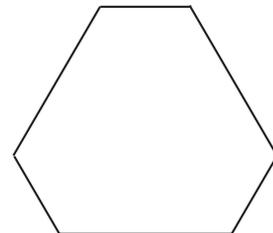
4.2



4.3



4.4



Having seen all the possible two dimensional nestohedra we will now look at some specific examples in other dimensions.

Example 4.2.1 (The Simplex). The n -dimensional simplex is a nestohedron. It has the building set consisting of the $n + 1$ singleton sets and $[n + 1]$ the set containing the first $n + 1$ natural numbers. The Minkowski sum is then the sum of $n + 1$ single points and the n -simplex, which is clearly the n -simplex. In two dimensions it is labelled above as 4.1.

One of the major calculations for any type of polytope is to find f_0 , the number of vertices each polytope has. The n -dimensional simplex has $n + 1$ vertices.

Example 4.2.2 (The Cube). The n -dimensional cube is also a nestohedron. It has two possible building sets, either the building set consisting of all the singleton sets and all sets of the form $\{1, 2, 3, \dots, i\}$ for $2 \leq i \leq n+1$ or the building set containing all the singleton sets and all pairs $\{2i-1, 2i\}$ for $1 \leq i \leq n$. Only the first one of these building sets is connected, although the second is graphical so is useful to use on some occasions. In two dimensions the first construction is the 4-gon and is labelled above as 4.2.

These two building sets produce the same polytope, because they are constructed in the same basic way as the cube. In both cases the $n+1$ -dimensional nestohedra can be obtained as the direct product of the n -dimensional nestohedra and the unit interval. Both building sets then produce the cubes because their 1 dimensional versions are the only possible 1 dimensional polytope, the unit interval.

Looking now at the number of vertices each polytope has, we see that this construction has 2^n vertices.

In future examples the words “all the singleton sets and” will be omitted from the definition of a building set, since the presence of these sets is mandated by condition 2.

The following examples appear in many places in the literature, particularly in [24] and [5].

Example 4.2.3 (The Associahedron). The n -dimensional associahedron is a nestohedron. This polytope is also known as the Stasheff polytope and gives a geometric representation to the number of possible products of $n+2$ numbers in a non-associative algebra. The representation is expressed as a bijection between vertices of the associahedron in such a way that any two vertices are connected by an edge if and only if they differ by precisely one set of brackets.

Historically these polytopes have been named as K_k for $k \geq 3$, but we will denote them as As^n where n is the dimension of the polytope for notational consistency with the other polytopes we are studying here. The relation between n and k is that $k = n + 2$.

This polytope was first introduced by Stasheff in [27] in 1963 as the parameter spaces of “higher associativities”. Stasheff did not initially give a realisation of the associahedron as a polytope, this was done independently by Milnor (unpublished), Haimen (1984) and Lee (1989). There have since been several other definitions found, all of which are shown to be the same by their isomorphic Stanley-Reisner rings. For a more detailed history of the associahedron I would advise the reader to look at [28] by Stasheff.

One way the associahedron can be defined is as the polytope with face poset which is isomorphic to the set of partial bracketings of $n + 2$ algebraic elements, for example

$$a_1 \cdot (a_2 \cdot a_3) \cdot a_4.$$

The incidence relation is that face A is contained in face B if the bracketing for A can be obtained from that for B by removing pairs of brackets. Two faces have non-empty intersection if there exists a third face which is contained in both of them.

Another definition of this polytope is as a representation of the number of ways to divide an $n + 3$ -gon by non-interesting lines between vertices (diagonals). A face of codimension k will correspond to a division of the $n + 3$ -gon by k diagonals. The incidence relation for these faces is defined analogously to the incidence relation for the previous definition.

The number of vertices of the n -dimensional associahedron is given by the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$.

As a nestohedron, it has the graphical building set that comes from the line graph on $n + 1$ nodes. This graph produces the building set which contains every subset of the form $\{i, i + 1, \dots, j - 1, j\}$ for $i < j$. The two-dimensional associahedron is labelled above as 4.3 and is the 5-gon.

Example 4.2.4 (The Permutohedron). The n -dimensional permutohedron is a nestohedron. The n -dimensional permutohedron is the polytope that represents the ways to permute $n + 1$ numbers: it can be defined as the polytope with corners with coordinates given by all the possible permutations of $n + 1$ distinct numbers. All of these

points become vertices and so the number of vertices of the n -dimensional permutohedron is the same as the number of elements of the symmetric group on n elements, $n!$.

This polytope was first studied by Schoute in [25] (1911). The term permutohedron was first coined by Guibaud and Rosenstiehl in [20] (1963).

As a nestohedron, it has the graphical building set that comes from the complete graph on $n + 1$ nodes. This graph produces the building set which contains every possible subset of $[n + 1]$. In two dimensions the permutohedron is labelled above as 4.4 and is the 6-gon.

Example 4.2.5 (The Cyclohedron). Our next example is called the cyclohedron. It is a graphical nestohedron from the cyclic graph on $n + 1$ nodes. This graph produces the building set which contains every subset of the form $\{i, i + 1, \dots, j - 1, j\} \bmod n + 1$. This polytope was also recognized by Stasheff in [27] and is also known as the Bott-Taubes polytope.

In two dimensions the cyclohedron is identical to the permutohedron and is shown in the above diagrams labelled as 4.4. This duplication is due to the low number of connected building sets on three elements and thus possible two dimensional nestohedra.

Example 4.2.6 (The Stellohedron). Our final example in this section only appears as a nestohedron. The n -dimensional stellohedron is a graphical nestohedron based on the star graph on $n + 1$ nodes. This graph produces the building set on $[n + 1]$ which contains every subset containing the element 1. Similarly to the cyclohedron, the 2 dimensional stellohedron is identical to one of our previous examples, in this case the 2 dimensional associahedron, and is shown in the above diagrams labelled as 4.3.

The number of vertices of the n -dimensional stellohedron is given in [24] as $\sum_{r=0}^n \frac{m!}{r!}$ which is an unnamed sequence, referred to only as A000522 from Sloan's On-Line Encyclopedia of Integer Sequences. This is a change from the famous sequences we have seen for our other examples.

4.3 Background

In this section we will summarise what is known about nestohedra. This consists of the papers by Posnikov [23], Zelevinski [29], Postnikov, Reiner and Williams [24], Buchstaber [5], Erkhovets [15] and Volodin [31]. The term nestohedron does not appear until [24], but for convenience I will use the terminology that we have used so far irrespective of whether the paper being discussed uses it.

In [23], nestohedra are introduced as generalised permutohedra, those polytopes which can be obtained from a permutohedron by parallel transport of the facets. The definition as a Minkowski sum is given as an alternative. This definition differs slightly from the one that we are using, in that it includes a set of non-negative parameters which act as multipliers on the terms of the Minkowski sum. Since we are dealing only with combinatorial polytopes and these parameters do not affect the combinatorial type of the sum they can be omitted. They are however important when considering non-combinatorial properties of nestohedra.

Theorem 4.3.1. [23, 6.3] A nestohedron is a generalised permutohedron.

Proof. Here we will use the alternative definition of a nestohedron. For $Y = \{y_I\}$ a set of non-negative parameters for each $I \subset [n]$ then we define a polytope,

$$P_n^Y(\{y_I\}) = \sum_{I \subset [n]} y_I \Delta_I$$

where Δ_I is as in the definition of nestohedra that we used earlier. For a building set B this agrees exactly with our definition 4.1.3 when $y_I = 1$ for $I \in B$ and 0 otherwise.

The theorem holds because with this definition the Minkowski sum of two nestohedra is the nestohedron defined by the coordinate-wise sum of their parameter sets. \square

A construction that appears in [23] is equivalent to our nestohedra. This construction is also in [24] and features prominently in [29].

Definition 4.3.2. [23, 7.3] A *nested set* $N \subset B$ where B is a building set is one where the following three conditions hold:

1. For any $I, J \in N$ we have that either $I \subset J$, $J \subset I$ or $I \cap J = \emptyset$.

2. If $J_1, \dots, J_k \in N$ is a collection of disjoint subsets with $k \geq 2$ then their union is not in B .
3. N contains all sets in B that are maximal under inclusion.

The *nested complex* $\mathcal{N}(B)$ is the poset of all nested sets of B ordered by inclusion.

We will use B_{\max} to denote the set of maximal elements of B under inclusion. The equivalence of the nested complex to a nestohedron is given in [23], the proof being rather technical so it will be omitted here.

Theorem 4.3.3. [23, 7.4] The face poset of a nestohedron N_B ordered by reverse inclusion is isomorphic to the nested complex defined on the same building set.

Postnikov in [23] then goes on to give an explicit expression of the face corresponding to each nested set, and to connect the nested sets to trees. This allows him to calculate the f -polynomial of a nestohedron (although he uses an equivalent polynomial) as:

Corollary 4.3.4. The f -polynomial of a nestohedron P_B with building set B is given by

$$f(P_B)(\alpha, t) = \sum_{N \in \mathcal{N}(B)} \alpha^{n-|N|} t^{|N|-1}.$$

In [23], Postnikov goes on to calculate geometrical properties of generalised permutohedra as real polytopes. The main properties he looks at are the volume, the possible subdivisions and the number of integer lattice points contained within the interior of the polytopes. Since these properties are not invariants of combinatorial polytopes, we shall now leave this paper to examine the work of Zelevinsky in [29].

Zelevinsky starts from the opposite direction to Postnikov. Beginning with a building set, B on $[n]$, he gives its nested complex a rank, $\text{rk}(B) = n - B_{\max}$. He then goes on to define a *link decomposition* of a nested complex.

Definition 4.3.5. For $C \in B - B_{\max}$ the link decomposition of C is

$$\mathcal{N}(B)_C = \{N' \subset B - B_{\max} - \{C\} : N' \cup \{C\} \in \mathcal{N}(B)\}.$$

There follows some study of these link decompositions with regards to two other concepts regarding building sets. For $C \subset [n]$ and B a building set on $[n]$ we have the *restriction*

$$B|_C = \{I \subset C : I \in B\}$$

and the *contraction*

$$C \setminus B = \{I \subset [n] - C : I \in B \text{ or } C \cup I \in B\}.$$

Zelevinsky goes on to show that

$$\mathcal{N}(B)_C \cong \mathcal{N}(B|_C \times C \setminus B)$$

as well as providing some properties of the rank of a nested set.

Zelevinsky then proceeds to realise a nested complex as a *nested fan* in a real vector space of dimension $\text{rk}(B)$. To do this he begins with a building set B on $[n]$. We can then define for any $I \subset [n]$ the vector

$$e_I = \sum_{i \in I} e_i \in \mathbb{Z}^n \subset \mathbb{R}^n$$

where the e_i are the standard basis vectors in \mathbb{Z}^n . Now we pass to the vector space of dimension $\text{rk}(B) = n - B_{\max}$ by means of a projection $\pi : \mathbb{R}^n \rightarrow V$ which takes \mathbb{R}^n to V , the quotient space of \mathbb{R}^n by the linear span of the vectors E_C for $C \in B_{\max}$. We insist that the standard lattice, L , in V is the image under π of the standard lattice \mathbb{Z}^n in \mathbb{R}^n . Finally we will denote $\pi(e_I)$ as \bar{e}_I .

We now have a set of rays in a real vector space of dimension $\text{rk}(B)$. We also know some properties of these rays.

Proposition 4.3.6. [29, 5.1 (1)] For every maximal nested set N , the vectors \bar{e}_I for $I \in N$ form a \mathbb{Z} -basis for L .

Proposition 4.3.7. [29, 5.1 (2)] Every vector $v \in L$ has a unique expansion

$$v = \sum_{I \in B - B_{\max}} c_I \bar{e}_I$$

such that all $c_I \geq 0$ and the set $\{I : c_I > 0\}$ is nested.

The properties ensure that the following fan is smooth, complete and simplicial in V .

Definition 4.3.8. [29, 5.2] The *nested fan* $\Delta(B)$ of a building set B consists of the fans $\mathbb{R}_{\geq} N$ for $N \in \mathcal{N}(B)$, where $\mathbb{R}_{\geq} N$ is the cone

$$\mathbb{R}_{\geq} N = \left\{ \sum_{I \in N} r_I \bar{e}_I \in V \mid r_I \geq 0 \right\}.$$

Having defined this nested fan, Zelevinsky introduces the polytopes that we know as nestohedra as *nested polytopes*.

Theorem 4.3.9. [29, 6.1] The nested fan $\Delta(B)$ is the normal fan of a simple complex polytope. This polytope can be explicitly expressed as the set of all n -tuples, (x_1, \dots, x_n) , of real numbers such that:

$$\sum_{i \in C} x_i = 0 \quad (C \in B_{\max}) \quad (4.5)$$

$$\sum_{i \in I} x_i \leq |I|(2^{|C|-1} - 2^{|I|-1}) \quad (I \subsetneq C \in B_{\max}) \quad (4.6)$$

Zelevinsky goes on to consider the case of graphical building sets, particularly how this change affects nested set complexes. He also shows that 2-dimensional faces of nested polytopes must have either 3, 4, 5 or 6 sides and that 2-dimensional faces of graphical nested polytopes cannot have 3 sides.

From our point of view it is important to emphasize that the two definitions by Postnikov and Zelevinsky actually produce polytopes of the same combinatorial type, since they both represent the nested set complex of the building set. Furthermore, Zelevinsky's construction of the nestohedra as the simple polytope with a certain normal fan gives us that:

Theorem 4.3.10. Nestohedra are Delzant polytopes.

Proof. We recall that a polytope is Delzant if at each vertex the normal vectors through each facet which meets the vertex form a \mathbb{Z} -basis for \mathbb{Z}^n . By property 4.3.6 the realisation of nestohedra as nested polytopes has this property. \square

We will now move on and summarize the results from the paper by Postnikov, Reiner and Williams [24]. This paper is more concerned with the combinatorial invariants of the polytopes it covers. We will only look here at what it says about nestohedra and shall begin with the simple results. It is important to note that it includes the definition of nestohedra that we have considered to be standard.

The first result of note on the combinatorial nature of nestohedra is that

Theorem 4.3.11 (6.5). [24] A nestohedron with building set B on $[n+1]$ is a simple polytope of dimension $n+1-B_{\max}$.

In particular this means that nestohedra with connected building sets on $[n+1]$ have dimension n . Postnikov et al. next examine when a nestohedron is flag. They do this by looking again at the nested set complex. Before stating the theorem on when nestohedra are flag we shall specify two conditions that are similar to those for a nested set.

Definition 4.3.12. A set $N \subset B - B_{\max}$ where B is a building set has property N if the following two conditions hold:

1. For any $I, J \in N$ we have that either $I \subset J$, $J \subset I$ or $I \cap J = \emptyset$.
2. If $I, J \in N$ are such that $I \cap J = \emptyset$ then their union is not in B .

We can then state the conditions for a nestohedron to be flag.

Proposition 4.3.13 (7.1). [24] For a building set B the following are equivalent.

1. The nestohedron P_B is flag.
2. The nested sets of B are precisely those which have property N .
3. If $J_1, \dots, J_k \in B$ with $l \geq 2$ are pairwise disjoint and their union is in B , then they can be reindexed so that both $J_1 \cup \dots \cup J_k$ and $J_{k+1} \cup \dots \cup J_l$ are in B .

In particular, graphical building sets have the second property and so we have the corollary that:

Corollary 4.3.14. [24, 7.2] All graphical nestohedra are flag.

The paper of Postnikov et al. goes on to show that the Gal conjecture holds for a certain class of nestohedra, *chordal nestohedra*, which are nestohedra with chordal building sets.

Definition 4.3.15. [24, 9.2] A *chordal* building set B on $[n]$ is one which satisfies the condition: for any $I = \{i_1 < \dots < i_r\} \in B$ and $s = 1, \dots, r$, the subset $\{i_s, i_{s+1}, \dots, i_r\}$ also belongs to B .

The name chordal comes from graph theory. A *chordal graph* is one with no induced k -cycles for $k > 4$. Equivalently a chordal graph, G , is one in which the nodes can be labelled by numbers in $[n]$ in such a way that any induced subgraph $G|_{\{i < j < k\}}$ that contains the edges (i, j) and (i, k) also contains the edge (j, k) .

Proposition 4.3.16. [24, 9.4] A graphical building set is chordal precisely when the graph is a perfectly labelled chordal graph.

These building sets are of particular importance because the main result of [24] is to show that:

Theorem 4.3.17. [24, 11.13] The Gal conjecture holds for chordal nestohedra.

I will not give the complete proof here since it takes up much of the paper and involves combinatorial arguments which we will make no further use of. We will however give an overview of some particular results which enable them to reach this end.

The paper starts by considering trees. A tree is a graph with no cycles or repeated edges. A rooted tree is a tree with a distinguished node. Such a tree induces a partial ordering on its nodes with $i < j$ if j lies on the unique path from i to the root. For a rooted tree, T , we denote by $T_{\leq i}$ the rooted tree consisting of i and its descendants.

They define a *B-tree* for a connected building set, B , as the rooted tree with the properties

1. For any i , we have $T_{\leq i} \in B$.

2. For $k \geq 2$ incomparable nodes $i_1, \dots, i_k \in [n]$ one has $\bigcup_{j=1}^k T_{\leq i_j} \notin B$.

The second condition needs only be stated with $k = 2$ for flag building sets. For a tree we can define the descent number, $\text{des}(T)$ to be the size of the minimum covering relation of the tree considered as a poset.

There is then a bijection between B -trees and maximal nested sets and we have the relation that

$$h(t, 1) = \sum_T t^{\text{des}(T)}$$

where T is any B -tree.

The next step is to develop another object from a connected building set. A B -permutation is a permutation ω on $[n]$ such that for any i we have that $\omega(i)$ and $\max\{\omega(1), \dots, \omega(i)\}$ are in the same connected component of the restricted building set $B|_{\{\omega(1), \dots, \omega(i)\}}$.

For a permutation we call i a *descent* if $\omega(i) > \omega(i+1)$. A *double descent* is when both i and $i+1$ are descents and a *final descent* is when $n-1$ is a descent. The *descent number* of a permutation $\text{des}(\omega)$ is the number of descents. We shall define $S_n(B)$ to be the set of B -permutations and $\tilde{S}_n(B)$ to be the set of B -permutations that do not include any double descents or final descents.

Postnikov et al. then proceed to show that there is a bijection between the set of B -trees and $S_n(B)$ such that $\text{des}(\omega) \leq \text{des}(T)$ with equality precisely for chordal building sets. This means that for chordal building sets we have the expression

$$h(t, 1) = \sum_{\omega \in S_n(B)} t^{\text{des}(\omega)}.$$

By studying the descent numbers of B -permutations they can then express the h -polynomial as

$$h(t, 1) = \sum_{\omega \in \tilde{S}_n(B)} t^{\text{peak}(\omega)-1} (t+1)^{n+1-2\text{peak}(\omega)},$$

which proves the Gal conjecture for chordal nestohedra.

In [15] this method was extended to cover the class of graphical nestohedra developed from the bipartite connected graphs $K_{m,n}$. The methods used in [5] will be explained in future sections when they are employed.

Lastly in this section we will consider the work in [31]. Volodin starts by reporting a result of Erokhovets that any nestohedron is combinatorially equivalent to a nestohedron with a connected building set. Let B, B_1, \dots, B_{n+1} be connected building sets on $[n+1], [k_1], \dots, [k_{n+1}]$. We define $B(B_1, \dots, B_{n+1})$ as the building set on $[k_1 + \dots + k_{n+1}]$ consisting of elements $S_i \in B_i$ and $\sqcup_{i \in S} [k_i]$ where $S \in B$.

Theorem 4.3.18. [31] Let B, B_1, \dots, B_{n+1} be connected building sets as above. Then the nestohedron $P_B \times P_{B_1} \times \dots \times P_{B_{n+1}}$ is combinatorially equivalent to the nestohedron $P_{B'}$ where $B' = B(B_1, \dots, B_{n+1})$.

This theorem gives the desired result inductively as follows.

Corollary 4.3.19. [31] Any nestohedron is combinatorially equivalent to a nestohedron with a connected building set.

Proof. For a nestohedron with two connected components $B = B_1 \sqcup B_2$ the previous theorem shows that $P_{B'} = P_B = P_{B_1} \times P_{B_2}$ when $B' = B_1(B_2, \{1\}, \dots, \{1\})$ as required. We will call B' the *connection* of B_1 and B_2 . For nestohedra with building sets with more than two connected components we can apply the two component case inductively to get the desired result. \square

Volodin then proves that the Gal conjecture holds for all flag nestohedra using cube shavings, successive truncations of a cube along faces of codimension two. These cube shavings can only increase the coefficients of the γ -polynomial and cannot stop a polytope being either simple or flag simple. Since the cube is flag simple and satisfies the Gal conjecture, any polytope which is obtainable using cube shavings is flag simple and satisfies the Gal conjecture. Volodin completes his work by showing that all flag nestohedra can be obtained using cube shavings. We will now look at how this is done.

Volodin starts this by showing that any connected building set which gives a flag nestohedron has a subset, B_I , which gives the n -dimensional cube.

To complete the proof that all flag nestohedra are attainable as cube shavings Volodin then shows that if $B' \subset B$ are building sets then P_B is attainable by shavings of $P_{B'}$.

Definition 4.3.20. Let B and B' be building sets on $[n + 1]$ with $B' \subset B$ and let $S \in B$. Then $B'(S)$, the *decomposition* of S by elements of B' , is $S = S_1 \sqcup \dots \sqcup S_k$ where $S_j \in B'$ and k is minimal.

Volodin then defines P_{cut} to be the polytope given by successive shavings of $P_{B'}$. These shavings are defined by extending the partial ordering of $B - B'$ by reverse inclusion to a total ordering. We denote the i th element under this ordering S^i . Identifying the facets of $P_{B'}$ with elements of B' , we have that the i th shaving is along face $G^i = \bigcap F_{S_j^i}$ where the S_j^i are the terms in the decomposition $B'(S^i)$.

Volodin then shows that P_{cut} is combinatorially equivalent to P_B . All that remains to show is that by taking a series of intermediary building sets between B_I and B only codimension two shavings need be used, which Volodin does, and we have that the Gal conjecture holds for all flag nestohedra.

4.4 Boundaries

Now we want to see what more we can say about these polytopes. In this section we will start by examining the key feature of \mathcal{P}_S , that it is a differential ring. So we will examine the boundary operator, d , of a single generalised nestohedron. In particular we would like to know if \mathcal{P}_N is closed under d . We can generalise a result from [24] to serve our purpose. To demonstrate the difference, we will state both here.

Theorem 4.4.1. [24] For a nestohedron, P_B , on a connected building set B , we have;

$$\frac{\partial}{\partial t} f(P_B) = \sum_{S \in B/[n+1]} f(P_{B|S}) \times f(P_{B-S})$$

where $B|S$ is the restriction of B to S and $B-S$ is the contraction of B by S .

This gives a formula for the f -polynomial of a nestohedron in terms of the f -polynomial of lower dimensional nestohedra. Fortunately with some effort we can extend this formula in a non-trivial way so that it does not require the f -polynomial.

Theorem 4.4.2. [17] For a nestohedron, P_B , on a connected building set B , we have;

$$d(P_B) = \sum_{S \in B/[n+1]} P_{B|S} \times P_{B-S}$$

where $B|S$ is the building set consisting of those sets in B which are subsets of S and $B-S$ is the building set consisting of sets in B with the elements of S removed.

Proof. We are looking at the facets of a Minkowski sum of faces of the standard simplex, Δ^n , which includes the standard simplex as one of the summands. All the summands will be simplices of dimension $m \leq n$ and all their faces will also be lower dimensional simplices. The standard definition of the m -dimensional simplex is

$$\Delta^m = \left\{ x = (x_1, \dots, x_m) : 0 \leq x_i \leq 1, \sum_{i=1}^m x_i = 1 \right\},$$

and the faces of Δ^m are subsets where some set of coordinates are minimised.

A facet of the sum will have a contribution from each summand. This contribution will be a face of the summand, and will frequently be the entire summand. Each face is defined by the set of coordinates on which it is minimised. If these sets are not restrictions of some set C the result is not a facet of the sum: it is either in the interior of the polytope or a face of lower dimension.

The facet, as the Minkowski sum of these faces, can be split up into the Minkowski sum of two other polytopes, X and Y . These are both Minkowski sums of faces, X the parts of those faces in $\text{span}\{e_i\}_{i \in C}$ and Y the parts of those faces in $\text{span}\{e_i\}_{i \notin C}$. Since A and B are orthogonal, we have that $X + Y = X \times Y$, the direct product.

We now examine the faces in two types, those where $\sum_{i \in C} x_i = 0$ and those where $\sum_{i \in C} x_i = 1$. It is easy to show that there are no other possibilities. A face, Δ_S , of the first type contributes 0 to X and Δ_{S-C} to Y . A face, Δ_S , of the second type contributes Δ_S to X and 0 to Y .

From the above we can clearly see that X is precisely $P_{B|S}$ and Y is P_{B-S} . This will produce a facet precisely when both sums contain the highest possible dimension simplices. This is only the case when Δ_S and Δ^n are present and distinct, which is when S and $[n+1]$ are distinct and both in B . So for a connected building set there is a facet for each element of B apart from $[n+1]$, and it is $P_{B|S} \times P_{B-S}$. \square

For a non-connected building set we can split it up into the product of its connected components and use the result from theorem 4.4.2 result on each component. This result shows that \mathcal{P}_N is a differential graded subring of both \mathcal{P}_S and \mathcal{P} .

This result can be restated in terms of graphs for graphical building sets, which means that \mathcal{P}_G , the ring of graphical nestohedra, is also a differential graded subring of \mathcal{P}_N , \mathcal{P}_S and \mathcal{P} .

Corollary 4.4.3. For a connected graph Γ on $n + 1$ nodes, we have

$$d(P(\Gamma)) = \sum_{*} P(\Gamma_G) \times P(\bar{\Gamma}_{G^c})$$

where

1. $G \subsetneq \{1, \dots, n + 1\}$
2. Γ_G is the subgraph of Γ with vertex set G
3. $\bar{\Gamma}_{G^c}$ is the simple graph with vertex set $\{1, \dots, n + 1\} - G$ and arcs between two vertices, i and j , if they are path connected in $\Gamma_{G \cup \{i, j\}}$
4. $*$ runs over all G such that Γ_G is connected.

When these results are taken together with formula 3.4.2, we are left with theorem 4.4.1.

In the remainder of this section we will look at the boundaries of some of the examples we saw in section 4.2.

Example 4.4.4 (Simplices). Firstly, we will look at the simplices. We recall that simplices are nestohedra with building set consisting of $[n + 1]$ and the singleton sets. We have that the facets have the form $P_{B|S} \times P_{B-S}$ for $S \in B/[n + 1]$. With this particular building set, S must be a singleton set and so $P_{B|S}$ is a point and P_{B-S} the $n - 1$ simplex. As there are $n + 1$ singleton sets we get the formula

$$d(\Delta^n) = (n + 1)\Delta^{n-1}.$$

Example 4.4.5 (Cubes). Our next example is that of the cubes. We recall that cubes are nestohedra with building set consisting of set of the form $[i + 1]$ for $i = 2, \dots, n$ and the singleton sets. We have that the facets have the form $P_{B|S} \times P_{B-S}$ for $S \in B/[n + 1]$. With this particular building set it is easy to see that for any S we must

have that $P_{B|S}$ is $|S|$ cube and P_{B-S} the $n - |S| - 1$ cube. As there are $2n$ sets in $B/[n + 1]$ we get the formula

$$d(I^n) = 2nI^{n-1}.$$

Example 4.4.6 (Permutohedra). Next we look at the permutohedra, which we recall are graphical nestohedra generated by the complete graphs. We have, for Γ the complete graph on $n + 1$ nodes

$$d(P(\Gamma)) = \sum_* P(\Gamma_G) \times P(\bar{\Gamma}_{G^c}),$$

where $*$ runs over all G such that Γ_G is connected. In this case, Γ_G is the complete graph on $|G|$ nodes for all G , also $\bar{\Gamma}_{G^c}$ is the complete graph on $(n + 1) - |G|$ nodes for all G . In addition we know that every possible subgraph is connected.

Collecting the subgraphs by cardinality, we are left with

$$d(Pe^n) = \sum_{i+j=n-1} \binom{n+1}{i+1} Pe^i \times Pe^j.$$

This formula usefully expresses the boundary of a permutohedron entirely in terms of other lower dimensional permutohedra.

Example 4.4.7 (Associahedra). Now we look at the associahedra, which we recall are graphical nestohedra generated by the line graphs. We have, for Γ the line graph on $n + 1$ nodes

$$d(P(\Gamma)) = \sum_* P(\Gamma_G) \times P(\bar{\Gamma}_{G^c}),$$

where $*$ runs over all G such that Γ_G is connected. With Γ being the line graph, we have that Γ_G is the line graph on $|G|$ nodes for all G . Also we have that $\bar{\Gamma}_{G^c}$ is the line graph on $(n + 1) - |G|$ nodes for all G , since any connected subgraph on removal partitions the remaining nodes into exactly two components, unless it contains an end of the graph. When the result is two components, we label the new ends i and j and it is clear that these nodes are path connected in $\Gamma_{G \cup \{i,j\}}$. However not every subgraph is connected, so we must count these carefully.

Grouping the subgraphs by cardinality, we are left with

$$d(As^n) = \sum_{i+j=n-1} (n - i + 1) As^i \times As^j.$$

This formula usefully expresses the boundary of a associahedron entirely in terms of other lower dimensional associahedra.

Our next two examples, the cyclohedron and the stellohedron, are more complicated. In these examples we find other types of nestohedra amongst the faces.

Example 4.4.8 (Cyclohedra). We recall that the cyclohedra are graphical nestohedra generated by the cyclic graphs. We have, for Γ the cyclic graph on $n + 1$ nodes

$$d(P(\Gamma)) = \sum_{*} P(\Gamma_G) \times P(\bar{\Gamma}_{G^c}),$$

where $*$ runs over all G such that Γ_G is connected. As was the case with the associahedron, Γ_G is the line graph on $|G|$ nodes for all G . Unlike in the previous two examples $\bar{\Gamma}_{G^c}$ is different. This time it is the cyclic graph on $(n + 1) - |G|$ nodes for all G , since any connected subgraph on removal leaves the remaining nodes forming a line graph. Labelling the new ends i and j , it is clear that these nodes are path connected in $\Gamma_{G \cup \{i,j\}}$. As with the second example not every subgraph is connected, so we must count these carefully.

Collecting the subgraphs together by cardinality, we are left with

$$d(Cy^n) = \sum_{i+j=n-1} (n+1)As^i \times Cy^j.$$

This formula usefully expresses the boundary of a cyclohedron in terms of other lower dimensional polytopes, but this time they are of two distinct types, cyclohedra and associahedra.

Example 4.4.9 (Stellohedra). Now we look at the stellohedra, which we recall are graphical nestohedra generated by the star graphs. We have, for Γ the line graph on $n + 1$ nodes,

$$d(P(\Gamma)) = \sum_{*} P(\Gamma_G) \times P(\bar{\Gamma}_{G^c})$$

where $*$ runs over all G such that Γ_G is connected. We are looking for the subsets, G , where Γ_G is connected. In this case there are two types, the singleton subsets and those sets containing the central element, which we shall fix as being the element 1.

Looking first at the singleton sets, $G = \{i\}$, for $i \neq 1$, we see that $P(\Gamma_G)$ is the 0-dimensional polytope and $P(\bar{\Gamma}_{G^c})$ is the star graph on n nodes. The product of these two polytopes is St^{n-1} and there are n such subgraphs.

Now looking at those subgraphs containing 1, we notice that every such graph is connected and that any two nodes i, j not in the subgraph are connected in Γ by the set $\{i, 1, j\}$. Thus $P(\bar{\Gamma}_{G^c})$ is the complete graph on $n - |G|$ nodes.

Combining these we are left with

$$d(St^n) = n \cdot St^{n-1} + \sum_{i=0}^{n-1} \binom{n}{i} St^i \times Pe^{n-i-1}.$$

This formula expresses the boundary of a stellohedron entirely in terms of other lower dimensional stellohedra and permutohedra.

4.5 Nestohedra and Toric Topology

At the beginning of this chapter we wanted a construction of polytopes with a canonical characteristic function. We have looked at the construction of nestohedra and seen that one particular construction of them is Delzant, which gives a canonical characteristic function. In this section we will look more closely at this characteristic function.

Definition 4.5.1. The *canonical characteristic function* of a nestohedron, P , shall be denoted Λ_P . It is a matrix with columns that are the outward pointing normal vectors to the facets defined as a nested polytope.

We recall that the nested polytope construction was as the normal polytope of the nested fan. The nested fan had a fan for each element in the nested set. By 4.3.3 we know that there is a isomorphism between faces of the nestohedron and nested sets under reverse inclusion. This means that there is a isomorphism between facets of the polytope and nested sets consisting of precisely one non-maximal element of the building set. In this section we shall let B be our building set on $[n + 1]$.

Clearly, nested sets consisting of precisely one non-maximal element of the building set have a natural isomorphism with non-maximal elements of the building set.

Thus we have a correspondence between facets and $S \in B - B_{\max}$. This correspondence is the same one that is set out in the proof of 4.4.2.

Theorem 4.5.2. Let P be a nestohedron with connected building set B . The column of Λ_P corresponding to a facet labelled by a set $I \in B - B_{\max}$ is $(v_1, \dots, v_n)^T$ where

$$v_i = \begin{cases} 1 & i \in I, n+1 \notin I \\ 0 & i \in I, n+1 \in I \\ 0 & i \notin I, n+1 \notin I \\ -1 & i \notin I, n+1 \in I \end{cases}$$

Proof. We have a facet of a nestohedron labelled by a non-maximal element of the building set I . The vector associated to it in the canonical characteristic function is the outward pointing normal vector of the facet using the construction as a nested polytope. This is given by \bar{e}_I where

$$e_I = \sum_{i \in I} e_i \in \mathbb{Z}^n \subset \mathbb{R}^{n+1}$$

and $\bar{e}_I = \pi(e_I)$. Here π is the projection $\mathbb{R}^n \rightarrow V$ that takes \mathbb{R}^n to V , the quotient space of \mathbb{R}^n by the linear span of the vectors E_C for $C \in B_{\max}$.

For a connected building set we have that $B_{\max} = [n+1]$ and so π is the projection along the diagonal $(1, \dots, 1)$. We can easily calculate the image of e_I under this map, because e_I is a vector containing only the entries 1 and 0. If $n+1 \notin I$ then the image $\bar{e}_I = e_I$. If $n+1 \in I$ then the image \bar{e}_I is on less the e_I in each coordinate. Combining this with the definition of e_I gives the expression in the statement of the theorem.

It should be noted at this point that either the entries in the normal vector of a facet of a nestohedron with a connected building set are all in $\{0, 1\}$ or they are all in the set $\{-1, 0\}$. \square

Since non-connected building sets are the unions of connected components, in these cases it is clear to see that π is the composite of π in the connected case for each component.

We shall now look at what happens to the characteristic function of a facet, F , of an n -dimensional nestohedron, P , with connected building set B .

Since F is a face of P , which has a canonical characteristic function and so a canonical quasi-toric manifold, there exists the facial sub-manifold of F . The facial sub-manifold is a quasi-toric manifold with characteristic function given as follows: we let V be the supporting hyperplane containing F . A facet, H , of F is a codimension two face of P that is the intersection of F with an adjacent facet G . Since P is Delzant, the normal vector to H can then be taken to be the projection of v_G , the normal vector to G , into the plane V . We shall denote this characteristic function $\Lambda_P|_F$. These characteristic functions can be related by the following theorem.

Theorem 4.5.3. For a face F of a nestohedron P we have that the sets of combinatorial data (F, Λ_F) and $(F, \Lambda_P|_F)$ are equivalent.

Proof. Let S be the element of B corresponding to F as in the proof of 4.4.2. Without loss of generality we can renumber the base set so that $S = \{1, \dots, i\}$, which gives us $v_F = (1, \dots, 1, 0, \dots, 0)$, the vector consisting of 1 in the first i positions and 0 thereafter. We apply a change of basis, π , to \mathbb{Z}^n which forces the i -th basis vector to be v_F and leaves the rest as the standard basis vectors. These actions do not affect the characteristic function up to equivalence. The projection into V with respect to this new basis is then simply given by dropping the i -th coordinate.

This construction means that for the nestohedron, F , we now have two ways to construct a characteristic function: firstly as a facet of P and secondly as a nestohedron in its own right. We will now see that these characteristic functions are equivalent.

Let us consider H as before, the facet of F given by its intersection with G some adjacent facet of P . We let $T = \{t_1, \dots, t_j\}$ be the element of B which corresponds to G .

By theorem 4.3.3 we know that H corresponds to a nested set that contains the nested sets corresponding to F and G . Thus we know that $\{S, T, [n+1]\}$ is a nested set, so in particular either $S \subset T$ or $T \subset S$ or $S \cap T = \emptyset$ with $S \cup T \notin B$. This means

that without loss of generality we can apply another renumbering of the base set so that $n + 1 \notin T$ and if $T \subset S$ we have that $i \notin T$.

We also have that H is a facet of F , the nestohedron that is the product of the connected nestohedra $P_{B|S}$ and P_{B-S} . A facet in the product is the product of a facet of one and the whole of the other. We know which one it is since by the proof of 4.4.2, H is the set of points maximising the sums $\sum_k x_{s_k}$ and $\sum_k x_{t_k}$. From this we can tell that if $S \subset T$ or $S \cap T = \phi$ then H is the product of $P_{B|S}$ and a facet of P_{B-S} and that facet is labelled by $T - S$. Alternatively if $T \subset S$ then H is the product a facet of $P_{B|S}$ and of P_{B-S} and that facet is labelled by T .

We can now calculate the normal vector to H as a facet of F by each method, in each of the three cases: when $S \subset T$, when $T \subset S$ and when $S \cap T = \phi$ with $S \cup T \notin B$.

If $S \subset T$ then $T = \{1, \dots, i, t_{i+1}, \dots, t_j\}$. Considering the normal vector given by the facial submanifold construction, v_G is given by the vector (v_1, \dots, v_n) where $v_k = 1$ if $k \in T$ and 0 otherwise. After applying the change of basis π this vector becomes $(\bar{v}_1, \dots, \bar{v}_n)$ where $\bar{v}_k = 1$ if $k = i$ or $k > i$ and $k \in T$ and 0 otherwise. Dropping the i -th coordinate then gives the vector $(\tilde{v}_1, \dots, \tilde{v}_n)$ where $\tilde{v}_k = 1$ if $k \geq i$ and $k + 1 \in T$ and 0 otherwise.

Now considering H as a facet of the nestohedron F , we see that the first $i - 1$ coordinates are the vector for S in a connected building set $B|S$, so are all 0. The remaining coordinates are the vector for the facet labelled by $T - S$ in $B - S$. Combining these two gives us the vector (v_1, \dots, v_n) where $v_k = 1$ if $k \geq i$ and $k + 1 \in T$.

If $T \subset S$, then in particular T contains no elements greater than or equal to i . Considering the vector given by the facial submanifold construction, v_G is given by the vector (v_1, \dots, v_n) where $v_k = 1$ if $k \in T$ and 0 otherwise. After applying the change of basis π this vector is unchanged. Dropping the i -th coordinate then gives the vector $(\tilde{v}_1, \dots, \tilde{v}_n)$ where $\tilde{v}_k = 1$ if $k \in T$ and 0 otherwise.

Now considering H as a facet of the nestohedron F , we see that the first $i - 1$ coordinates are the vector for T in a connected building set on $B|S$, since $i \notin T$ that

is given by $v_k = 1$ if $k \in T$ and 0 otherwise. The remaining coordinates are the vector for the facet labelled by $[n + 1] - S$ in $B - S$ which are all 0. Combining these gives us the normal vector (v_1, \dots, v_n) where $v_k = 1$ if $k \geq i$ and $k + 1 \in T$.

If $S \cap T = \emptyset$ with $S \cup T \notin B$, then in particular T contains no elements less than or equal to i . Considering the vector given by the facial submanifold construction, v_G is given by the vector (v_1, \dots, v_n) where $v_k = 1$ if $k \in T$ and 0 otherwise. After applying the change of basis π this vector is unchanged. Dropping the i -th coordinate then gives the vector $(\tilde{v}_1, \dots, \tilde{v}_n)$ where $\tilde{v}_k = 1$ if $k \geq i$ and $k + 1 \in T$ and 0 otherwise.

Now considering H as a facet of the nestohedron F , we see that the first $i - 1$ coordinates are the vector for S in a connected building set on $B|S$, so are all 0. The remaining coordinates are the vector for the facet labelled by T in $B - S$, which combined give us the vector (v_1, \dots, v_n) where $v_k = 1$ if $k \geq i$ and $k + 1 \in T$.

In each of the three cases above the vector given for any facet of F by the two independent constructions of combinatorial data are identical. Thus the two constructions give the same vector for any facet of F and we have seen that for F a facet of P , the characteristic functions Λ_F and $\Lambda_P|_F$ differ only by some changes of coordinates. Thus (F, Λ_F) and $(F, \Lambda_P|_F)$ are equivalent.

We have seen this is true for facets of a nestohedron, so inductively it then holds for faces of any codimension. \square

This theorem leads us to a corollary about the quasi-toric manifolds.

Corollary 4.5.4. Let F be a face of a nestohedron P , then there is an inclusion

$$M_F \hookrightarrow M_P$$

where M_F is the quasi-toric manifold with combinatorial data (F, Λ_F) and M_P is the quasi-toric manifold with combinatorial data (P, Λ_P)

Proof. Since M_P is a quasi-toric manifold we have an embedding of the facial submanifold of F into M_P . The facial submanifold of F is a quasi-toric manifold with combinatorial data $(F, \Lambda_P|_F)$. Since $(F, \Lambda_P|_F)$ is equivalent to (F, Λ_F) there is a weakly equivariant homeomorphism between the facial submanifold of F and M_F . Composing the homeomorphism and the embedding gives the map we require. \square

4.6 Bigraded Betti Numbers

To finish this chapter we will take a look at another combinatorial invariant of nestohedra, bigraded Betti numbers. These were not covered in chapter 3 because they will only feature in this section.

Bigraded Betti numbers are a way of looking at the information encoded in the Stanley-Reisner ring of a polytope.

Definition 4.6.1. [9] The *bi-graded Betti numbers* $\beta^{-i,2j}(P)$ of a polytope P^n are defined by

$$\beta^{-i,2j}(P) := \dim_k \operatorname{Tor}_{k[v_1, \dots, v_m]}^{-i, 2j}(k(P), k)$$

where $k(P)$ is the Stanley-Reisner ring of P .

We are able to use the index $2j$ in this expression because as $\deg(v_i) = 2$ for all i , we know that for odd j we have $\dim_k \operatorname{Tor}_{k[v_1, \dots, v_m]}^{-i, j}(k(P), k) = 0$.

There are two tools which help us calculate the bigraded Betti numbers of polytopes. Firstly we have a theorem which relates bigraded Betti numbers to the homology groups of subsets of its facets.

Theorem 4.6.2. [9, 3.27] We have, for a polytope P with m numbered facets

$$\beta^{-i, 2j}(P) = \sum_{\sigma \subset [m]: |\sigma|=j} \deg_k \tilde{H}_{j-i-1}(P_\sigma)$$

where P_σ is the cell complex consisting of the facets which are in σ . We assume $\tilde{H}_{-1}(\phi) = k$.

The other tool we have at our disposal relates the bigraded Betti numbers to the moment angle complex which is defined by the polytope. This is one reason the bigraded Betti numbers are of particular interest in toric topology.

Theorem 4.6.3. [9, 7.6] We have an isomorphism of graded algebras

$$H^*(\mathcal{Z}_P) = \operatorname{Tor}_{k[v_1, \dots, v_m]}(k(P), k).$$

Since the right hand side is bigraded, we can use this to define a bigrading on the left hand side.

What then can we say about a nestohedron, P^n , with connected building set B ? We can look at the cell complexes P_σ from theorem 4.6.2. We will now demonstrate that we can write any P_σ as $Pe_{\bar{\sigma}}^n$, which will reduce the problem of calculating the bigraded Betti numbers of any nestohedron to that of calculating the bigraded Betti numbers of the permutohedron. For the remainder of this section we will assume $P^n \neq Pe^n$, as the result is trivial in that case.

We recall that there is a direct way to label the facets of P by the elements of $B - [n + 1]$ from the proof of 4.4.2. We will denote by F_ω the facet labelled by ω .

Under this labelling we notice that for the permutohedron the incidence relation between two facets is that two facets, F_ω and F_τ , of Pe^n have non-empty intersection if and only if $\omega \subset \tau$ or $\tau \subset \omega$. We can extend this to our general nestohedron P on B . Two facets, F_ω and F_τ , of P have non-empty intersection if $\omega \subset \tau$, $\tau \subset \omega$ or $v = \omega \cup \tau$ is not in B .

Using this we can define a map from cell complexes of facets of P to cell complexes of facets of Pe^n by $P_\sigma \mapsto Pe_{\bar{\sigma}}^n$ where

$$\bar{\sigma} = \sigma \cup \{v \in [n + 1] : v \notin B, \omega \subset v \exists \omega \in \sigma\}.$$

Theorem 4.6.4. We have that $\tilde{H}_{j-i-1}(P_\sigma) = \tilde{H}_{j-i-1}(Pe_{\bar{\sigma}}^n)$.

Proof. In order to prove this we will construct a homeomorphism between the spaces P_σ and $Pe_{\bar{\sigma}}^n$ using the construction of the representatives of nestohedra given by 4.1.3.

We recall that a k -simplex can be defined to be the points $t_1v_1 + \dots + t_kv_k$ where $\sum_{i=1}^k t_i = 1$ and the t_i are non-negative. We can then naturally define the centre of the simplex as the point where $t_1 = \dots = t_k$. This extends to give the centre of a nestohedron as the Minkowski sum of the centres of the simplices which are used to construct the nestohedron. We can then translate P and Pe^n so that their centres coincide.

To get our map from P_σ to $Pe_{\bar{\sigma}}^n$, we project, by π , the points in P_σ outwards from the centre onto Pe^n , which by construction is necessarily larger than P as it includes more summands. We have that $\pi(P_\sigma) \subset Pe_{\bar{\sigma}}^n$ and that every facet in $Pe_{\bar{\sigma}}^n$ contains a point in $\pi(P_\sigma)$. We can now smoothly pull the image out so it fits the boundaries of

any facet if appears in, since if two adjacent facets contain a point in $\pi(P_\sigma)$ then so does the common face between them.

Thus we have the required homeomorphism and $\tilde{H}_{j-i-1}(P_\sigma) = \tilde{H}_{j-i-1}(Pe_{\bar{\sigma}}^n)$. \square

Chapter 5

Families of Polytopes

We ended the Motivation chapter by posing four questions to which we would like to find answers. In this chapter we will examine an answer to the first and second questions:

Question 1. Can we find families of polytopes that can underlie quasi-toric manifolds with similar properties?

Question 2. What machinery can we develop to study the combinatorics of such families?

In order to do this we will make a formal definition and then work on extending the machinery developed in chapter 3 to the new context in which we are working. We will also give some examples of families and look at what data it is interesting to know about them. The nestohedra we have seen in the previous chapter will feature heavily in the examples of families we see here.

5.1 Families

In this section we will be answering the first of the two questions we are looking at in this chapter. The question asks for families of polytopes that can underlie quasi-toric manifolds with similar properties. In order to answer this we should look in more detail at what we mean by similar properties.

Our examples of quasi-toric manifolds with similar properties were the motivating example, complex projective spaces and Bott towers. For linguistic simplicity we shall refer to these as families. We notice that all three have precisely one quotient space in each dimension. This is because while the Bott towers are not limited to a single quasi-toric manifold in each dimension, all those of dimension $2n$ have the same quotient space. We also observe that members of each family appear as face manifolds of higher dimensional members of the same family: Bott towers as face manifolds of Bott towers, complex projective spaces as face manifolds of complex projective space and so on.

This last observation is actually more connected with combinatorial data than polytopes alone. However when we are dealing with Delzant polytopes and their canonical characteristic functions, this condition can be restated as: members of a family appear as facets of higher dimensional members of that family and the characteristic function as a facet agrees with the canonical characteristic function. We notice that for nestohedra the second part of this condition holds automatically from 4.5.3.

We can thus make a formal definition of a *family of polytopes* which takes into account these observations. Because we are dealing with polytopes rather than combinatorial data, we drop the condition about characteristic functions.

Definition 5.1.1. A *family of polytopes*, $\Psi = \{P^n\}_{n \in \mathbb{N}}$, is a collection of polytopes, one in each dimension, such that P^{n+1} has a facet combinatorially equivalent to P^n for all $n \geq 1$.

From now on we will regard \mathbb{N} as including 0, making it the set of possible dimensions of a polytope. It is worth noting that inductively a polytope $P^n \in \Psi$ contains all lower dimensional members of Ψ as faces. We will refer to polytopes in a family as *members* of the family. An easy way to check whether a set of polytopes, $\Psi = \{P^n\}_{n \in \mathbb{N}}$, is a family is to find a formula for $d(P^n)$ in terms of n and check that it contains P^{n-1} .

Let us now look for some examples. Our first examples of families will be the

ones underlying the families of quasi-toric manifolds that motivated our definition, the simplices (complex projective spaces) and the cubes (Bott towers).

Example 5.1.2 (Simplices). We will denote by $\Delta = \{\Delta^n\}_{n \in \mathbb{N}}$ the family consisting of all simplices. It is a family because we have the formula $d(\Delta^n) = (n+1)\Delta^{n-1}$, so Δ^n has a facet combinatorially equivalent to Δ^{n-1} .

Example 5.1.3 (Cubes). We will denote by $I = \{I^n\}_{n \in \mathbb{N}}$ the family consisting of all cubes. It is a family because we have the formula $d(I^n) = 2nI^{n-1}$, so I^n has a facet combinatorially equivalent to I^{n-1} .

Looking for further examples of families we will look at the examples of nestohedra we discussed in the last chapter: Examples 4.2.3 to 4.2.6 turn out to be families.

Example 5.1.4 (Associahedra). The associahedra from example 4.2.3 form a family which we will denote by $As = \{As^n\}_{n \in \mathbb{N}}$. We know it is a family because we have the formula

$$d(As^n) = \sum_{i+j=n-1} (n-i)As^i \times As^j$$

from example 4.4.7, which in particular contains As^{n-1} when $i = 0$ and $j = n - 1$ for any n .

Example 5.1.5 (Permutohedra). The permutohedra from example 4.2.4 form a family which we will denote by $Pe = \{Pe^n\}_{n \in \mathbb{N}}$. We know it is a family because we have the formula

$$d(Pe^n) = \sum_{i+j=n-1} \binom{n+1}{i+1} Pe^i \times Pe^j$$

from example 4.4.6, which in particular contains Pe^{n-1} when $i = 0$ and $j = n - 1$ for any n .

Example 5.1.6 (Cyclohedra). The cyclohedra from example 4.2.5 form a family which we will denote by $Cy = \{Cy^n\}_{n \in \mathbb{N}}$. We know it is a family because we have the formula

$$d(Cy^n) = \sum_{i+j=n-1} (n+1)As^i \times Cy^j$$

from example 4.4.8, which in particular contains Cy^{n-1} when $i = 0$ and $j = n - 1$ for any n .

Example 5.1.7 (Stellohedra). The stellohedra from example 4.2.6 form a family which we will denote by $St = \{St^n\}_{n \in \mathbb{N}}$. We know it is a family because we have the formula

$$d(St^n) = n \cdot St^{n-1} + \sum_{i=0}^{n-1} \binom{n}{i} St^i \times Pe^{n-i-1}$$

from example 4.4.9, which in particular contains St^{n-1} when $i = n - 1$ for any n .

Families do not have to be nestohedra, although this construction has yielded useful examples so far. Indeed we can find a family which includes any polytope.

Example 5.1.8. For a polytope P^n we can choose a face of each dimension i less than n , which we will call Q^i , in such a way that

$$Q^0 \subset Q^1 \subset \dots \subset Q^{n-1}.$$

Then the polytopes $Q^0, Q^1, \dots, Q^{n-1}, P^n, P^n \times I, P^n \times I^2, \dots$ form a family which contains P^n . The inclusion property for polytopes of dimension greater than n follows from the fact that facets of a product $A \times B$ are of the form $A \times F$ with F a facet of B or $F \times B$ with F a facet of A .

However, such families are not the most interesting examples. For a more complicated example that is not made up of nestohedra, or indeed simple polytopes, we will look at the permuto-associahedron that can be found in section 9.3 of [30].

Example 5.1.9. The n -dimensional permuto-associahedron is a polytope that has face poset corresponding to the partial bracketings of orderings of $n + 1$ distinct numbers. An explicit construction is given in [30]. These polytopes are not nestohedra, nor are they simple. However for any n it is clear that the n dimensional permuto-associahedron has a facet combinatorially equivalent to $n - 1$ dimensional permuto-associahedron, so these polytopes do form a family.

Now that we have a definition of a family of polytopes, we have provided an answer to question 1, which asked us to find families of polytopes. In order to answer

question 2, and develop machinery to study the combinatorics of families, we need to fit these families into the existing machinery that we have for studying polytopes. We shall follow two approaches in order to accomplish this task.

5.2 Family Subrings

The first approach to studying families we shall consider is to look at a family as a subset of the ring of polytopes \mathcal{P} . This approach gives us natural extensions of several definitions.

Definition 5.2.1. A family Ψ has a property **A** if and only if every polytope in Ψ has property **A**.

In particular this definition holds for being simple, simplicial, flag and flag simple. It also holds for being a family of nestohedra. However it is important to note that this does not extend the definitions of the invariant polynomials of section 3.4 to families.

When looking at families as subsets of the differential graded ring \mathcal{P} , or indeed the smaller differential graded rings \mathcal{P}_S , \mathcal{P}_N and \mathcal{P}_G , we notice that the families themselves are not subrings. It would be useful to have differential graded subrings corresponding in some way to families. As such we shall define:

Definition 5.2.2. For a family, Ψ , the *family subring*, $R(\Psi)$, is the subring generated by elements of Ψ under the operations $+$, \times and d . It is smallest subring of \mathcal{P} which contains every polytope in the family and is closed under d .

We will normally express a family subring as

$$R(\Psi) = \mathbb{Z}[a_i, i = 1, \dots]$$

where the a_i generate $R(\Psi)$ under $+$ and \times only. By definition the members of Ψ are included amongst the a_i . We will call the a_i the *generators* of the family subring.

Since \mathcal{P}_S , \mathcal{P}_N and \mathcal{P}_G are closed under the differential operator, it is useful to note that if $\Psi \subset \mathcal{P}_S$ then $R(\Psi) \subset \mathcal{P}_S$, if $\Psi \subset \mathcal{P}_N$ then $R(\Psi) \subset \mathcal{P}_N$ and if $\Psi \subset \mathcal{P}_G$ then $R(\Psi) \subset \mathcal{P}_G$. We shall refer to a family as being in \mathcal{P} , \mathcal{P}_S , \mathcal{P}_N or \mathcal{P}_G as appropriate.

We saw at the end of the last section that “the associahedra and cyclohedra” is not a family, but “the associahedra” and “the cyclohedra” are families. However by examining the formulas for the differential operators it is easy to see that there is a strong relationship between the two families, since the associahedra are the only other polytopes that appear in the formula which gives the differential operator of the cyclohedra. Such a relationship does not exist between the cyclohedra and, for example, the permutohedra. The family subrings clearly show these relationships since $R(As) \subset R(Cy)$ but $R(Pe) \not\subset R(Cy)$.

Let us now examine some of the families that we have seen so far. It is useful to note for clarity that when we say “the subring generated by the family” we mean $\mathbb{Z}[\psi : \psi \in \Psi]$ and not the family subring. This is an important distinction as $\mathbb{Z}[\psi : \psi \in \Psi]$ is by definition a subset of $R(\Psi)$, but it is not necessarily closed under d .

Example 5.2.3 (Cubes). First we will look at the family I of cubes. We recall that the formula for the boundary is that

$$d(I^n) = (2n)I^{n-1}.$$

This means that the subring generated by the family is closed under d and so this is the family subring. In this case however the members of the family can all be generated by the single element I^1 under multiplication, so

$$R(I) = \mathbb{Z}[I^1].$$

Example 5.2.4 (Simplices). Secondly we will look at the family Δ of simplices. We recall that the formula for the boundary is that

$$d(\Delta^n) = (n + 1)\Delta^{n-1}.$$

Again this means that the subring generated by the the family is closed under d and so the family subring is

$$R(\Delta) = \mathbb{Z}[\Delta^n : n \in \mathbb{N}].$$

Example 5.2.5 (Permutohedra). The family Pe of permutohedra has the formula

$$d(Pe^n) = \sum_{i+j=n-1} \binom{n+1}{i+1} Pe^i \times Pe^j$$

for its boundaries. Again this means that the subring generated by the family is closed under d and so the family subring is

$$R(Pe) = \mathbb{Z}[Pe^n : n \in \mathbb{N}].$$

Example 5.2.6 (Associahedra). The family As of associahedra has the formula

$$d(As^n) = \sum_{i+j=n-1} (n-i+1) As^i \times As^j$$

for its boundaries. Again this means that the subring generated by the family is closed under d and so the family subring is

$$R(As) = \mathbb{Z}[As^n : n \in \mathbb{N}].$$

Up to now we have only seen examples where the subring of the family is the same as the subring generated by the family. Now we have two example where that is not the case.

Example 5.2.7 (Cyclohedra). The family Cy of cyclohedra had the formula

$$d(Cy^n) = \sum_{i+j=n-1} (n+1) As^i \times Cy^j$$

for its boundaries. This means that the subring generated by the family is not closed under d , since it also contains associahedra. Consequently the subring of the family is

$$R(Cy) = R(As)[Cy^n : n = 2, 3, \dots].$$

Example 5.2.8 (Stellohedra). The family St of stellohedra had the formula

$$d(St^n) = n \cdot St^{n-1} + \sum_{i=0}^{n-1} \binom{n}{i} St^i \times Pe^{n-i-1}$$

for its boundaries. This means that the subring generated by the family is not closed under d , since it also contains permutohedra. Consequently the subring of the family is

$$R(St) = R(Pe)[St^n : n = 2, 3, \dots]$$

In three of these examples the family subring is generated by the members of the family, in one the family subring is finitely generated and in the final two there are generators that are not part of the family. Two of these types of family subring are of interest.

Proposition 5.2.9. A family subring will be finitely generated if and only if the members of the family are all products of a finite number of polytopes.

Proof. Let Ψ be a family where the members of the family are all products of a finite number of polytopes. We will call these polytopes $A_i^{n_i}$. We consider the subring, R , generated by the terms $B_{ij} = d^j(A_i^{n_i})$ for $j = 0, \dots, n$. Since the faces of a product are the products of faces of the factors, we know that $R(\Psi)$ is contained in R . Furthermore we know that the polytopes B_{ij} are contained in $R(\Psi)$ so we know that $R(\Psi) = R$ and so $R(\Psi)$ must be finitely generated.

For the reverse implication let Ψ be a family with a finitely generated family subring. Certainly the family subring contains all the elements of the family, so the family is finitely generated under addition and multiplication. However each element is a connected polytope and so cannot be generated as the formal sum of two polytopes. Hence the members of the family are all products of a finite number of polytopes. \square

The other type of family subring that is of special interest here are those family subrings that are generated by precisely the members of the family.

Definition 5.2.10. A family which has family subring that is generated by precisely the members of the family will be called an *independent family*.

The reason these families are of special interest is illustrated by the examples we saw earlier: the two family subrings that are not independent both contained family subrings that are independent. In the next section we will see why this type of family is both useful and important.

5.3 Generating Functions

The other approach we will take to study families will use those parts of chapter 3 that are not utilised in section 5.2. Here we will focus on extending the various polynomials that were developed for polytopes to families.

Since many results about polytopes are stated in terms of their f -polynomials, or indeed their f -vectors, it is then sensible to try to produce a concept of the f -polynomial of a family. This will have the effect of simplifying proofs regarding families. Instead of having to prove that a property holds for many polynomials we will merely have to prove that the property holds for just one polynomial. Naturally, the same logic applies to results stated in terms of other invariants. Consequently we would also like to find a single version of the other invariants for entire families.

In [5] there are six “specifically chosen generating series” for collections of polytopes which are families, although the notion of a family does not appear. In this section we will generalise this notion to all families. The generating series are used to calculate f - and h -polynomials by methods which we will generalise to become 5.3.12 and 5.3.9. Following this example, we will begin by replacing the family with a single formal series of polytopes.

Definition 5.3.1. For a family of polytopes, Ψ , we define the *generating function* as the formal power series

$$\Psi(x) := \sum_{n=0}^{\infty} s_n P^n x^{n+q}$$

in $\mathcal{P} \otimes \mathbb{Q}[[x]]$. In this series, the parameters $s_n \in \mathbb{Q}$ and $q \in \mathbb{N}$ are chosen appropriately for the family Ψ in question, in order to simplify later equations.

The purpose of s_n and q is “to simplify later equations”; To see why we do this, we will of course need to define those equations. However, by asserting that these parameters will depend on the boundary operator, d , we can now give two immediate examples of generating functions.

Example 5.3.2. We shall first consider the family I of cubes. We recall that we have the formula $d(I^n) = 2nI^{n-1}$ for the boundary of the n -cube. Because of this

formula we will choose $s_n = 1/n!$ and set q to be 0. These choices give us $\sum I^n x^n / n!$ as the generating function. This example is included in [5].

Example 5.3.3. We shall also look the family Δ of simplices. We recall that we have the formula $d(\Delta^n) = (n+1)\Delta^{n-1}$ for the boundary of the n -simplex. Because of this formula we will choose $s_n = 1/(n+1)!$ and set q to be 1. These choices give us $\sum \Delta^n x^{n+1} / (n+1)!$ as the generating function. This example is included in [5].

It will not always be possible to choose parameters based on a general formula for $d(P^n)$, but this will always be our preferred method.

The advantage that generating functions have over families is that we can extend the machinery that exists on \mathcal{P} by linearity to $\mathcal{P} \otimes \mathbb{Q}[[x]]$. We will do this explicitly here for the derivative d , the f -, h -, γ - and F -polynomials. The extensions of d , and the f -, h - and γ -polynomials were used in [5].

Definition 5.3.4. The boundary operator $d : \mathcal{P} \rightarrow \mathcal{P}$ defines a map $d : \mathcal{P} \otimes \mathbb{Q}[[x]] \rightarrow \mathcal{P} \otimes \mathbb{Q}[[x]]$ which replaces any polytope P^n by $d(P^n)$.

In particular for a family with generating function, $\Psi(x)$, we will call the series

$$d\Psi(x) = \sum_{n=0}^{\infty} s_n d(P^n) x^{n+q}$$

the *differential series* of Ψ .

Definition 5.3.5. The f -polynomial defines a map $f : \mathcal{P} \otimes \mathbb{Q}[[x]] \rightarrow \mathbb{Q}[\alpha, t][[x]]$ which replaces any polytope P^n by its f -polynomial $f(P^n)$. For Ξ an element of $\mathcal{P} \otimes \mathbb{Q}[[x]]$, we will use the notation Ξ_f for $f(\Xi)$.

In particular for a family with generating function, $\Psi(x)$, we will call the series

$$\Psi_f(\alpha, t, x) = \sum_{n=0}^{\infty} s_n f(P^n)(\alpha, t) x^{n+q}$$

the *f-series* of Ψ .

Definition 5.3.6. The h -polynomial defines a map $h : \mathcal{P} \otimes \mathbb{Q}[[x]] \rightarrow \mathbb{Q}[\alpha, t][[x]]$ which replaces any polytope P^n by its h -polynomial $h(P^n)$. For Ξ an element of $\mathcal{P} \otimes \mathbb{Q}[[x]]$, we will use the notation Ξ_h for $h(\Xi)$.

In particular for a family with generating function, $\Psi(x)$, we will call the series

$$\Psi_h(\alpha, t, x) = \sum_{n=0}^{\infty} s_n h(P^n)(\alpha, t) x^{n+q}$$

the *h-series* of Ψ .

Definition 5.3.7. The γ -polynomial defines a map $\gamma : \mathcal{P} \otimes \mathbb{Q}[[x]] \rightarrow \mathbb{Q}[\tau][[x]]$ which replaces any polytope P^n by its γ -polynomial $\gamma(P^n)$. For Ξ an element of $\mathcal{P} \otimes \mathbb{Q}[[x]]$, we will use the notation Ξ_γ for $\gamma(\Xi)$.

In particular for a family with generating function, $\Psi(x)$, we will call the series

$$\Psi_\gamma(\alpha, t, x) = \sum_{n=0}^{\infty} s_n \gamma(P^n)(\tau) x^{n+q}$$

the *γ -series* of Ψ .

Definition 5.3.8. The F -polynomial defines a map $F : \mathcal{P} \otimes \mathbb{Q}[[x]] \rightarrow \mathbb{Q}[t][[x]]$ which replaces any polytope P^n by its F -polynomial $F(P^n)$. For Ξ an element of $\mathcal{P} \otimes \mathbb{Q}[[x]]$, we will use the notation Ξ_F for $F(\Xi)$.

In particular for a family with generating function, $\Psi(x)$, we will call the series

$$\Psi_F(\alpha, t, x) = \sum_{n=0}^{\infty} s_n F(P^n)(t) x^{n+q}$$

the *F-series* of Ψ .

Now that we have extended the f -, h -, γ - and F -polynomials to our generating functions we should look at what we can say about them. We will look at the results in section 3.4 and restate them in terms of generating functions.

Proposition 5.3.9. [5] We have that the change of coordinates that transforms the f -series into the h -series is

$$\Psi_f(\alpha, t, x) = \Psi_h(\alpha - t, t, x).$$

Proof. We know that this coordinate change converts $f(P^n)$ into $h(P^n)$ for any P^n by 3.4.5. By comparing coefficients of x we see that this holds for families. \square

Proposition 5.3.10. We have that the change of coordinates that transforms the h -series into the γ -series is

$$a^q \Psi_\gamma(\tau, z) = \Psi_h(\alpha, t, x)$$

for $a = \alpha + t$, $b = \alpha t$, $\tau = \frac{b}{a^2}$ and $z = ax$.

Proof. We have from 3.4.7 that $h(P^n)(\alpha, t) = a^n \gamma(P)(\tau)$. If we then rewrite the γ -polynomial for the family in terms of x we get

$$\Psi_\gamma(\tau, z) = \sum_{n=0}^{\infty} s_n \gamma(P^n)(\tau) (ax)^{n+q}$$

which gives the required result by again comparing coefficients of x . \square

We will now look at three results concerning families in \mathcal{P}_S .

Theorem 5.3.11 (Dehn-Sommerville relations). For a family, Ψ , of simple polytopes

$$\Psi_h(\alpha, t, x) = \Psi_h(t, \alpha, x).$$

Proof. We know that this holds for each polytope in Ψ by theorem 3.4.6. By comparing coefficients of x we see that this holds for families. \square

Theorem 5.3.12. For a family, Ψ , of simple polytopes

$$(d\Psi)_f(\alpha, t, x) = \frac{\partial}{\partial t} \Psi_f(\alpha, t, x).$$

Proof. We know that this holds for each polytope in Ψ by theorem 3.4.2. By comparing coefficients of x we see that this holds for families. \square

Theorem 5.3.13. For a family, Ψ , of simple polytopes

$$(d\Psi)_F(t, x) = \frac{\partial}{\partial t} \Psi_F(t, x).$$

Proof. We know that this holds for each polytope in Ψ by theorem 3.4.10. By comparing coefficients of x we see that this holds for families. \square

If we can calculate $d\Psi(x)$ for a family then we can use theorems 5.3.12 and 5.3.13 to calculate $\Psi_f(\alpha, t, x)$ and $\Psi_F(t, x)$ respectively. This is because we have well defined initial conditions for the appropriate differential equations.

Theorem 5.3.14. The initial conditions for the equation

$$(d\Psi)_f(\alpha, t, x) = \frac{\partial}{\partial t} \Psi_f(\alpha, t, x)$$

which defines $\Psi_f(\alpha, t, x)$ are given by

$$\Psi_f(\alpha, 0, x) = \sum_{n=0}^{\infty} s_n \alpha^n x^{n+q}.$$

Proof. We know that for any n -dimensional polytope, P^n , the f -polynomial has the form $\alpha^n + t(\phi(\alpha, t))$ for some function $\phi(\alpha, t)$ because P^n has precisely one n -dimensional face, P^n . Thus when $t = 0$ we have $f(P^n) = \alpha^n$. Applying this to each element of Ψ gives the required series. \square

Theorem 5.3.15. The initial conditions for the equation

$$(d\Psi)_F(t, x) = \frac{\partial}{\partial t} \Psi_F(t, x)$$

which defines $\Psi_F(t, x)$ are given by

$$\Psi_F(0, x) = \Psi(x).$$

Proof. We know from the definition of the F -polynomial that for any polytope, P^n , we have $F(P^n)(0) = P^n$. Combining this with the generating function gives

$$\begin{aligned} \Psi_F(0, x) &= \sum_{n=0}^{\infty} s_n F(P^n)(0) x^{n+q} \\ &= \sum_{n=0}^{\infty} s_n P^n x^{n+q} \\ &= \Psi(x) \end{aligned}$$

as required. \square

In order to use these partial differential equations to calculate the f - and F -series we must know the f - or F - polynomials of any polytope which appears in $d\Psi(x)$. Inductively we see that in order to use this method, we need to know the f - or F - polynomials for any polytope in the family subring that is not in the family. Since the family subrings of independent families contain only polytopes which are

products and sums of members of the family, we will be able to use this method for independent families without needing to use outside data. It is for this reason that independent families are important.

Finally we will look at what generating functions can tell us about the Gal conjecture. To do this we will rewrite it in terms of generating functions. We recall that the Gal conjecture is

Conjecture 5.3.16 (Gal '05). For any flag simple polytope, P , the γ -series $\gamma(P)$ has non-negative coefficients.

In order to restate this conjecture we shall make two definitions. Firstly we shall fix a grading on the ring $\mathbb{Q}[\alpha, t][[x]]$ where $\deg(1) = 0$, $\deg(\alpha) = \deg(t) = -2$ and $\deg(x) = 2$. Then, with $a = \alpha + t$ and $b = \alpha t$ as before,

Definition 5.3.17. A *Gal series* in $\mathbb{Q}[\alpha, t][[x]]$ is an element $\psi(\alpha, t, x)$, such that,

1. $\psi(\alpha, t, x) = \psi(t, \alpha, x) = \widehat{\psi}(a, b, x)$.
2. ψ is homogenous under the above grading.
3. $\widehat{\psi}(a, b, x)$ has all non-negative coefficients.

We note that by the nature of the h -series, $\Psi_h(\alpha, t, x)$ satisfies the first two conditions for any family of simple polytopes. We can now restate the Gal conjecture for families.

Conjecture 5.3.18 (Gal conjecture for families). For any family of flag simple polytopes, Ψ , we have that $\Psi_h(\alpha, t, x)$ is a Gal series.

To show that $\Psi_h(\alpha, t, x)$ is a Gal series we will employ the following theorem.

Theorem 5.3.19. Let Ψ be a family of polytopes. If $\Psi_h(\alpha, t, x)$ is such that:

- $\frac{\partial \Psi_h(\alpha, t, x)}{\partial x} \Big|_{x=0}$ is a Gal series
- $\frac{\partial \Psi_h(\alpha, t, x)}{\partial x}$ is a homogeneous polynomial

$$F(a, b, \Psi_h(\alpha, t, x), S_{1,h}(\alpha, t, x), \dots, S_{k,h}(\alpha, t, x)) \in \mathbb{Z}[\alpha, t][[x]]$$

with non-negative coefficients, where the $S_{i,h}(x)$, for $i = 1, \dots, k$, are Gal series.

Then $\Psi_h(\alpha, t, x)$ is a Gal series and the polytopes in Ψ satisfy the Gal conjecture.

Proof. Let

$$\frac{\partial \Psi_h}{\partial x}(\alpha, t, x, y) = F(a, b, \Psi_h(\alpha, t, x, y), S_{i,h}(\alpha, t, x)).$$

Then by applying the standard substitutions that give the γ -series we have that

$$\frac{\partial \Psi_\gamma}{\partial z}(\tau, z) = F(1, \tau, \Psi_\gamma(\tau, z), S_{i,\gamma}(\tau, z)). \quad (5.1)$$

We have

$$\begin{aligned} \Psi_\gamma(\tau, z) &= \sum_{n=0}^{\infty} s_n \gamma(\Psi^n) z^{n+q} \\ S_{i,\gamma}(\tau, z) &= \sum_{n=0}^{\infty} S_i^n(\alpha, t) z^n \\ \frac{\partial \Psi_\gamma}{\partial z}(\tau, z) &= \sum_{n \in \mathbb{N}} (n+q) s_n \gamma(\Psi^n) z^{n+q-1} \end{aligned}$$

where $s(\Psi^n)$ are known and positive and $S_i^n(\alpha, t)$ is a homogeneous symmetric polynomial.

Examining equation (5.1) term by term in z gives an identity for $\gamma(\Psi^n)$. This identity is expressed as a polynomial with non-negative coefficients in variables τ , $\gamma(\Psi^m)$ for $m < n$ and S_i^m , $i = 1, \dots, k$ for $m \leq n$. Since $S_{i,\gamma}^m$ has non-negative coefficients of τ , we get that $\gamma(\Psi^n)$ will have all non-negative co-efficient of τ , if $\gamma(\Psi^m)$ has all non-negative coefficients of τ for all $m < n$.

Since $\frac{\partial \Psi_h}{\partial x}|_{x=0}$ is Gal, so is $\frac{\partial \Psi_\gamma}{\partial x}|_{x=0} = \gamma(\Psi^1)$. Thus, by induction, $\gamma(\Psi^n)$ has non-negative coefficients for all n . Consequently $\Psi_\gamma(\tau, z) = \hat{\Psi}_h(a, b, x)$ has all non-negative coefficients and $\Psi_h(\alpha, t, x, y)$ meets the third property for being a Gal series. Since it is the h -series of $\Psi(x, y)$, it automatically meets the other two conditions and so is a Gal series. \square

Chapter 6

Calculating Combinatorial Invariants

We ended the Motivation chapter by posing four questions to which we would like to find answers. In this chapter we will attempt to answer to the final question:

Question 4. Can we calculate the combinatorial invariants of these polytopes?

The polytopes in question were the polytopes with canonical combinatorial data we constructed to answer the third question. In chapter 4 we constructed nestohedra to have this property. Since we have seen in chapter 5 that we can form families of polytopes we will instead answer the larger question:

Question 6. Can we calculate the combinatorial invariants of these families of nestohedra?

In chapter 5, we developed the machinery to be able to answer this question, generating functions. In this chapter we will use this machinery to calculate combinatorial invariants of the families of nestohedra that we have seen so far. We will also introduce some further families and perform these calculations on those families. The combinatorial invariants we will calculate will be the f -, h - and F -series and we will show that the γ -series satisfy the Gal conjecture. We will start with some general results about families of nestohedra.

6.1 Families of Nestohedra

In this section we will look at what we know about families of nestohedra in general. We will start with a consequence of theorem 4.4.2, about the building sets which define families of nestohedra.

Theorem 6.1.1. Let $\{B_n\}_{n \in \mathbb{N}}$ be a collection of connected building sets each on $[n + 1]$. Then the nestohedra $P^n = P_{B_n}$ form a family if and only if B_{n-1} is the connection of $B_n|S$ and $B_n - S$ for some S in B_n .

Proof. If the condition on the building set is met then by the definition of connection in the proof of 4.3.18 we know that $P^{n-1} = P_{B_n|S} \times P_{B_n-S}$ which by theorem 4.4.2 is a facet of P^n . Then there is exactly one polytope in each dimension and P^{n-1} is combinatorially equivalent to a facet of P^n for all n so these polytopes are a family.

For the converse, we know from the definition of a family that P^{n-1} is a facet of P^n and from theorem 4.4.2 that facets of P^n have the form $P_{B_n|S} \times P_{B_n-S}$ for some S in B_n . Thus P^{n-1} is $P_{B_n|S} \times P_{B_n-S}$ for some S in B_n . From the proof of 4.3.18 we know that the connection of $B_n|S$ and $B_n - S$ is the connected building set which gives this nestohedron, so B_{n-1} is of this form. \square

We can formulate a corollary of this about graphical nestohedra.

Corollary 6.1.2. If $\{\Gamma_n\}_{n \in \mathbb{N}}$ is a ordered set of simple graphs on $n + 1$ nodes such that Γ_{n+1} can be obtained from Γ_n by the addition of one node and arcs connected to that node then the polytopes P_{Γ_n} form a family of graphical nestohedra.

Proof. We label the additional node that takes us from Γ_n to Γ_{n+1} as $n + 2$, then the building set $B(\Gamma_n)$ is the connection of $B(\Gamma_{n+1})|S$ and $B(\Gamma_{n+1}) - S$ for some $S = [n + 1]$ in $B(\Gamma_{n+1})$ because $B(\Gamma_{n+1}) - S = \{\{n + 2\}\}$. By the previous theorem, the polytopes form a family. \square

Staying with graphical nestohedra we can show that the number of independent families in \mathcal{P}_G is two.

Theorem 6.1.3. The only families in \mathcal{P}_G which are independent are Pe and As .

Proof. We have seen that these two families have this property.

For a family to meet the conditions of this theorem, the family subring must have at most one 2-dimensional generator. We saw in section 4.2 that there are only four 2-dimensional nestohedra; P_3^2 , P_4^2 , P_5^2 and P_6^2 . We have that P_3^2 is not a graphical nestohedron, P_4^2 is I^2 the product of two 1-dimensional graphical nestohedra, P_5^2 is the graphical nestohedron produced from the line graph on 3 nodes and P_6^2 is the graphical nestohedron produced from the complete graph on 3 nodes. Thus the 2-dimensional generator must be either P_5^2 or P_6^2 .

We will now use 4.4.3 to look at the boundaries of higher dimensional members of the family, Ψ . While we don't know that the graphs in Ψ are connected we know there is a graph, Γ with n -nodes amongst the components of a nestohedra of dimension greater than or equal to n in Ψ , because otherwise there is no generator of dimension n . Furthermore the nestohedron with graph Γ must be in $R(\Psi)$.

We know that the facets of the nestohedra with graph Γ , have the form $P(\Gamma_G) \times P(\bar{\Gamma}_{G^c})$ for some $G \subset [n+1]$. In order for there to be only one 2-dimensional generator then graphs of the form Γ_G and $\bar{\Gamma}_{G^c}$ with three nodes must all be either the line graph on 3 nodes or the complete graph on 3 nodes. Furthermore this must remain true after repeated applications of d .

Let us examine two cases, the case when Γ contains a node of order greater than or equal to three and the case when it does not. If Γ does not contain a node of order greater than or equal to three then the highest order a node in Γ has is two and Γ is either the line graph on n nodes or the cyclic graph on n nodes. We have seen that the cyclic graph on n nodes does not meet the conditions on subgraphs and the line graph does so Γ must be the line graph on n nodes.

If Γ does not contain a node of order greater than or equal to three then we will fix the label of this node as 1 and three of the connected nodes as 2, 3 and 4. We know that $\bar{\Gamma}_{\{2,3,4\}}$ is the complete graph on 3 nodes and so every subgraph of 3 nodes must be the complete graph. Since Γ is connected we can show by induction that the distance between any two nodes is 1 and thus that Γ must be the complete graph on n nodes.

Thus we have that the n -dimensional generator of the family subring is either As^n or Pe^n for all n . Since the generators are also the members of the family, the family is either Pe or As . \square

This theorem allows us to consider both the permutohedra and associahedra as limit cases for graphical nestohedra.

In the following sections we will look at some examples of families of nestohedra and their generating functions, beginning with the six major examples of families we have seen before; I , Δ , Pe , As , Cy and St . These are also the six families for which “specially chosen generating series” were chosen in [5], which used the formula for $d(P^n)$ obtained from theorem 4.4.2 and presented in examples 4.4.4 through 4.4.9. In [5] the generating functions for these families are given, the series $d\Psi(x)$ are found and the f - and h -series are calculated. We shall now cover these examples in detail.

6.2 Cubes

We shall begin with the family of cubes. In the last chapter we fixed the generating function $I(x) = \sum I^n x^n / n!$. We can then give a formula for $dI(x)$.

Theorem 6.2.1. [5] The relation $dI(x) = 2xI(x)$ holds.

Proof. We have seen before that $d(I^n) = 2nI^{n-1}$, thus we have that

$$\begin{aligned} dI(x) &= \sum 2nI^{n-1} \frac{x^n}{n!} \\ &= \sum 2xI^{n-1} \frac{x^{n-1}}{(n-1)!} \\ &= 2xI(x), \end{aligned}$$

as required. \square

We can now calculate the f -series of I .

Corollary 6.2.2. [5] We have that $I_f(\alpha, t, x) = e^{(\alpha+2t)x}$

Proof. Using theorems 5.3.12 and 5.3.14 and the relation $dI(x) = 2xI(x)$ we have that $I_f(\alpha, t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} I_f(\alpha, t, x) = 2xI_f(\alpha, t, x)$$

with the initial conditions $I_f(\alpha, 0, x) = e^{\alpha x}$. Solving this differential equation yields the result. \square

From this we can calculate the h -series using proposition 5.3.9.

Corollary 6.2.3. [5] The h -series of I is

$$I_h(\alpha, t, x) = e^{(\alpha+t)x}.$$

We can also calculate the F -series of I .

Corollary 6.2.4. We have that $I_F(t, x) = e^{2tx}I(x)$

Proof. Using theorems 5.3.13 and 5.3.15 and the relation $dI(x) = 2xI(x)$ we have that $I_F(t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} I_F(t, x) = 2xI_F(t, x)$$

with the initial conditions $I_F(0, x) = I(x)$. Solving this differential equation yields the result. \square

We have that I is a family of flag polytopes so we can check that it satisfies the Gal conjecture for families.

Theorem 6.2.5. We have that $I_h(\alpha, t, x)$ is a Gal series.

Proof. We will use theorem 5.3.19. For that we must find $\frac{\partial I_h(\alpha, t, x)}{\partial x}$. We have that $I_h(\alpha, t, x) = e^{(\alpha+t)x}$, so

$$\frac{\partial I_h(\alpha, t, x)}{\partial x} = (\alpha + t)e^{(\alpha+t)x} = aI_h(\alpha, t, x).$$

Which meets the second condition of theorem 5.3.19. We also have $\frac{\partial I_h(\alpha, t, x)}{\partial x}|_{x=0} = a$ which is a Gal Series so the first condition of theorem 5.3.19 is met and $I_h(\alpha, t, x)$ is a Gal series. \square

6.3 Simplices

Next we look at Δ , the family of simplices. In the last chapter we fixed the generating function $\sum \Delta^n x^{n+1}/(n+1)!$. We can then give a formula for $d\Delta(x)$.

Theorem 6.3.1. [5] The relation $d\Delta(x) = x\Delta(x)$ holds.

Proof. We have seen before that $d(\Delta^n) = (n+1)\Delta^{n-1}$, thus we have that

$$\begin{aligned} d\Delta(x) &= \sum (n+1)\Delta^{n-1} \frac{x^{n+1}}{(n+1)!} \\ &= \sum x\Delta^{n-1} \frac{x^n}{n!} \\ &= x\Delta(x), \end{aligned}$$

as required. □

We can now calculate the f -series of Δ .

Corollary 6.3.2. [5] We have that $\Delta_f(\alpha, t, x) = \frac{1}{\alpha}(e^{(\alpha+t)x} - e^{tx})$

Proof. Using theorems 5.3.12 and 5.3.14 and the relation $d\Delta(x) = x\Delta(x)$ we have that $\Delta_f(\alpha, t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} \Delta_f(\alpha, t, x) = x\Delta_f(\alpha, t, x)$$

with the initial conditions $\Delta_f(\alpha, 0, x) = \frac{1}{\alpha}(e^{\alpha x} - 1)$. Solving this differential equation yields the result. □

From this we can calculate the h -series using proposition 5.3.9.

Corollary 6.3.3. [5] The h -series of Δ is

$$\Delta_h(\alpha, t, x) = \frac{(e^{\alpha x} - e^{tx})}{\alpha - t}.$$

We can also calculate the F -series of Δ .

Corollary 6.3.4. We have that $\Delta_F(t, x) = e^{tx}\Delta(x)$

Proof. Using theorems 5.3.13 and 5.3.15 and the relation $d\Delta(x) = 2x\Delta(x)$ we have that $\Delta_F(t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t}\Delta_F(t, x) = x\Delta_F(t, x)$$

with the initial conditions $\Delta_F(0, x) = \Delta(x)$. Solving this differential equation yields the result. \square

We know that Δ is not a family of flag polytopes, so we will not look at whether it satisfies the Gal conjecture or not.

6.4 Permutohedra

Next we look at Pe , the family of permutohedra. We recall from example 4.4.6 that we have the formula

$$d(Pe^n) = \sum_{i+j=n-1} \binom{n+1}{i+1} Pe^i \times Pe^j.$$

Bearing this in mind we will set the generating function to be

$$Pe(x) = \sum_{n=0}^{\infty} Pe^n \frac{x^{n+1}}{(n+1)!}$$

We can then give a formula for $dPe(x)$.

Theorem 6.4.1. [5] The relation $dPe(x) = (Pe(x))^2$ holds.

Proof. Using the above formula we have that

$$\begin{aligned} dPe(x) &= \sum_{n=0}^{\infty} \left(\sum_{i+j=n-1} \binom{n+1}{i+1} Pe^i \times Pe^j \right) \frac{x^{n+1}}{(n+1)!} \\ &= Pe(x)^2 \end{aligned}$$

as required. \square

We can now calculate the f -series of Pe .

Corollary 6.4.2. [5] We have that $Pe_f(\alpha, t, x) = \frac{e^{\alpha x} - 1}{\alpha - t(e^{\alpha x} - 1)}$.

Proof. Using theorems 5.3.12 and 5.3.14 and the relation $dPe(x) = (Pe(x))^2$, we have that $Pe_f(\alpha, t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} Pe_f(\alpha, t, x) = (Pe_f(\alpha, t, x))^2$$

with the initial conditions $Pe_f(\alpha, 0, x) = \frac{1}{\alpha}(e^{\alpha x} - 1)$. Solving this differential equation yields the result. \square

From this we can calculate the h -series using proposition 5.3.9.

Corollary 6.4.3. [5] The h -series of Pe is

$$Pe_h(\alpha, t, x) = \frac{e^{\alpha x} - e^{tx}}{\alpha e^{tx} - t e^{\alpha x}}.$$

We can also calculate the F -series of Pe .

Corollary 6.4.4. We have that $Pe_F(t, x) = \frac{Pe(x)}{1 - tPe(x)}$

Proof. Using theorems 5.3.13 and 5.3.15 and the relation $dPe(x) = (Pe(x))^2$ we have that $Pe_F(t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} Pe_F(t, x) = (Pe_F(t, x))^2$$

with the initial conditions $Pe_F(0, x) = Pe(x)$. Solving this differential equation yields the result. \square

We have that Pe is a family of flag polytopes so we can check that it satisfies the Gal conjecture for families.

Theorem 6.4.5. We have that $Pe_h(\alpha, t, x)$ is a Gal series.

Proof. We use theorem 5.3.19. For that we must find $\frac{\partial Pe_h(\alpha, t, x)}{\partial x}$. We have that $Pe_h(\alpha, t, x) = \frac{e^{\alpha x} - e^{tx}}{\alpha e^{tx} - t e^{\alpha x}}$, so

$$\frac{\partial Pe_h(\alpha, t, x)}{\partial x} = e^{(\alpha+t)x} \left(\frac{(\alpha - t)}{\alpha e^{tx} - t e^{\alpha x}} \right)^2,$$

let us call the expression in the brackets $P(\alpha, t, x)$. We have from 5.3.19 that $Pe_h(\alpha, t, x)$ is a Gal series if $P(\alpha, t, x)$ is and the initial conditions are. The initial conditions are

$$\frac{\partial Pe_h(\alpha, t, x)}{\partial x} \Big|_{x=0} = \left(\frac{\alpha - t}{\alpha - t} \right)^2 = 1$$

which is a Gal series. All that then remains is to show that $P(\alpha, t, x)$ is a Gal Series, for which we will use 5.3.19 again. We have that

$$\begin{aligned}\frac{\partial P(\alpha, t, x)}{\partial x} &= \frac{-(\alpha - t)\alpha t(e^{tx} - e^{\alpha x})}{(\alpha e^{tx} - t e^{\alpha x})^2} \\ &= \alpha t P(\alpha, t, x) P e_h(\alpha, t, x).\end{aligned}$$

with initial conditions

$$\left. \frac{\partial P(\alpha, t, x)}{\partial x} \right|_{x=0} = \frac{-(\alpha - t)\alpha t(1 - 1)}{(\alpha - t)^2} = 0$$

which gives us that $P(\alpha, t, x)$ is a Gal series if $P e_h(\alpha, t, x)$ is a Gal series.

This presents us with a problem, in that we cannot use 5.3.19 for either series without the other. However, proceeding inductively on each series in turn, as we did for a single series in the proof of 5.3.19, we can show that both series are indeed Gal series. In particular $P e_h(\alpha, t, x)$ is a Gal series as required. \square

6.5 Stellohedra

Next we look at St , the family of stellohedra. We recall from example 4.4.9 that we have the formula

$$d(St^n) = n \cdot St^{n-1} + \sum_{i=0}^{n-1} \binom{n}{i} St^i \times P e^{n-i-1}.$$

Bearing this in mind we will set the generating function to be

$$St(x) = \sum_{n=0}^{\infty} St^n \frac{x^n}{n!}$$

We can then give a formula for $dSt(x)$.

Theorem 6.5.1. [5] The relation $dSt(x) = (x + P e(x))St(x)$ holds.

Proof. Using the above formula we have that

$$\begin{aligned}dSt(x) &= \sum_{n=0}^{\infty} \left(n \cdot St^{n-1} + \sum_{i=0}^{n-1} \binom{n}{i} St^i \times P e^{n-i-1} \right) \frac{x^n}{n!} \\ &= (x + P e(x)) St(x).\end{aligned}$$

as required. \square

We can now calculate the f -series of St . Since St is not an independent family, its family subring contains Pe , we will need to recall that $Pe_f(\alpha, t, x) = \frac{e^{\alpha x} - 1}{\alpha - t(e^{\alpha x} - 1)}$.

Corollary 6.5.2. [5] We have that $St_f(\alpha, t, x) = e^{(\alpha+t)x} \frac{\alpha}{\alpha - t(e^{\alpha x} - 1)}$

Proof. Using theorems 5.3.12 and 5.3.14 and the relation $dSt(x) = (x + Pe(x))St(x)$, we have that $St_f(\alpha, t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} St_f(\alpha, t, x) = (x + Pe_f(\alpha, t, x))St_f(\alpha, t, x)$$

with the initial conditions $St_f(\alpha, 0, x) = e^{\alpha x}$. Solving this differential equation yields the result. \square

From this we can calculate the h -series using proposition 5.3.9.

Corollary 6.5.3. [5] The h -series of St is

$$St_h(\alpha, t, x) = e^{(\alpha+t)x} \frac{\alpha - t}{\alpha e^{tx} - t e^{\alpha x}}.$$

We can also calculate the F -series of St .

Corollary 6.5.4. We have that $St_F(t, x) = St(x) \frac{e^{tx}}{1 - tPe(x)}$

Proof. Using theorems 5.3.13 and 5.3.15 and the relation $dSt(x) = (x + Pe(x))St(x)$, we have that $St_F(t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} St_F(t, x) = (x + Pe_F(t, x))St_F(t, x)$$

with the initial conditions $St_F(0, x) = St(x)$. Solving this differential equation yields the result. \square

We have that St is a family of flag polytopes so we can check that it satisfies the Gal conjecture for families.

Theorem 6.5.5. We have that $St_h(\alpha, t, x)$ is a Gal series.

Proof. We will use lemma 5.3.19 for this. We begin by calculating the partial derivative of the h -series, which we found earlier, with respect to x .

$$\begin{aligned} St_h(\alpha, t, x) &= e^{(\alpha+t)x} \frac{(\alpha - t)}{\alpha e^{tx} - t e^{\alpha x}} \\ \frac{\partial}{\partial x} St_h(\alpha, t, x) &= (\alpha + t) e^{(\alpha+t)x} \frac{(\alpha - t)}{\alpha e^{tx} - t e^{\alpha x}} + e^{(\alpha+t)x} \frac{(\alpha - t)(\alpha e^{tx} - t e^{\alpha x})}{(\alpha e^{tx} - t e^{\alpha x})^2} \\ &= (\alpha + t) St_h(\alpha, t, x) + \alpha t Pe_h(\alpha, t, x) St_h(\alpha, t, x). \end{aligned}$$

since $Pe_h(\alpha, t, x)$ is a Gal series, $St_h(\alpha, t, x)$ fits the conditions of 5.3.19 if and only if $St_h(\alpha, 0, x)$ has non-negative coefficients.

$$St_h(\alpha, 0, x) = \sum_n \alpha^n \frac{x^n}{n!} = e^{\alpha x},$$

which does have non-negative coefficients so $St_h(\alpha, t, x)$ is a Gal series. □

6.6 Associahedra

Next we look at As , the family of associahedra. We recall from example 4.4.7 that we have the formula

$$d(As^n) = \sum_{i+j=n-1} (n - i + 1) As^i \times As^j.$$

Bearing this in mind we will set the generating function to be

$$As(x) = \sum_{n=0}^{\infty} As^n x^{n+2}$$

We can then give a formula for $dAs(x)$.

Theorem 6.6.1. [5] The relation $dAs(x) = As(x) \frac{\partial}{\partial x} As(x)$ holds.

Proof. Using the above formula we have that

$$\begin{aligned} dAs(x) &= \sum_{n=0}^{\infty} \left(\sum_{i+j=n-1} (n - i + 1) As^i \times As^j \right) x^{n+2} \\ &= \sum_{n=0}^{\infty} \sum_{i+j=n-1} As^i x^{i+2} \times (j + 2) As^j x^{j+1} \\ &= As(x) \frac{\partial}{\partial x} As(x) \end{aligned}$$

as required. □

At this point we will quote two results about the standard solution to a well known partial differential equation, the Hopf equation.

Lemma 6.6.2. [8, 2.1] The solution to the Hopf equation

$$U_t = UU_x$$

with initial conditions $U(0, x) = \varphi(x)$ is the solution to

$$U = \varphi(x + tU).$$

Lemma 6.6.3. [8] By using conservation laws we can rewrite the solution to the Hopf equation $U_t = UU_x$ with initial conditions $U(0, x) = \varphi(x)$ in the alternative form of

$$U(t, x) = \sum_{i \geq 0} \frac{1}{(i+1)!} t^i \frac{d^i}{dx^i} (\varphi(x)^{i+1}).$$

We can now calculate the f -series of As .

Theorem 6.6.4. [5] We have that $As_f(\alpha, t, x) = U$ where U is the solution of $(x + tU)(x + (\alpha + t)U) = U$.

Proof. Using theorems 5.3.12 and 5.3.14 and the relation $dAs(x) = As(x) \frac{\partial}{\partial x} As(x)$, we have that $As_f(\alpha, t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} As_f(\alpha, t, x) = As_f(\alpha, t, x) \frac{\partial}{\partial x} As_f(\alpha, t, x)$$

with the initial conditions $As_f(\alpha, 0, x) = \frac{x^2}{1-\alpha x}$.

This quasilinear differential equation is identifiable as the Hopf equation. We can therefore use lemma 6.6.2 to show that $As_f(\alpha, t, x)$ solves the equation

$$As_f(\alpha, t, x) = \frac{(x + tAs_f(\alpha, t, x))^2}{1 - \alpha(x + tAs_f(\alpha, t, x))}$$

This statement can be rewritten as

$$As_f(\alpha, t, x) = x^2 + (2tx + \alpha x)As_f(\alpha, t, x) + t(t + \alpha)As_f(\alpha, t, x)$$

which factorizes as required. □

Alternatively, we can use lemma 6.6.3 to find the f -series of As .

Theorem 6.6.5. We have that

$$As_f(\alpha, t, x) = \sum_{n \geq 0} \sum_{i=0}^n \sum_{j=0}^k \frac{(i+j+1)!(2i+2)!n!}{(i+1)!j!(i-j)!(i+j)!(i+j)!(n-(i+j))!} \alpha^{n-i} t^i x^{n+2}$$

where $k = \min\{i, n-i\}$.

Proof. As before we have that $As_f(\alpha, t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} As_f(\alpha, t, x) = As_f(\alpha, t, x) \frac{\partial}{\partial x} As_f(\alpha, t, x)$$

with the initial conditions $As_f(\alpha, 0, x) = \frac{x^2}{1-\alpha x}$. From 6.6.3 we have that

$$As_f(\alpha, t, x) = \sum_{i \geq 0} \frac{1}{(i+1)!} t^i \frac{\partial^i}{\partial x^i} (\varphi(\alpha, x)^{i+1}).$$

where $\varphi(\alpha, x) = \frac{x^2}{1-\alpha x}$. We can calculate $\frac{\partial^i}{\partial x^i} (\varphi(\alpha, x)^{i+1})$ as follows

$$\begin{aligned} \frac{\partial^i}{\partial x^i} \left(\left(\frac{x^2}{1-\alpha x} \right)^{i+1} \right) &= \sum_{j=0}^i \binom{i}{j} \left(\frac{\partial^{i-j}}{\partial x^{i-j}} x^{2i+2} \right) \left(\frac{\partial^j}{\partial x^j} (1-\alpha x)^{-(i+1)} \right) \\ &= \sum_{j=0}^i \binom{i}{j} \left(\frac{(2i+2)!}{(i+j)!} x^{i+j+2} \right) \left(\frac{(i+j+1)! \alpha^j}{i!(1-\alpha x)^{i+j+1}} \right) \\ &= \sum_{j=0}^i \alpha^j x^{i+j+2} \frac{(i+j+1)!(2i+2)!}{j!(i-j)!(i+j)!} (1-\alpha x)^{-(i+j+1)} \end{aligned}$$

We can use the binomial theorem to give us

$$\begin{aligned} &= \sum_{j=0}^i \alpha^j x^{i+j+2} \frac{(i+j+1)!(2i+2)!}{j!(i-j)!(i+j)!} \sum_{l \geq 0} \binom{i+j+l}{i+j} (\alpha x)^l \\ &= \sum_{j=0}^i \sum_{l \geq 0} \frac{(i+j+1)!(2i+2)!(i+j+l)!}{j!(i-j)!(i+j)!l!} \alpha^{j+l} x^{i+j+l+2} \end{aligned}$$

We can put this back in to get

$$\begin{aligned} As_f(\alpha, t, x) &= \sum_{i \geq 0} \frac{1}{(i+1)!} t^i \sum_{j=0}^i \sum_{l \geq 0} \frac{(i+j+1)!(2i+2)!(i+j+l)!}{j!(i-j)!(i+j)!l!} \alpha^{j+l} x^{i+j+l+2} \\ &= \sum_{i \geq 0} \sum_{j=0}^i \sum_{l \geq 0} \frac{(i+j+1)!(2i+2)!(i+j+l)!}{(i+1)!j!(i-j)!(i+j)!l!} \alpha^{j+l} t^i x^{i+j+l+2} \end{aligned}$$

using the substitution $n = i + j + l$ gives, with $k = \min\{i, n-i\}$

$$As_f(\alpha, t, x) = \sum_{n \geq 0} \sum_{i=0}^n \sum_{j=0}^k \frac{(i+j+1)!(2i+2)!n!}{(i+1)!j!(i-j)!(i+j)!(i+j)!(n-(i+j))!} \alpha^{n-i} t^i x^{n+2}$$

as required. \square

From this we can calculate the h -series using proposition 5.3.9.

Corollary 6.6.6. We have that $As_h(\alpha, t, x) = U$ where U is the solution of $(x + tU)(x + \alpha U) = U$.

Proof. This expression is obtained by applying proposition 5.3.9 to the expression for $As_f(\alpha, t, x)$ given by theorem 6.6.4. \square

Corollary 6.6.7. The h -series, $As_h(\alpha, t, x)$, of As is

$$\sum_{n \geq 0} \sum_{p=0}^n \sum_{i=0}^{n-p} \sum_{j=0}^k \frac{(i+j+1)!(2i+2)!n!(n-i)!}{(i+1)!j!(i-j)!(i+j)!^2(n-(i+j))!(n-i-p)!p!} \alpha^p t^{n-p} x^{n+2}$$

where $k = \min\{i, n-i\}$.

Proof. This formula is obtained by applying proposition 5.3.9 to the formula

$$As_f(\alpha, t, x) = \sum_{n \geq 0} \sum_{i=0}^n \sum_{j=0}^k \frac{(i+j+1)!(2i+2)!n!}{(i+1)!j!(i-j)!(i+j)!^2(n-(i+j))!} \alpha^{n-i} t^i x^{n+2}$$

where $k = \min\{i, n-i\}$ given by theorem 6.6.5, giving

$$\begin{aligned} & \sum_{n \geq 0} \sum_{i=0}^n \sum_{j=0}^k \frac{(i+j+1)!(2i+2)!n!}{(i+1)!j!(i-j)!(i+j)!^2(n-(i+j))!} (\alpha - t)^{n-i} t^i x^{n+2} \\ &= \sum_{n \geq 0} \sum_{i=0}^n \sum_{j=0}^k \frac{(i+j+1)!(2i+2)!n!}{(i+1)!j!(i-j)!(i+j)!^2(n-(i+j))!} \sum_{p=0}^{n-i} \binom{n-i}{p} \alpha^p t^{n-p} x^{n+2} \\ &= \sum_{n \geq 0} \sum_{i=0}^n \sum_{j=0}^k \sum_{p=0}^{n-i} \frac{(i+j+1)!(2i+2)!n!(n-i)!}{(i+1)!j!(i-j)!(i+j)!^2(n-(i+j))!(n-i-p)!p!} \alpha^p t^{n-p} x^{n+2} \\ &= \sum_{n \geq 0} \sum_{p=0}^n \sum_{i=0}^{n-p} \sum_{j=0}^k \frac{(i+j+1)!(2i+2)!n!(n-i)!}{(i+1)!j!(i-j)!(i+j)!^2(n-(i+j))!(n-i-p)!p!} \alpha^p t^{n-p} x^{n+2} \end{aligned}$$

as required. \square

We can also calculate the F -series of As .

Theorem 6.6.8. We have that

$$As_F(t, x) = \sum_{i, j \geq 0} \frac{1}{(i+1)!} t^i \frac{(j+2i+2)!}{(j+i+2)!} \sum_{l_1 + \dots + l_{i+1} = j} As^{l_1} \dots As^{l_{i+1}} x^{j+i+2}.$$

Proof. Using theorems 5.3.13 and 5.3.15 and the relation $dAs(x) = As(x) \frac{\partial}{\partial x} As(x)$ we have that $As_F(t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} As_F(t, x) = As_F(t, x) \frac{\partial}{\partial x} As_F(t, x)$$

with the initial conditions $As_F(0, x) = As(x)$. Solving this differential equation yields the result, it is the Hopf equation again so we use the above lemmas once more.

From lemma 6.6.2 we have that $As_F(t, x)$ solves the equation

$$As_F(t, x) = As(x + tAs_F(t, x)).$$

from lemma 6.6.3 we get that

$$As_F(t, x) = \sum_{i \geq 0} \frac{1}{(i+1)!} t^i \frac{d^i}{dx^i} (As(x)^{i+1}).$$

Since $As(x)$ is known and a simple function, this expression can be identified as

$$As_F(t, x) = \sum_{i, j \geq 0} \frac{1}{(i+1)!} t^i \frac{(j+2i+2)!}{(j+i+2)!} \sum_{l_1 + \dots + l_{i+1} = j} As^{l_1} \dots As^{l_{i+1}} x^{j+i+2}.$$

□

By feeding the expression for $As_F(t, x)$ back into the differential equation and equating coefficients of x we get the following identity as a corollary.

Corollary 6.6.9. For any ordered sequence of positive integers k_1, \dots, k_n with $\sum_{i=1}^j k_i = m_j$ and $m_n = m$ we have the identity

$$\sum_{i=1}^{n1} \frac{1}{i} \binom{m_i + 2i}{i-1} \binom{m - m_i + 2(r-i+1)}{r-i+1} = \frac{n}{m+2s+3} \binom{m+2n+3}{n+1}.$$

We have that As is a family of flag polytopes so we can check that it satisfies the Gal conjecture for families.

Theorem 6.6.10. We have that $As_h(\alpha, t, x)$ is a Gal series.

Proof. We will use theorem 5.3.19. For that we must find $\frac{\partial As_h(\alpha, t, x)}{\partial x}$. We have from 6.6.6 that $As_h(\alpha, t, x)$ solves the equation $(x+tU)(x+\alpha U) = U$. Differentiating this

with respect to x gives

$$\begin{aligned} \frac{\partial}{\partial x} ((x + tU)(x + \alpha U)) &= \frac{\partial}{\partial x} U \\ \frac{\partial}{\partial x} (x^2 + x(\alpha + t)U + \alpha tU^2) &= \frac{\partial}{\partial x} U \\ 2x + (\alpha + t) \left(U + x \frac{\partial}{\partial x} U \right) + 2\alpha tU \frac{\partial}{\partial x} U &= \frac{\partial}{\partial x} U \\ 2x + (\alpha + t)U &= \frac{\partial}{\partial x} U - (x(\alpha + t) + 2\alpha tU) \frac{\partial}{\partial x} U \\ 2x + (\alpha + t)U &= \frac{\partial}{\partial x} U (1 - (x(\alpha + t) + 2\alpha tU)) \\ (2x + (\alpha + t)U) (1 - (x(\alpha + t) + 2\alpha tU))^{-1} &= \frac{\partial}{\partial x} U. \end{aligned}$$

We can then apply a binomial expansion to give

$$\frac{\partial}{\partial x} U = (2x + \alpha U) \left(\sum_{k=0}^{\infty} (\alpha x + 2tU)^k \right)$$

which meets the conditions of theorem 5.3.19. We must now check that $\frac{\partial As_h(\alpha, t, x)}{\partial x} |_{x=0}$ is a Gal series. This time we differentiate the formula for $As_h(\alpha, t, x)$ from corollary 6.6.7, giving $\frac{\partial}{\partial x} As_h(\alpha, t, x)$ as

$$\sum_{n \geq 0} \sum_{p=0}^n \sum_{i=0}^{n-p} \sum_{j=0}^k \frac{(n+2)(i+j+1)!(2i+2)!n!(n-i)!}{(i+1)!j!(i-j)!(i+j)!^2(n-(i+j))!(n-i-p)!p!} \alpha^p t^{n-p} x^{n+1}.$$

When $x = 0$ this expression is 0. This means that both conditions of theorem 5.3.19 are met and $As_h(\alpha, t, x)$ is a Gal series. \square

6.7 Cyclohedra

As the last of the six examples in [5] we look at Cy , the family of cyclohedra. We recall from example 4.4.7 that we have the formula

$$d(Cy^n) = \sum_{i+j=n-1} (n+1)As^i \times Cy^j.$$

Bearing this in mind we will set the generating function to be

$$Cy(x) = \sum_{n=0}^{\infty} Cy^n \frac{x^{n+1}}{n+1}.$$

We can then give a formula for $dAs(x)$.

Theorem 6.7.1. [5] The relation $dCy(x) = As(x)\frac{\partial}{\partial x}Cy(x)$ holds.

Proof. Using the above formula we have that

$$\begin{aligned} dCy(x) &= \sum_{n=0}^{\infty} \left(\sum_{i+j=n-1} (n+1)As^i \times Cy^j \right) \frac{x^{n+1}}{n+1} \\ &= \sum_{n=0}^{\infty} \sum_{i+j=n-1} As^i x^{i+2} \times Cy^j x^j \\ &= As(x)\frac{\partial}{\partial x}Cy(x) \end{aligned}$$

as required. □

As we did for the associahedra we will now quote a result about the standard solution to a particular partial differential equation.

Lemma 6.7.2. [5, 20] The solution to the equation

$$\frac{\partial}{\partial t}U = V\frac{\partial}{\partial x}U$$

where V is a known solution of the Hopf equation and U has known initial conditions $U(0, x) = \varphi(x)$, is the solution to

$$U = \varphi(x + tV).$$

We can now calculate the f -series of Cy . As was the case with the stellohedra the family of cyclohedra is not independent as the ring $R(Cy)$ contains the associahedra. We therefore need to recall the results from the previous section.

Theorem 6.7.3. [5] We have that $Cy_f(\alpha, t, x) = U$ where U is the solution of $e^{-\alpha U} = 1 - \alpha(x + tAs_f(\alpha, t, x))$.

Proof. Using theorems 5.3.12 and 5.3.14 and the relation $dCy(x) = As(x)\frac{\partial}{\partial x}Cy(x)$, we have that $Cy_f(\alpha, t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t}Cy_f(\alpha, t, x) = As_f(\alpha, t, x)\frac{\partial}{\partial x}Cy_f(\alpha, t, x)$$

with the initial conditions

$$Cy_f(\alpha, 0, x) = \sum_{n=0}^{\infty} \alpha^n \frac{x^{n+1}}{n+1} = -\frac{1}{\alpha} \ln(1 - \alpha x).$$

Solving this differential equation will yield the result.

We use lemma 6.7.2 to show that $Cy_f(\alpha, t, x)$ solves the equation

$$Cy_f(\alpha, t, x) = -\frac{1}{\alpha} \ln(1 - \alpha(x + tAs_f(\alpha, t, x)))$$

This statement can be rewritten as

$$e^{-\alpha Cy_f(\alpha, t, x)} = 1 - \alpha(x + tAs_f(\alpha, t, x)).$$

Due to the convoluted nature of $As_f(\alpha, t, x)$ we will leave our expression for $Cy_f(\alpha, t, x)$ in this form. \square

From this we can calculate the h -series using proposition 5.3.9.

Corollary 6.7.4. The h -series of Cy is U where U is the solution of $e^{(t-\alpha)U} = 1 - (\alpha - t)(x + tAs_h(\alpha, t, x))$.

We can also calculate the F -series of Cy .

Theorem 6.7.5. We have that

$$Cy_F(t, x) = Cy(x + tAs_F(t, x)).$$

Proof. Using theorems 5.3.13 and 5.3.15 and the relation $dCy(x) = Cy(x) \frac{\partial}{\partial x} Cy(x)$ we have that $Cy_F(t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} Cy_F(t, x) = As_F(t, x) \frac{\partial}{\partial x} Cy_F(t, x)$$

with the initial conditions $Cy_F(0, x) = Cy(x)$. Solving this differential equation yields the result. Again we use lemma 6.7.2 we have that $Cy_F(t, x)$ solves the equation

$$Cy_F(t, x) = Cy(x + tAs_F(t, x))$$

as required. \square

6.8 Multi-Parameter Families

So far, we have used families which are controlled by a single index, the dimension of the polytope. However we will now define a generalisation of this notion, *multi-parameter families*.

Definition 6.8.1. A *multi-parameter family of polytopes* is a collection of polytopes at least one in each dimension, defined by a set of integer parameters (i_1, \dots, i_k) such that any path in the parameter space which starts at $(0, \dots, 0)$ and at each step precisely one parameter increases by one, defines a family of polytopes.

We will call the families defined by paths in the parameter space *subfamilies*. The parameter space will generally be \mathbb{N}^k , possibly with some restrictions. These restriction will generally be about when a parameter is allowed to be 0. We will allow repetitions of polytopes within these families to simplify the restrictions placed on the parameters.

The family subring of a multi-parameter family is defined in exactly the same way as for a standard family. However, we must make a change to the definition of generating function for the wider definition of multi-parameter families.

Definition 6.8.2. For a multi-parameter family of polytopes, Ψ , with k parameters, we define the *generating function* as the formal power series

$$\Psi(x) := \sum_{n=0}^{\infty} s_{i_1, \dots, i_k} P^n x_1^{i_1+q_1} \dots x_k^{i_k+q_k}$$

in $\mathcal{P} \otimes \mathbb{Q}[[x_1, \dots, x_k]]$. In this series, the parameters $s_{i_1, \dots, i_k} \in \mathbb{Q}$ and $q_1, \dots, q_k \in \mathbb{N}$ are chosen appropriately to simplify later equations.

Despite the change to the definition of generating function all the results we developed in section 5.3 still hold.

What we need now is some examples of multi-parameter families to work on. For the rest of the chapter we will look at two multi-parameter families of nestohedra that are given by the joins of graphs.

Definition 6.8.3. For two graphs X^m and Y^n we obtain the *join*, $X + Y = \Gamma_{X,m,Y,n}$, from $X \cup Y$ by adding in all edges between X and Y . We shall denote the resultant nestohedra as $P_{X,m,Y,n}$.

The two multi-parameter families we will look at will be denoted by $P_{:,:,}$ and $P_{\nabla, :,}$.

We will have $P_{:,:,}$ to be the multi-parameter family defined by the complete bipartite graph, the join of two groups of nodes with no edges between nodes in the same group. The parameters for this family are the number of nodes in each group and the restrictions are that no group may be empty if the other contains more than one node. We will denote $K_{m,n}$ by $\Gamma_{:,m, :,n}$ for notational consistency and the related nestohedra by $P_{:,m, :,n}$.

We will have $P_{\nabla, :,}$ denoting the the multi-parameter family defined by the join of two graphs one of which has nodes with no edges between them and the other is the complete graph. The parameters for this family are the number of nodes in each graph and the restrictions are that no group may be empty if the other contains more than one node. The graph denoted $\Gamma_{:,m, :,n}$ will have m nodes in the complete graph and n in the empty graph. The related nestohedra will be denoted by $P_{:,m, :,n}$.

We have not yet shown that these are multi-parameter families. We will do so now.

Theorem 6.8.4. We have that $P_{:,:,}$ and $P_{\nabla, :,}$ are multi-parameter families of graphical nestohedra.

Proof. Let us consider $P_{:,:,}$ first. If we take a path in the parameter space of $P_{:,:,}$ that starts at $(0,0)$ and at each step precisely one parameter increases by one the we see this defines an ordered set, $\{\Gamma_n\}_{n \in \mathbb{N}}$, of graphs on $n+1$ nodes such that Γ_{n+1} can be obtained from Γ_n by the addition of one node and arcs connected to that node. Thus by 6.1.2 this path defines a family of polytopes. Since this is true for any path, $P_{:,:,}$ is a multi-parameter family of graphical nestohedra.

This holds similarly for $P_{\nabla, :,}$. □

We will give these families names, primarily for the purpose of naming the two sections in which we present the calculations of their invariants. We shall call $P_{\cdot,\cdot,\cdot}$ the family of *bipartohedra* and $P_{\nabla,\cdot,\cdot}$ the family of *helianthahedra*, because the graphs remind me of sunflowers.

To perform these calculations we need to have generating functions for these families. We define these generating functions to be;

$$P_{\nabla,\cdot,\cdot}(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P_{\nabla,k,\cdot,\cdot,l} \frac{x^k y^l}{k! l!}$$

$$P_{\cdot,\cdot,\cdot}(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P_{\cdot,\cdot,k,\cdot,\cdot,l} \frac{x^k y^l}{k! l!}.$$

6.9 Helianthahedra

We will first look at $P_{\nabla,\cdot,\cdot}$ the multi-parameter family that we have called helianthahedra. The first thing we need to do is apply theorem 4.4.2 to these nestohedra so we can find the relation for d . However we will first draw attention to two particular subfamilies.

The family $P_{\nabla,1,\cdot,\cdot,i}$ is St , the family of stellohedra. However the generating function $P_{\nabla,1,\cdot,\cdot,i}(x, y) = ySt(x)$. This means that we know the f -, h - and F -series from our earlier work. We recall that $P_{f,\nabla,1,\cdot,\cdot,i}(\alpha, t, x) = ye^{(\alpha+t)x} \frac{\alpha}{\alpha - t(e^{\alpha x} - 1)}$, that $P_{h,\nabla,1,\cdot,\cdot,i}(\alpha, t, x) = ye^{(\alpha+t)x} \frac{\alpha - t}{\alpha e^{tx} - t e^{\alpha x}}$ and that $P_{h,\nabla,1,\cdot,\cdot,i}(t, x) = P_{h,\nabla,1,\cdot,\cdot,i}(x) \frac{e^{tx}}{1 - tPe(x)}$. These will prove useful in our later work.

The other important subfamily is $P_{\nabla,i,\cdot,\cdot,1}$ which is Pe . However in this case the relation between generating function restricted to the subfamily and the generating function of Pe cannot be expressed simply.

We will now attempt to find a formula for d of a general member of this family, which we find using theorem 4.4.2.

Theorem 6.9.1. We have that

$$dP_{\nabla,s,\cdot,\cdot,t} = tP_{\nabla,s,\cdot,\cdot,t-1} + \sum_{i=1}^s \sum_{j=0}^t \binom{s}{i} \binom{t}{j} P_{\nabla,i,\cdot,\cdot,j} \times Pe^{n-i-j}.$$

Before we look at the generating function of this family we can look at its family subring.

Theorem 6.9.2. The family subring of the heliathahedra is

$$R(P_{\nabla, \cdot}) = \mathbb{Z}[P_{\nabla, i, \cdot, j} : (i, j) \in X]$$

where X is the parameter space of the multi-parameter family.

Proof. We see from theorem 6.9.1 that the formula of $dP_{\nabla, s, \cdot, t}$ contains only the polytopes $P_{\nabla, i, \cdot, j}$ and Pe^n for some values of i , j , and n . We have already mentioned that Pe is a subfamily of $P_{\nabla, \cdot}$ and so the subring generated by the members of the family is closed under d , so the family subring has the required form. \square

In the previous section we set the generating family of this function as

$$P_{\nabla, \cdot}(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P_{\nabla, k, \cdot, l} \frac{x^k}{k!} \frac{y^l}{l!}.$$

We can then give a formula for $dP_{\nabla, \cdot}(x, y)$.

Theorem 6.9.3. The relation $dP_{\nabla, \cdot}(x, y) = P_{\nabla, \cdot}(x, y) (y + Pe(x + y))$ holds.

Proof. Using 6.9.1 and the definition of the generating function, we get that

$$\begin{aligned} dP_{\nabla, \cdot} &= d \sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1} P_{\nabla, k, \cdot, l} \frac{x^k}{k!} \frac{y^l}{l!} \\ &= \sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1} l P_{\nabla, k, \cdot, l-1} \frac{x^k}{k!} \frac{y^l}{l!} \\ &\quad + \sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1} \sum_{i=1}^k \sum_{j=0}^l \binom{k}{i} \binom{l}{j} P_{\nabla, i, \cdot, j} Pe^{n-i-j} \frac{x^k}{(k)!} \frac{y^l}{(l)!}. \end{aligned}$$

We can notice that the first sum is

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1} l P_{\nabla, k, \cdot, l-1} \frac{x^k}{k!} \frac{y^l}{l!} &= \sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1} y P_{\nabla, k, \cdot, l-1} \frac{x^k}{k!} \frac{y^{l-1}}{(l-1)!} \\ &= y P_{\nabla, \cdot}(x, y). \end{aligned}$$

Now turning our attention to the second sum.

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1} \sum_{i=1}^k \sum_{j=0}^l \binom{k}{i} \binom{l}{j} P_{\nabla, i, \cdot, j} Pe^{n-i-j} \frac{x^k}{(k)!} \frac{y^l}{(l)!} \\ &= \sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1} \sum_{i=1}^k \sum_{j=0}^l \frac{k!}{i!(k-i)!} \frac{l!}{j!(l-j)!} P_{\nabla, i, \cdot, j} Pe^{n-i-j} \frac{x^k}{(k)!} \frac{y^l}{(l)!} \end{aligned}$$

We set $g = k - i$, $h = l - j$.

$$\begin{aligned} &= \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{i=1}^k \sum_{j=0}^l \frac{1}{i!g!} \frac{1}{j!h!} P_{\nabla, i, \dots, j} P e^{g+h-1} x^k y^l \\ &= \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{i+g=k, i \geq 1} \sum_{j+h=l, j \geq 0} \frac{1}{i!g!} \frac{1}{j!h!} P_{\nabla, i, \dots, j} P e^{g+h-1} x^i x^g y^j y^h \end{aligned}$$

and utilising the identity $\sum_{i=0}^{\infty} \sum_{j+k=i+1, j \geq l} a_{jk} = \sum_{j=l}^{\infty} \sum_{k=0}^{\infty} a_{jk}$, we have

$$= \sum_{k=1}^{\infty} \sum_{i+g=k, i \geq 1} \sum_{l=0}^{\infty} \sum_{j+h=l, j \geq 0} P_{\nabla, i, \dots, j} P e^{g+h-1} \frac{x^i}{i!} \frac{x^g}{g!} \frac{y^j}{j!} \frac{y^h}{h!}$$

We can then split the terms if i, j from those in g, h and get some known functions.

$$\begin{aligned} &= \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} P_{\nabla, i, \dots, j} \frac{x^i}{i!} \frac{y^j}{j!} P e^{g+h-1} \frac{x^g}{g!} \frac{y^h}{h!} \\ &= \left(\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} P_{\nabla, i, \dots, j} \frac{x^i}{i!} \frac{y^j}{j!} \right) \left(\sum_{g=0}^{\infty} \sum_{h=0}^{\infty} P e^{g+h-1} \frac{x^g}{g!} \frac{y^h}{h!} \right) \\ &= P_{\nabla, \cdot, \cdot}(x, y) P e(x + y). \end{aligned}$$

Recombining this we get

$$dP_{\nabla, \cdot, \cdot}(x, y) = P_{\nabla, \cdot, \cdot}(x, y) (y + P e(x + y)).$$

□

We can now calculate the f -series of $P_{\nabla, \cdot, \cdot}$. To shorten the written expressions we will use the function $\eta(z) = \frac{e^{\alpha z} - 1}{\alpha}$.

Corollary 6.9.4. We have that

$$P_{f, \nabla, \cdot, \cdot}(x, y) = e^{(\alpha+t)y} \frac{\eta(x)}{1 - t\eta(x + y)}.$$

Proof. As we have before we use theorems 5.3.12 and 5.3.14 and the relation $dP_{\nabla, \cdot, \cdot}(x, y) = P_{\nabla, \cdot, \cdot}(x, y) (y + P e(x + y))$, we have that $P_{f, \nabla, \cdot, \cdot}(\alpha, t, x, y)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} P_{f, \nabla, \cdot, \cdot}(\alpha, t, x, y) = P_{f, \nabla, \cdot, \cdot}(\alpha, t, x, y) (y + P e_f(\alpha, t, x + y))$$

with the initial conditions $P_{f, \nabla, \cdot, \cdot}(\alpha, 0, x, y) = e^{\alpha y} \eta(x)$.

We then solve this partial differential equation.

$$\begin{aligned} \frac{\partial}{\partial t} P_{f,\nabla,;\cdot}(\alpha, t, x, y) &= P_{f,\nabla,;\cdot}(\alpha, t, x, y) (y + Pe_f(\alpha, t, x + y)) \\ &= P_{f,\nabla,;\cdot}(\alpha, t, x, y) \left(y - \frac{\partial}{\partial t} \ln(1 - t\eta(x + y)) \right) \\ \frac{\frac{\partial}{\partial t} P_{f,\nabla,;\cdot}(\alpha, t, x, y)}{P_{f,\nabla,;\cdot}(\alpha, t, x, y)} &= \left(y - \frac{\partial}{\partial t} \ln(1 - t\eta(x + y)) \right) \\ \frac{\partial}{\partial t} \ln(P_{f,\nabla,;\cdot}(\alpha, t, x, y)) &= y - \frac{\partial}{\partial t} \ln(1 - t\eta(x + y)) \\ P_{f,\nabla,;\cdot}(\alpha, t, x, y) &= e^{yt} \frac{1}{1 - t\eta(x + y)} c. \end{aligned}$$

We then use the initial conditions to get

$$P_{f,\nabla,;\cdot}(x, y) = e^{(\alpha+t)y} \frac{\eta(x)}{1 - t\eta(x + y)}$$

as required. □

From this we can calculate the h -series using proposition 5.3.9.

Corollary 6.9.5. The h -series of $P_{\nabla,;\cdot}$ is

$$P_{h,\nabla,;\cdot}(x, y) = e^{\alpha y} e^{ty} \frac{e^{\alpha x} - e^{tx}}{\alpha e^{tx} e^{ty} - t e^{\alpha x} e^{\alpha y}}.$$

We can also calculate the F -series of $P_{\nabla,;\cdot}$.

Corollary 6.9.6. We have that $P_{F,\nabla,;\cdot}(t, x, y) = P_{\nabla,;\cdot}(x, y) \frac{e^{tx}}{1 - tPe(x)}$.

Proof. Using formula 6.9.1 and equation 5.3.13 gives us the partial differential equation

$$\frac{\partial}{\partial t} P_{F,\nabla,;\cdot}(t, x, y) = (y + Pe_F(t, x + y)) P_{F,\nabla,;\cdot}(t, x, y),$$

with initial conditions

$$\frac{\partial}{\partial t} P_{\nabla,1,;\cdot}(0, x, y) = P_{\nabla,1,;\cdot}(x, y).$$

Rearranging the partial differential equation gives

$$\frac{\partial P_{F,\nabla,;\cdot}(t, x, y)}{P_{F,\nabla,;\cdot}(t, x, y)} = (y + Pe_F(t, x + y)) \partial t,$$

we then substitute the known expression for $Pe_F(x + y)$, giving

$$\frac{\partial P_{F,\nabla,;\cdot}(t, x, y)}{P_{F,\nabla,;\cdot}(t, x, y)} = \left(y + \frac{Pe(x + y)}{1 - tPe(x + y)} \right) \partial t,$$

finally solving this expression yields

$$P_{F,\nabla,\cdot}(t, x, y) = \frac{e^{ty}}{1 - tPe(x+y)}C,$$

where C is found using the initial conditions to be $P_{\nabla,\cdot}(x)$. \square

We have that $P_{\nabla,\cdot}$ is a family of flag polytopes so we can check that it satisfies the Gal conjecture for families.

Theorem 6.9.7. We have that $P_{h,\nabla,\cdot}(\alpha, t, x, y)$ is a Gal series.

Proof. We know that the h -series of $P_{\nabla,\cdot}$ is

$$P_{h,\nabla,\cdot}(x, y) = e^{\alpha y} e^{ty} \frac{e^{\alpha x} - e^{tx}}{\alpha e^{tx} e^{ty} - t e^{\alpha x} e^{\alpha y}}.$$

So, we calculate the partial derivative or the h -series with respect to x ,

$$\begin{aligned} P_{h,\nabla,\cdot}(\alpha, t, x, y) &= e^{(\alpha+t)y} \frac{e^{\alpha x} - e^{tx}}{\alpha e^{t(x+y)} - t e^{\alpha(x+y)}} \\ \frac{\partial}{\partial x} P_{h,\nabla,\cdot}(\alpha, t, x) &= e^{(\alpha+t)y} \frac{\alpha e^{\alpha x} - t e^{tx}}{\alpha e^{t(x+y)} - t e^{\alpha(x+y)}} - e^{(\alpha+t)y} \frac{(e^{\alpha x} - e^{tx})(\alpha t e^{t(x+y)} - \alpha t e^{\alpha(x+y)})}{(\alpha e^{t(x+y)} - t e^{\alpha(x+y)})^2} \\ &= e^{(\alpha+t)y} \phi_h(\alpha, t, x) + \alpha t P_{h,\nabla,\cdot}(\alpha, t, x, y) P e_h(\alpha, t, x + y). \end{aligned}$$

where $\phi_h(\alpha, t, x, y) = \frac{\alpha e^{\alpha x} - t e^{tx}}{\alpha e^{t(x+y)} - t e^{\alpha(x+y)}}$. By 5.3.19, $P_{h,\nabla,\cdot}(\alpha, t, x, y)$ is Gal if $\phi_h(\alpha, t, x, y)$ is Gal. We now apply lemma 5.3.19 to $\phi_h(\alpha, t, x, y)$ differentiating with respect to y rather than x .

$$\begin{aligned} \frac{\partial}{\partial y} \frac{\alpha e^{\alpha x} - t e^{tx}}{\alpha e^{t(x+y)} - t e^{\alpha(x+y)}} &= \frac{-(\alpha e^{\alpha x} - t e^{tx})(\alpha t e^{t(x+y)} - \alpha t e^{\alpha(x+y)})}{(\alpha e^{t(x+y)} - t e^{\alpha(x+y)})^2} \\ &= \alpha t P e_h(\alpha, t, x + y) \phi_h(\alpha, t, x, y). \end{aligned}$$

and we have that

$$\begin{aligned} \left. \frac{\partial \phi_h(\alpha, t, x, y)}{\partial y} \right|_{y=0} &= \frac{\alpha e^{\alpha x} - t e^{tx}}{\alpha e^{tx} - t e^{\alpha x}} \\ &= 1 + (\alpha + t) P e_h(\alpha, t, x). \end{aligned}$$

which is Gal. By repeated application of the theorem 5.3.19, ϕ_γ is Gal. Thus, by the same theorem, $P_{h,\nabla,\cdot}(\alpha, t, x, y)$ is a Gal series. \square

6.10 Bipartohedra

Finally, we will first look at $P_{\cdot,\cdot,\cdot}$: the multi-parameter family that we have called bipartohedra. The first thing we need to do is apply theorem 4.4.2 to these nestohedra so we can find the relation for d . However we will first draw attention to one particular subfamily.

The family $P_{\cdot,1,\cdot,i}$ is St , the family of stellohedra. However the generating function $P_{\cdot,1,\cdot,i}(x, y) = ySt(x)$. This means that we know the f -, h - and F -series from our earlier work. We recall that $P_{f,\cdot,1,\cdot,i}(\alpha, t, x) = ye^{(\alpha+t)x} \frac{\alpha}{\alpha-t(e^{\alpha x}-1)}$, that $P_{h,\cdot,1,\cdot,i}(\alpha, t, x) = ye^{(\alpha+t)x} \frac{\alpha-t}{\alpha e^{tx}-te^{\alpha x}}$ and that $P_{f,\cdot,1,\cdot,i}(t, x) = P_{h,\cdot,1,\cdot,i}(x) \frac{e^{tx}}{1-tPe(x)}$. These will prove useful in our later work.

We will now attempt to find a formula for d of a general member of this family, which we find using theorem 4.4.2.

Theorem 6.10.1. We have that, for $s, t \geq 2$,

$$\begin{aligned} dP_{\cdot,s,\cdot,t} &= sP_{\cdot,s-1,\nabla,t} + tP_{\nabla,s,\cdot,t-1} + \sum_{i=1}^{s-1} \sum_{j=1}^{t-1} \binom{s}{i} \binom{t}{j} P_{\cdot,i,\cdot,j} \times Pe^{s+t-i-j-1} \\ &+ \sum_{i=1}^{s-1} \binom{s}{i} P_{\cdot,i,\cdot,t} \times Pe^{s-i-1} + \sum_{j=1}^{t-1} \binom{t}{j} P_{\cdot,s,\cdot,j} \times Pe^{t-j-1}. \end{aligned}$$

when either $s < 2$ or $t < 2$ there are only two possible outcomes. Since, if $s = 0$ then we must have $t = 1$ for the graph to be connected and vice versa, we must have either $s = 1$ or $t = 1$ or both. Here we notice that $P_{\cdot,1,\cdot,k} = P_{\nabla,1,\cdot,k} = St^k$.

Before we look at the generating function of this family we can look at its family subring.

Theorem 6.10.2. The family subring of the bipartohedra is

$$R(P_{\cdot,\cdot,\cdot}) = R(P_{\nabla,\cdot,\cdot})[P_{\nabla,i,\cdot,j} : (i, j) \in X]$$

where X is the parameter space of the multi-parameter family.

Proof. We see from theorem 6.10.1 that the formula of $dP_{\cdot,s,\cdot,t}$ contains polytopes $P_{\cdot,s,\cdot,t}$, $P_{\nabla,i,\cdot,j}$ and Pe^n for some values of s , t , i , j and n . We have already know that $R(P_{\nabla,\cdot,\cdot})$ contains Pe , so the family subring has the required form. \square

In the previous section we set the generating family of this function as

$$P_{\cdot,\cdot,\cdot}(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P_{\cdot,\cdot,k,\cdot,l} \frac{x^k y^l}{k! l!}.$$

We can then give a formula for $dP_{\cdot,\cdot,\cdot}(x, y)$.

Theorem 6.10.3. The relation $dP_{\cdot,\cdot,\cdot}(x, y) = xP_{\nabla,\cdot,\cdot}(y, x) + yP_{\nabla,\cdot,\cdot}(x, y) + P_{\cdot,\cdot,\cdot}(x, y)Pe(x+y) - (x+y)Pe(x+y)$ holds.

Proof. We use 6.10.1 and combine this with the definition of the generating function, giving us that

$$\begin{aligned} P_{\cdot,\cdot,\cdot}(x, y) &= \sum_{n=0}^{\infty} \sum_{k+l=n+1, k, l \geq 2} P_{\cdot,\cdot,k,\cdot,l} \frac{x^k y^l}{k! l!} + \sum_{n=0}^{\infty} \sum_{k=n, l=1} P_{\cdot,\cdot,k,\cdot,1} \frac{x^k y}{k!} \\ &\quad + \sum_{n=0}^{\infty} \sum_{l=n, k=1} P_{\cdot,\cdot,1,\cdot,l} x \frac{y^l}{l!} - P_{\cdot,\cdot,1,\cdot,1} xy \\ &= \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} P_{\cdot,\cdot,k,\cdot,l} \frac{x^k y^l}{k! l!} + \sum_{k=0}^{\infty} P_{\cdot,\cdot,k,\cdot,1} \frac{x^k y}{k!} + \sum_{l=0}^{\infty} P_{\cdot,\cdot,1,\cdot,l} x \frac{y^l}{l!} - P_{\cdot,\cdot,1,\cdot,1} xy \\ &= \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} P_{\cdot,\cdot,k,\cdot,l} \frac{x^k y^l}{k! l!} + P_{\nabla,1,\cdot,\cdot}(x, y) + P_{\nabla,1,\cdot,\cdot}(y, x) - I^1 xy. \end{aligned}$$

which we have because the formula in 6.10.1 was given for $s, t \geq 2$.

We can then use this to give a formula for the $dP_{\cdot,\cdot,\cdot}(x, y)$, using the known formula from theorem 6.10.1 and the formula for $dP_{\cdot,\cdot,i,\cdot,l}(x, y)$.

$$\begin{aligned} dP_{\cdot,\cdot,\cdot}(x, y) &= d \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} P_{\cdot,\cdot,k,\cdot,l} \frac{x^k y^l}{k! l!} + dP_{\nabla,1,\cdot,\cdot}(x, y) + dP_{\nabla,1,\cdot,\cdot}(y, x) - d(I^1)xy \\ &= \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \left(kP_{\cdot,\cdot,k-1,\nabla,l} + lP_{\nabla,\cdot,k,\cdot,l-1} \right. \\ &\quad \left. + \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \binom{k}{i} \binom{l}{j} P_{\cdot,\cdot,i,\cdot,j} \times Pe^{k+l-i-j-1} \right. \\ &\quad \left. + \sum_{i=1}^{k-1} \binom{k}{i} P_{\cdot,\cdot,i,\cdot,l} \times Pe^{k-i-1} + \sum_{j=1}^{l-1} \binom{l}{j} P_{\cdot,\cdot,k,\cdot,j} \times Pe^{l-j-1} \right) \frac{x^k y^l}{k! l!} \\ &\quad + dP_{\nabla,1,\cdot,\cdot}(x, y) + dP_{\nabla,1,\cdot,\cdot}(y, x) - d(I^1)xy \end{aligned}$$

Expanding out the brackets gives an eight part expression for $dP_{\cdot, \cdot, \cdot}(x, y)$.

$$\begin{aligned}
 dP_{\cdot, \cdot, \cdot}(x, y) &= \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} k P_{\cdot, k-1, \nabla, l} + \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} l P_{\nabla, k, \cdot, l-1} \\
 &+ \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \binom{k}{i} \binom{l}{j} P_{\cdot, i, \cdot, j} \times P e^{k+l-i-j-1} \\
 &+ \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} \sum_{i=1}^{k-1} \binom{k}{i} P_{\cdot, i, \cdot, l} \times P e^{k-i-1} \\
 &+ \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} \sum_{j=1}^{l-1} \binom{l}{j} P_{\cdot, k, \cdot, j} \times P e^{l-j-1} \\
 &+ (x + P e(x)) P_{\nabla, 1, \cdot, \cdot}(x, y) + (y + P e(y)) P_{\nabla, 1, \cdot, \cdot}(y, x) - d(I^1)xy.
 \end{aligned}$$

We would like to have all eight of these parts in in terms of other known generating functions as the last three terms are. Let us now consider each of the first five parts in turn. Taking the first part, we notice that by changing one of the indices we can take out a factor of x ,

$$\begin{aligned}
 \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} k P_{\cdot, k-1, \nabla, l} &= x \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^{k-1} y^l}{(k-1)! l!} P_{\cdot, k-1, \nabla, l} \\
 &= x \sum_{k=1}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} P_{\cdot, k, \nabla, l}
 \end{aligned}$$

In order to get this to look more like known generation functions we want the indices k and l to start from 1 and 0. So we have

$$\begin{aligned}
 \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} k P_{\cdot, k-1, \nabla, l} &= x \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \frac{x^k y^l}{k! l!} P_{\cdot, k, \nabla, l} \\
 &- x \sum_{k=1}^{\infty} \frac{x^k}{k!} y P_{\cdot, k, \nabla, 1} - x \sum_{l=1}^{\infty} \frac{y^l}{l!} P_{\cdot, 0, \nabla, l}.
 \end{aligned}$$

Since we know that $P_{\cdot, 0, \nabla, l} = P e^{l-1}$ we have

$$\sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} k P_{\cdot, k-1, \nabla, l} = x P_{\nabla, \cdot, \cdot}(y, x) - x(P_{\nabla, 1, \cdot, \cdot}(x, y) - y) - x P e(y).$$

It is obvious to spot that the second part is the same as the first part with x and

y reversed. Proceeding to the third part, setting $g = k - i$ and $h = l - j$ gives us

$$\begin{aligned}
& \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \binom{k}{i} \binom{l}{j} P_{:,i,:,j} \times Pe^{k+l-i-j-1} \\
&= \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \sum_{l=2}^{\infty} \sum_{j=1}^{l-1} P_{:,i,:,j} \times Pe^{k+l-i-j-1} \frac{x^i}{i!} \frac{x^{k-i}}{(k-i)!} \frac{y^j}{j!} \frac{y^{l-j}}{(l-j)!} \\
&= \sum_{g=1}^{\infty} \sum_{i=1}^{\infty} \sum_{h=1}^{\infty} \sum_{j=1}^{\infty} P_{:,i,:,j} \times Pe^{g+h-1} \frac{x^i}{i!} \frac{x^g}{g!} \frac{y^j}{j!} \frac{y^h}{h!}
\end{aligned}$$

We notice now that the terms in indices i and j are independent of those in indices g and h so we can split them up and then rewrite in terms of known generating functions as

$$\begin{aligned}
&= \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_{:,i,:,j} \frac{x^i y^j}{i! j!} \right) \left(\sum_{g=1}^{\infty} \sum_{h=1}^{\infty} Pe^{g+h-1} \frac{x^g y^h}{g! h!} \right) \\
&= (P_{:, :, :}(x, y) - (x + y)) \left(Pe(x + y) - \sum_{g=1}^{\infty} Pe^{g-1} \frac{x^g}{g!} - \sum_{h=1}^{\infty} Pe^{h-1} \frac{y^h}{h!} \right) \\
&= (P_{:, :, :}(x, y) - (x + y)) (Pe(x + y) - Pe(x) - Pe(y)).
\end{aligned}$$

Again we notice the similarities between the fourth and fifth parts. We examine the fourth in detail, again with $g = k - i$.

$$\begin{aligned}
& \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} \sum_{i=1}^{k-1} \binom{k}{i} P_{:,i,:,l} \times Pe^{k-i-1} \\
&= \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \sum_{l=2}^{\infty} P_{:,i,:,l} \times Pe^{k-i-1} \frac{x^{k-i}}{(k-i)!} \frac{x^i}{i!} \frac{y^l}{l!} \\
&= \sum_{g=1}^{\infty} \sum_{i=1}^{\infty} \sum_{l=2}^{\infty} P_{:,i,:,l} \times Pe^{g-1} \frac{x^g}{g!} \frac{x^i}{i!} \frac{y^l}{l!}.
\end{aligned}$$

As before we notice now that the terms in indices i and l are independent of those in index g so we can split them up,

$$= \left(\sum_{i=1}^{\infty} \sum_{l=2}^{\infty} P_{:,i,:,l} \frac{x^i y^l}{i! l!} \right) \left(\sum_{g=1}^{\infty} Pe^{g-1} \frac{x^g}{g!} \right).$$

The bounds of the first sum do not match any known generating functions so we add include additional terms to make this the case and then write in terms of known

generating functions,

$$\begin{aligned}
 &= \left(\sum_{n=0}^{\infty} \sum_{i+l=n+1, i, l \geq 0} P_{:,i,::,l} \frac{x^i y^l}{i! l!} - \sum_{i=1}^1 \sum_{l=0}^0 P_{:,i,::,l} \frac{x^i y^l}{i! l!} \right. \\
 &\quad \left. - \sum_{i=0}^{\infty} \sum_{l=1}^1 P_{:,i,::,l} \frac{x^i y^l}{i! l!} \right) \left(\sum_{g=1}^{\infty} P e^{g-1} \frac{x^g}{g!} \right) \\
 &= (P_{:,::}(x, y) - x - P_{\nabla,1,::}(x, y)) (Pe(x)).
 \end{aligned}$$

We go back to our eight part expression

$$\begin{aligned}
 dP_{:,::}(x, y) &= \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} k P_{:,k-1,\nabla,l} + \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} l P_{\nabla,k,::,l-1} \\
 &\quad + \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} \binom{k}{i} \binom{l}{j} P_{:,i,::,j} \times P e^{k+l-i-j-1} \\
 &\quad + \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} \sum_{i=1}^{k-1} \binom{k}{i} P_{:,i,::,l} \times P e^{k-i-1} \\
 &\quad + \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{x^k y^l}{k! l!} \sum_{j=1}^{l-1} \binom{l}{j} P_{:,k,::,j} \times P e^{l-j-1} \\
 &\quad + (x + Pe(x)) P_{\nabla,1,::}(x, y) + (y + Pe(y)) P_{\nabla,1,::}(y, x) - d(I^1)xy,
 \end{aligned}$$

which we now know can be rewritten as

$$\begin{aligned}
 dP_{:,::}(x, y) &= x P_{\nabla,::}(y, x) - x(P_{\nabla,1,::}(x, y) - y) - xPe(y) + yP_{\nabla,::}(x, y) \\
 &\quad - y(P_{\nabla,1,::}(y, x) - x) - yPe(x) - d(I^1)xy \\
 &\quad + (P_{:,::}(x, y) - (x + y)) (Pe(x + y) - Pe(x) - Pe(y)) \\
 &\quad + (P_{:,::}(x, y) - x - P_{\nabla,1,::}(x, y)) (Pe(x)) + (x + Pe(x)) P_{\nabla,1,::}(x, y) \\
 &\quad + (P_{:,::}(x, y) - y - P_{\nabla,1,::}(y, x)) (Pe(y)) + (y + Pe(y)) P_{\nabla,1,::}(y, x).
 \end{aligned}$$

Which can be simplified to give that

$$dP_{:,::}(x, y) = xP_{\nabla,::}(y, x) + yP_{\nabla,::}(x, y) + P_{:,::}(x, y)Pe(x + y) - (x + y)Pe(x + y)$$

as required. \square

We can now calculate the f -series of $P_{\nabla,::}$. To shorten the written expressions we will use the function $\eta(z) = \frac{e^{\alpha z} - 1}{\alpha}$.

Corollary 6.10.4. We have that

$$P_{f,;\dots}(\alpha, t, x, y) = \frac{e^{(\alpha+t)x}\eta(y) + e^{(\alpha+t)y}\eta(x) + \alpha\eta(y)\eta(x) - e^{\alpha x}\eta(y) - e^{\alpha y}\eta(x)}{1 - t\eta(x+y)} + (x+y).$$

Proof. By the identity 5.3.12 we have

$$\begin{aligned} \frac{\partial}{\partial t} P_{f,;\dots}(x, y) &= xP_{f,\nabla,;\dots}(y, x) + yP_{f,\nabla,;\dots}(x, y) \\ &+ P_{f,;\dots}(x, y)Pe_f(x+y) - (x+y)Pe_f(x+y) \\ &= xe^{(\alpha+t)x} \frac{\eta(y)}{1 - t\eta(x+y)} + ye^{(\alpha+t)y} \frac{\eta(x)}{1 - t\eta(x+y)} \\ &+ P_{f,;\dots}(x, y) \frac{\eta(x+y)}{1 - t\eta(x+y)} - (x+y) \frac{\eta(x+y)}{1 - t\eta(x+y)}. \end{aligned}$$

To solve this we shall start by setting $\hat{P} = P_{f,;\dots} - (x+y)$, then we have

$$\frac{\partial}{\partial t} \hat{P}(x, y) = \frac{xe^{(\alpha+t)x}\eta(y) + ye^{(\alpha+t)y}\eta(x) + \hat{P}(x, y)\eta(x+y)}{1 - t\eta(x+y)}.$$

If we now set $\hat{P} = P_1P_2$ and $P_1 = \frac{c_1}{1-t\eta(x+y)}$ then we get, by application of the quotient rule and integrating,

$$P_2 = e^{(\alpha+t)x}\eta(y) + e^{(\alpha+t)y}\eta(x) + c_2.$$

combining all these we have

$$\begin{aligned} P_2(\alpha, t, x, y) &= e^{(\alpha+t)x}\eta(y) + e^{(\alpha+t)y}\eta(x) + c_2 \\ \hat{P}(\alpha, t, x, y) &= \frac{c_1}{1 - t\eta(x+y)} (e^{(\alpha+t)x}\eta(y) + e^{(\alpha+t)y}\eta(x) + c_2) \\ P_{f,;\dots}(\alpha, t, x, y) &= \frac{c_1}{1 - t\eta(x+y)} (e^{(\alpha+t)x}\eta(y) + e^{(\alpha+t)y}\eta(x) + c_2) \\ &+ (x+y). \end{aligned}$$

Examining the initial conditions given by 5.3.14, we have that

$$\begin{aligned}
 P_{f, \dots}(\alpha, 0, x, y) &= \sum_{n=0}^{\infty} \sum_{k+l=n+1, k, l > 0} \alpha^n \frac{x^k y^l}{k! l!} + x + y \\
 &= \sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1, l \geq 0} \alpha^n \frac{x^k y^l}{k! l!} - \sum_{n=0}^{\infty} \sum_{k+l=n+1, k > 0, l=0} \alpha^n \frac{x^k}{k!} + x + y \\
 &= \sum_{n=0}^{\infty} \sum_{k+l=n+1, k \geq 1, l \geq 0} \alpha^n \frac{x^k y^l}{k! l!} - \sum_{k=1}^{\infty} \alpha^{k+1} \frac{x^k}{k!} + x + y \\
 &= P_{f, \nabla, \dots}(\alpha, 0, x, y) - \eta(x) + x + y \\
 &= e^{\alpha y} \eta(x) - \eta(x) + x + y \\
 &= \alpha \eta(y) \eta(x) + x + y
 \end{aligned}$$

so, setting $c_1 = 1$, we have

$$\begin{aligned}
 \alpha \eta(y) \eta(x) + x + y &= \frac{1}{1} (e^{\alpha x} \eta(y) + e^{\alpha y} \eta(x) + c_2) + (x + y) \\
 c_2 &= \alpha \eta(y) \eta(x) - e^{\alpha x} \eta(y) - e^{\alpha y} \eta(x).
 \end{aligned}$$

Then we have

$$\begin{aligned}
 P_{f, \dots}(\alpha, t, x, y) &= \frac{1}{1 - t \eta(x + y)} (e^{(\alpha+t)x} \eta(y) + e^{(\alpha+t)y} \eta(x) \\
 &\quad + \alpha \eta(y) \eta(x) - e^{\alpha x} \eta(y) - e^{\alpha y} \eta(x)) + (x + y)
 \end{aligned}$$

as required. □

From this we can calculate the h -series.

Corollary 6.10.5. The h -series of P_{\dots} is

$$\begin{aligned}
 P_{h, \dots}(\alpha, t, x, y) &= \frac{1}{\frac{\alpha e^{tx} e^{ty} - t e^{\alpha x} e^{\alpha y}}{(\alpha - t)}} \left(\frac{e^{\alpha y} e^{\alpha x} e^{tx} - e^{\alpha x} e^{tx} e^{ty} + e^{\alpha x} e^{\alpha y} e^{ty} - e^{\alpha y} e^{tx} e^{ty}}{(\alpha - t)} \right. \\
 &\quad \left. + \frac{e^{tx} e^{ty} - e^{\alpha x} e^{\alpha y}}{(\alpha - t)} \right) + (x + y).
 \end{aligned}$$

Proof. Using proposition 5.3.9 we have that

$$\begin{aligned}
P_{h, \cdot, \cdot, \cdot}(\alpha, t, x, y) &= P_{f, \cdot, \cdot, \cdot}((\alpha - t), t, x, y) \\
&= \frac{1}{1 - t \frac{e^{(\alpha-t)(x+y)} - 1}{(\alpha-t)}} \left(e^{((\alpha-t)+t)x} \frac{e^{(\alpha-t)(y)} - 1}{(\alpha-t)} + e^{((\alpha-t)+t)y} \frac{e^{(\alpha-t)(x)} - 1}{(\alpha-t)} \right. \\
&\quad \left. + (\alpha - t) \frac{e^{(\alpha-t)(y)} - 1}{(\alpha-t)} \frac{e^{(\alpha-t)(x)} - 1}{(\alpha-t)} - e^{(\alpha-t)x} \frac{e^{(\alpha-t)(y)} - 1}{(\alpha-t)} \right. \\
&\quad \left. - e^{(\alpha-t)y} \frac{e^{(\alpha-t)(x)} - 1}{(\alpha-t)} \right) + (x + y) \\
&= \frac{1}{\frac{\alpha e^{tx} e^{ty} - t e^{\alpha x} e^{\alpha y}}{(\alpha-t)}} \left(\frac{e^{\alpha y} e^{\alpha x} e^{tx} - e^{\alpha x} e^{tx} e^{ty} + e^{\alpha x} e^{\alpha y} e^{ty} - e^{\alpha y} e^{tx} e^{ty}}{(\alpha-t)} \right. \\
&\quad \left. + \frac{e^{tx} e^{ty} - e^{\alpha x} e^{\alpha y}}{(\alpha-t)} \right) + (x + y).
\end{aligned}$$

□

We can also calculate the F -series of $P_{\nabla, \cdot, \cdot}$.

Corollary 6.10.6. We have that $P_{F, \nabla, \cdot, \cdot}(t, x, y) = \frac{e^{ty}}{1 - tPe(x+y)} P_{\nabla, \cdot, \cdot}(x, y)$.

Proof. Using formula 6.9.1 and equation 5.3.13 gives us the partial differential equation

$$\frac{\partial}{\partial t} P_{F, \nabla, \cdot, \cdot}(t, x, y) = (y + Pe_F(t, x + y)) P_{F, \nabla, \cdot, \cdot}(t, x, y),$$

with initial conditions

$$\frac{\partial}{\partial t} P_{\nabla, 1, \cdot, \cdot}(0, x, y) = P_{\nabla, 1, \cdot, \cdot}(x, y).$$

Rearranging the partial differential equation gives

$$\frac{\partial P_{F, \nabla, \cdot, \cdot}(t, x, y)}{P_{F, \nabla, \cdot, \cdot}(t, x, y)} = (y + Pe_F(t, x + y)) \partial t,$$

we then substitute the known expression for $Pe_F(x + y)$, giving

$$\frac{\partial P_{F, \nabla, \cdot, \cdot}(t, x, y)}{P_{F, \nabla, \cdot, \cdot}(t, x, y)} = \left(y + \frac{Pe(x + y)}{1 - tPe(x + y)} \right) \partial t,$$

finally solving this expression yields

$$P_{F, \nabla, \cdot, \cdot}(t, x, y) = \frac{e^{ty}}{1 - tPe(x + y)} C,$$

where C is found using the initial conditions to be $P_{\nabla, \cdot, \cdot}(x, y)$.

□

We have that $P_{\cdot, \cdot, \cdot}$ is a family of flag polytopes so we can check that it satisfies the Gal conjecture for families.

Theorem 6.10.7. We have that $P_{h, \cdot, \cdot, \cdot}(\alpha, t, x, y)$ is a Gal series.

Proof. We know that the h -series of $P_{\nabla, \cdot, \cdot}$ is

$$P_{h, \cdot, \cdot, \cdot}(\alpha, t, x, y) = \frac{1}{\frac{\alpha e^{tx} e^{ty} - t e^{\alpha x} e^{\alpha y}}{(\alpha - t)}} \left(\frac{e^{\alpha y} e^{\alpha x} e^{tx} - e^{\alpha x} e^{tx} e^{ty} + e^{\alpha x} e^{\alpha y} e^{ty} - e^{\alpha y} e^{tx} e^{ty}}{(\alpha - t)} + \frac{e^{tx} e^{ty} - e^{\alpha x} e^{\alpha y}}{(\alpha - t)} \right) + (x + y).$$

We look at the partial derivative of $P_{h, \cdot, \cdot, \cdot}(\alpha, t, x, y)$ with respect to x , and we have,

$$\begin{aligned} \frac{\partial}{\partial x} P_{h, \cdot, \cdot, \cdot}(\alpha, t, x, y) &= \frac{(\alpha - t)}{\alpha t e^{tx} e^{ty} - \alpha t e^{\alpha x} e^{\alpha y}} \left(\frac{e^{\alpha y} e^{\alpha x} e^{tx} - e^{\alpha x} e^{tx} e^{ty} + e^{\alpha x} e^{\alpha y} e^{ty} - e^{\alpha y} e^{tx} e^{ty}}{(\alpha - t)} \right. \\ &\quad \left. + \frac{e^{tx} e^{ty} - e^{\alpha x} e^{\alpha y}}{(\alpha - t)} + \frac{\alpha e^{tx} e^{ty} - t e^{\alpha x} e^{\alpha y}}{(\alpha - t)} (x + y) \right) \\ &\quad + \frac{(\alpha - t)}{\alpha e^{tx} e^{ty} - t e^{\alpha x} e^{\alpha y}} \frac{1}{(\alpha - t)} \left((\alpha + t) e^{\alpha y} e^{\alpha x} e^{tx} - (\alpha + t) e^{\alpha x} e^{tx} e^{ty} \right. \\ &\quad \left. + \alpha e^{\alpha x} e^{\alpha y} e^{ty} - t e^{\alpha y} e^{tx} e^{ty} + t e^{tx} e^{ty} - \alpha e^{\alpha x} e^{\alpha y} \right. \\ &\quad \left. + (\alpha e^{tx} e^{ty} - t e^{\alpha x} e^{\alpha y}) + (\alpha t e^{tx} e^{ty} - \alpha t e^{\alpha x} e^{\alpha y}) (x + y) \right) \\ &= \alpha t P e_h(\alpha, t, x + y) P_{h, \cdot, \cdot, \cdot} + (\alpha + t) P_{h, \nabla, \cdot, \cdot}(\alpha, t, y, x) \\ &\quad + (\alpha + t + \alpha t(x + y)) P e_h(\alpha, t, x + y) + e^{(\alpha + t)y} \frac{(\alpha e^{\alpha x} - t e^{tx})}{\alpha e^{t(x+y)} - t e^{\alpha(x+y)}}. \end{aligned}$$

So by our lemma, $P_{h, \cdot, \cdot, \cdot}$ is a Gal series if $\phi_h(\alpha, t, x, y) = \frac{\alpha e^{\alpha x} - t e^{tx}}{\alpha e^{t(x+y)} - t e^{\alpha(x+y)}}$ is a Gal series. We showed that this series was Gal in the previous section, so $P_{\gamma, \cdot, \cdot, \cdot}$ is a Gal series by 5.3.19. \square

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