

# *KO*-THEORY OF THOM COMPLEXES

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# Contents

<b>Abstract</b>	<b>5</b>
<b>Declaration</b>	<b>6</b>
<b>Copyright</b>	<b>7</b>
<b>Acknowledgements</b>	<b>8</b>
<b>1 Introduction</b>	<b>9</b>
<b>2 Notation and Prerequisites</b>	<b>14</b>
2.0.1 Thom Algebras . . . . .	16
2.0.2 Euler, Chern and Thom Classes . . . . .	17
2.0.3 Atiyah-Hirzebruch Spectral Sequence . . . . .	22
2.0.4 Inverse Limits . . . . .	23
<b>3 Complex Projective Space</b>	<b>26</b>
3.1 Integral Cohomology and Complex $K$ -theory . . . . .	27
3.2 Real $K$ -theory . . . . .	28
3.2.1 Group Structure . . . . .	28
3.2.2 Basis Elements . . . . .	30
3.2.3 Multiplicative Relations . . . . .	31
3.2.4 Finite Projective Spaces . . . . .	32
3.2.5 Infinite Projective Space . . . . .	34
3.2.6 Characteristic Classes . . . . .	35

3.3	Thom Algebra . . . . .	36
<b>4</b>	<b>Stunted Projective Spaces</b>	<b>40</b>
4.1	Complex $K$ -theory . . . . .	41
4.2	Real $K$ -theory . . . . .	41
4.2.1	The $KO$ -orientable case . . . . .	41
4.2.2	Group Structure . . . . .	43
4.2.3	Basis Elements . . . . .	44
4.2.4	Multiplicative Relations . . . . .	46
4.3	Thom Algebra . . . . .	48
<b>5</b>	<b>Line Bundles over Complex Projective Space</b>	<b>54</b>
5.1	Integral Cohomology and Complex $K$ -theory . . . . .	55
5.2	Real $K$ -theory . . . . .	56
5.2.1	The $KO$ -orientable case . . . . .	56
5.2.2	Characteristic Classes . . . . .	57
5.2.3	Group Structure . . . . .	57
5.2.4	Basis Elements . . . . .	58
5.2.5	Multiplicative Relations . . . . .	58
5.3	Thom Algebra . . . . .	60
<b>6</b>	<b>Wedges of 2-spheres</b>	<b>66</b>
6.1	Integral Cohomology and Complex $K$ -Theory . . . . .	67
6.2	Real $K$ -theory . . . . .	68
6.2.1	The $KO$ -orientable case . . . . .	68
6.2.2	Group Structure . . . . .	69
6.2.3	Basis elements . . . . .	70
6.2.4	Multiplicative Relations . . . . .	72
6.3	Thom Algebra . . . . .	74
<b>7</b>	<b>Complex 2-plane Grassmannians</b>	<b>75</b>
7.1	Complex $K$ -theory . . . . .	78

7.1.1	Preliminary Relations . . . . .	80
7.2	Real $K$ -theory . . . . .	83
7.2.1	Pontryagin Classes . . . . .	83
7.2.2	Basis Elements . . . . .	85
7.2.3	Multiplicative Relations . . . . .	89
7.2.4	$KO_*$ -algebras . . . . .	94
7.2.5	Characteristic Classes . . . . .	96
7.2.6	Thom Algebra . . . . .	96
7.2.7	Comments on Torsion . . . . .	98
<b>8</b>	<b>Products of Complex Projective Spaces</b>	<b>100</b>
8.1	Complex $K$ -theory . . . . .	101
8.2	Real $K$ -theory . . . . .	102
8.2.1	Group Structure . . . . .	102
8.2.2	Basis Elements . . . . .	104
8.2.3	Multiplicative Relations . . . . .	108
8.2.4	Products of Finite Projective Spaces . . . . .	109
8.2.5	Product of Infinite Projective Spaces . . . . .	114
8.3	Thom Algebra . . . . .	116
	<b>Bibliography</b>	<b>123</b>

# Abstract

Let  $\theta$  over  $X$  be a vector bundle. We wish to determine  $KO^*(T\theta)$ , the real  $K$ -theory of  $T\theta$ , the Thom complex associated with  $\theta$ . If  $\theta$  is *Spin*, we may appeal to the Thom isomorphism theorem, which identifies  $KO^*(T\theta)$  as a free module over  $KO^*(X_+)$  with one generator  $t^{KO} \in KO^*(T\theta)$ . Our aim is to shed some light on the case where  $\theta$  is **not** *Spin*.

For several families of complex vector bundles  $\theta$  over  $X$ , where  $X$  is a CW-complex with cells in only even dimensions, we compute  $KO^*(T\theta)$  and  $KO^*(X)$ . We interpret our results in light of the action

$$\delta^*: KO^*(X_+) \otimes_{KO_*} KO^*(T\theta) \rightarrow KO^*(T\theta)$$

exhibiting  $KO^*(T\theta)$  as a graded commutative algebra over  $KO^*(X_+)$ . We aim to find general information about  $KO^*(T\theta)$  in terms of  $KO^*(X_+)$ .

Our computations are informed by the Thom isomorphism in complex  $K$ -theory and Bott's exact sequence linking real and complex  $K$ -theory.

# Declaration

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# Chapter 1

## Introduction

Amongst the applications of the theory of vector bundles are topological  $K$ -theories (including real and complex  $K$ -theory) and Thom complexes. Since their creation, both  $K$ -theory and Thom complexes have proved to be vital tools for algebraic topologists, demonstrated by the extent to which their use has permeated the subject.

Topological  $K$ -theory was first defined by Atiyah and Hirzebruch [6], building on ideas of Grothendieck, and initiated the study of generalised cohomology theories. Bott's celebrated periodicity theorems [10] [11] identify that both real and complex  $K$ -theories are periodic, of orders 8 and 2 respectively. Early applications included Adams' solution of the vector fields on spheres problem [1] and later Adams and Atiyah's "postcard proof" of the Hopf invariant 1 problem [2]. The techniques of  $K$ -theory spread to diverse areas of mathematics, including algebraic  $K$ -theory, and the  $K$ -theory of  $C^*$ -algebras. Much recent work has focused on Morava  $K$ -theories, of which rational cohomology and complex  $K$ -theory may be considered examples.

Thom complexes are a basic construction in cobordism theory; the set of spaces  $MU(n)$  and natural maps  $\Sigma^2 MU(n) \rightarrow MU(n+1)$  forming the spectrum for the complex cobordism cohomology theory. The Thom isomorphism theorem for ordinary cohomology was generalised by Atiyah, Bott and Shapiro [7], who prove the corresponding Thom isomorphism theorems for both real and complex  $K$ -theories.

In this thesis, we aim to understand the real  $K$ -theory of the Thom complex of an  $n$ -dimensional vector bundle  $\theta$  over  $X$  in terms of the real  $K$ -theory of  $X$ . If  $\theta$

is orientable with respect to  $KO$ -theory, or is *Spin*, then we may refer to the Thom isomorphism theorem for  $KO$ -theory [7]: there exists a Thom class  $t^{KO} \in KO^n(T\theta)$ , and  $KO^*(T\theta)$  is a free  $KO^*(X_+)$ -module on one generator,  $t^{KO}$ . Assuming we have some knowledge of  $KO^*(X_+)$ , this result provides us with similar information about  $KO^*(T\theta)$ . In particular, we have an isomorphism of groups  $KO^{i+n}(T\theta) \cong KO^i(X_+)$ . Given a basis  $\{x_1, \dots, x_k\}$  of  $KO^i(X_+)$ , it follows that  $\{x_1 \cdot t^{KO}, \dots, x_k \cdot t^{KO}\}$  is a basis of  $KO^{i+n}(T\theta)$ . Multiplication is only slightly more complicated: given a pair of elements  $y_i \in KO^i(T\theta)$  and  $y_j \in KO^j(T\theta)$ , then  $y_i = y'_i \cdot t^{KO}$  and  $y_j = y'_j \cdot t^{KO}$  where  $y'_i$  and  $y'_j$  are elements of  $KO^*(X_+)$  of dimensions  $i - n$  and  $j - n$  respectively. So the product  $y_i y_j \in KO^*(T\theta)$  is equal to  $y'_i y'_j \cdot (t^{KO})^2$ , and given that we know the multiplicative structure of  $KO^*(X_+)$ , we need only compute  $(t^{KO})^2$  to understand  $KO^*(T\theta)$  as an algebra over the coefficient ring.

This brings us to the topic of this thesis: if  $\theta$  is *not* orientable with respect to  $KO$ -theory, then there is no Thom class  $t^{KO} \in KO^n(T\theta)$  in general, and we currently have little understanding of  $KO^*(T\theta)$ , particularly in comparison with the orientable case, where we are able to use Atiyah, Bott and Shapiro's methods. Our objective is to answer the following question: given  $KO^*(X_+)$  and a bundle  $\theta \rightarrow X$ , what can we say about  $KO^*(T\theta)$ ? We proceed by computing  $KO^*(T\theta)$  and  $KO^*(X_+)$  for examples of  $\theta \rightarrow X$  and exploring the relationship between the two  $KO_*$ -algebras.

We restrict our attention to complex vector bundles over spaces  $X$  with cells in only even dimensions. There are several reasons for this. The Thom complex of such a vector bundle will also have cells in only even dimensions, and both the ordinary cohomology and complex  $K$ -theory of such spaces are particularly tractable. Since every complex vector bundle is orientable with respect to both ordinary cohomology and complex  $K$ -theory, the corresponding Thom isomorphism theorems inform our real  $K$ -theory computations.

The world of complex geometry gives us a wealth of interesting and useful examples of such spaces. We can easily generate families of closely related vector bundles which contain examples both of *Spin* bundles, and of bundles which are not *Spin*. For example, given any complex vector bundle  $\theta \rightarrow X$  which is not  $KO$ -orientable,

it is easy to show that the Whitney sum of  $n$  copies of  $\theta$  is  $KO$ -orientable if and only if  $n$  is even. Such vector bundles are therefore a sensible class of objects with which to begin our investigation.

We interpret our results in light of the action (2.4)

$$\delta^*: KO^*(X_+) \otimes_{KO_*} KO^*(T\theta) \rightarrow KO^*(T\theta)$$

exhibiting  $KO^*(T\theta)$  as a graded commutative algebra over  $KO^*(X_+)$ . We refer to this description of  $KO^*(T\theta)$  as the *Thom algebra* of  $\theta$ . When  $\theta$  is *Spin*,  $KO^*(T\theta)$  is a free module over  $KO^*(X_+)$  with a single generator  $t^{KO} \in KO^*(T\theta)$ . When  $\theta$  is not *Spin*,  $KO^*(T\theta)$  is a module over  $KO^*(X_+)$ , but is in general a module with many generators and relations.

We introduce characteristic classes for complex line and 2-plane bundles (Sections 3.2.6 and 7.2.5 respectively) by pulling back the generators of the  $KO_*$ -algebras  $KO^*(\mathbb{C}P_+^\infty)$  and  $KO^*(BU(2)_+)$  along the appropriate classifying map.

We now summarise the contents of each chapter.

Chapter 2 establishes our notation and several results that will be referred to repeatedly throughout the rest of the thesis.

In Chapter 3, we extend Fujii's [17] partial computation of the  $KO_*$ -algebra  $KO^*(\mathbb{C}P^n)$  and compute the  $KO$ -theory of  $n$ -dimensional complex projective space. Our results confirm Fujii's [17] and Civan and Ray's [14] results on the  $KO_*$ -algebras  $KO^*(\mathbb{C}P_+^n)$ . Since the Thom complex of the canonical line bundle  $\eta(n)$  over  $\mathbb{C}P^n$  is homeomorphic to  $\mathbb{C}P^{n+1}$ , we compare  $KO^*(\mathbb{C}P^{n+1})$  with  $KO^*(\mathbb{C}P_+^n)$  to provide our first examples of Thom algebras. The Thom algebra of  $\eta(n)$  over  $\mathbb{C}P^n$  has four generators when  $n$  is odd or infinite, and five generators when  $n$  is even. The space  $\mathbb{C}P^\infty$  classifies all complex line bundles  $\phi$  via homotopy classes of maps  $\phi: X \rightarrow \mathbb{C}P^\infty$ , and we introduce the Fujii classes of a complex line bundle by applying  $\phi^*$  to the generators of the  $KO_*$ -algebra  $KO^*(\mathbb{C}P_+^\infty)$ .

Chapters 4 and 5 are concerned with other complex bundles over  $\mathbb{C}P^n$ . In Chapter 4 we consider the Whitney sum of  $m$  copies of the canonical bundle over  $\mathbb{C}P^n$ , and in Chapter 5 the tensor product of the same. In each chapter we first examine the

case where  $m$  is even, and the bundles are *Spin*. The majority of both chapters concerns the computation of the Thom algebras in case when  $m$  is odd. In Chapter 5, we replicate Civan and Ray's [14] computations of the Thom algebras of  $\eta^2$  over  $\mathbb{C}P^\infty$  and  $\eta^{2a}$  over  $\mathbb{C}P^2$ . There is a marked contrast in the complexity of the result depending on the parity of  $m$ . When  $m$  is odd, the Thom algebras of  $m\eta(n)$  and  $\eta(n)^m$  over  $\mathbb{C}P^n$  each have four generators when  $n$  is odd or infinite, and five generators when  $n$  is even.

Chapter 6 concerns line bundles  $\phi$  over 2-dimensional complexes. As we are interested in spaces with cells only in even dimensions, our base space is a wedge of 2-spheres. This provides a contrast with previous chapters, where  $X$  consisted of at most a single cell in each even dimension. When  $\phi$  is not *Spin*, the Thom algebra of  $\phi$  over  $X$  has a number of generators varying with the number of 2-cells in  $X$ ; if  $X$  is a wedge of  $n$  spheres, then the Thom algebra has  $n + 3$  generators.

In Chapter 7 we progress from bundles derived from line bundles to study the canonical 2-plane bundle  $\eta_2$  over  $BU(2)$ . We make use of Hoggar's calculations of the graded abelian groups  $KO^i(G_2(\mathbb{C}^n)_+)$  [23]. The Thom algebra of  $\eta_2$  over  $BU(2)$  has five generators. The space  $BU(2)$  classifies all complex 2-plane bundles  $\theta$  via homotopy classes of maps  $\theta: X \rightarrow BU(2)$ , and we introduce the Fujii classes for an arbitrary complex 2-plane bundle by applying  $\theta^*$  to the generators of the  $KO_*$ -algebra  $KO^*(BU(2)_+)$ .

In Chapter 8, we examine the product of two complex projective spaces. The Thom complex of the product of two canonical bundles is homotopy equivalent to the smash product of two complex projective spaces, and we obtain  $KO^*(T(\eta \times \eta))$  as an ideal of  $KO^*(\mathbb{C}P^m \times \mathbb{C}P^n)$ . The Thom algebra of  $\eta(m) \times \eta(n)$  over  $\mathbb{C}P^m \times \mathbb{C}P^n$  has eight generators when either of  $m$  or  $n$  is odd or if  $m$  and  $n$  are infinite, and nine generators when both  $m$  and  $n$  are even. Our calculations highlight a further complexity of dealing with real  $K$ -theory: the absence of a Künneth Theorem.

We believe that the computations of real  $K$ -theory algebras are original unless stated, although Yamaguchi has recently computed the  $KO_*$ -algebras  $KO^*(\mathbb{C}P_+^n)$  and  $KO^*(\mathbb{C}P^m \times \mathbb{C}P_+^n)$  independently [32]. Within each chapter we discuss existing

results before commencing calculations. The consideration of these algebras as Thom algebras is a new point of view. The ordinary cohomology and complex  $K$ -theory algebras we discuss are well known, and are restated for ease of reference.

Readers should note that Chapter 3 and subsequent chapters follow a reasonably rigid format. Our hope is that this will assist the reader in wading through lengthy computations. We begin each chapter with a brief survey of the complex  $K$ -theory of  $X$  and of  $T\theta$  with reference to the Thom isomorphism. The ordinary cohomology and complex  $K$ -theory of the spaces we study is well known, and in each case we are able to make use of the Thom isomorphism theorem to state the complex  $K$ -theory of  $T\theta$ . When relevant, typically when discussing a set of vector bundles of which some subset are  $KO$ -orientable, we next study the real  $K$ -theory of those bundles which are  $Spin$ . As we observed above, we can refer to the Thom isomorphism, and our computations simply involve determining the square of  $t^{KO}$ .

Our main computations, making up the bulk of each chapter, concern the cases where  $\theta$  is not  $Spin$ . Where  $T\theta$  is a finite CW-complex, we begin by computing the graded abelian groups  $KO^*(T\theta)$ , using either long exact sequences or the Atiyah-Hirzebruch spectral sequence. In either case, the exact sequence due to Bott (2.1) is invaluable. We then find bases for these groups, typically in terms of elements defined using the complex  $K$ -theory and the realification homomorphism. By computing multiplicative relations we determine  $KO^*(T\theta_+)$  as an algebra over the coefficient ring. Finding additive bases and multiplicative relations are both achieved using the complexification homomorphism and working with the image of elements in complex  $K$ -theory. Again these calculations are informed by Bott's sequence. Where  $T\theta$  an infinite dimensional CW-complex defined as the union of the finite dimensional CW-complexes examined earlier, we deduce  $KO^*(T\theta)$  using inverse limits. We may discuss some examples of the Fujii classes, which we introduce in Chapters 3 and 7. Finally, we compare  $KO^*(T\theta)$  with  $KO^*(X_+)$  to deduce the Thom algebra of  $\theta$ .

# Chapter 2

## Notation and Prerequisites

Throughout this thesis, we restrict our attention to complex vector bundles  $\theta$  over  $X$  of complex dimension  $n$ , where  $X$  is a pointed CW-complex of finite type. We further assume that  $X$  is *even*, in the sense that  $X$  has cells in only even dimensions. Consequently the Thom complex of  $\theta$ , which we denote  $T\theta$ , is also even.

For any even space  $X$ , it is straightforward to show that the abelian groups  $H^i(X)$  and  $K^i(X)$  are isomorphic to the trivial group when  $i$  is odd, and are free abelian when  $i$  is even [20]. The following result, due to Hoggar [23], shows that the graded abelian group  $KO^*(X)$  is also particularly simple for an even space  $X$ .

**Proposition 2.0.1.** *Let  $X$  be a finite CW-complex with cells only in even dimensions. Then  $KO^{2i+1}(X) \cong \mathbb{Z}_2^a$ , for some non-negative integer  $a$ , and  $KO^{2i}(X) \cong KO^{2i+1}(X) \oplus \mathbb{Z}^b$ .*

We use  $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$  throughout to denote the group of integers modulo 2.

We will always work with reduced cohomology theories  $D^*$ , and we denote the reduced and unreduced cohomology algebras of a space  $X$  by  $D^*(X)$  and  $D^*(X_+)$  respectively. We shall generally discuss the cases  $D = H, K$  or  $KO$ , where  $H$  is integral cohomology.

The coefficient ring  $K_*$  for complex  $K$ -theory is isomorphic to  $\mathbb{Z}[z, z^{-1}]$ , where  $z \in K_2$  is the element represented by the complex Hopf line bundle over  $S^2$  ([30, 13.92], for example). For real  $K$ -theory, the coefficient ring  $KO_*$  is isomorphic to

$\mathbb{Z}[e, \alpha, \beta^{\pm 1}]/(2e, e^3, e\alpha, \alpha^2 - 4\beta)$ , where  $e$ ,  $\alpha$  and  $\beta$  are the elements represented respectively by the canonical bundle over  $S^1$ , the symplectic bundle over  $S^4$ , and the canonical bundle over  $S^8$  ([30, 13.94], for example).

There are natural additive homomorphisms

$$r: K^*(X) \longrightarrow KO^*(X)$$

$$c: KO^*(X) \longrightarrow K^*(X)$$

$$-: K^*(X) \longrightarrow K^*(X)$$

namely: realification, complexification, and complex conjugation respectively [1]. Complexification and conjugation are also multiplicative. These homomorphisms are related by the formulae

$$rc(x) = 2x, \quad cr(x) = x + \bar{x}, \quad \overline{\bar{x}} = x.$$

Realification is not multiplicative, and we frequently make use of the following result.

**Proposition 2.0.2.** *If complexification  $c: KO^{s+t}(X) \rightarrow K^{s+t}(X)$  is injective, then the relation*

$$r(x)r(y) = r(x(y + \bar{y}))$$

*holds for any  $x \in K^s(X)$ ,  $y \in K^t(X)$ .*

*Proof.* We complexify the left hand side:

$$c(r(x)r(y)) = (x + \bar{x})(y + \bar{y}) = xy + x\bar{y} + \overline{xy + x\bar{y}} = cr(xy + x\bar{y}).$$

Injectivity of  $c$  completes the proof. □

We next describe the interaction of realification and complexification with the coefficients.

**Lemma 2.0.3.** *The image of  $z^i \in K^{-2i}(S^0)$  under the realification homomorphism is as follows.*

$$r(z^i) = \begin{cases} 2\beta^{\frac{i}{4}} & i \equiv 0 \pmod{4}, \\ e^2\beta^{\frac{i-1}{4}} & i \equiv 1 \pmod{4}, \\ \alpha\beta^{\frac{i-2}{4}} & i \equiv 2 \pmod{4}, \\ 0 & i \equiv 3 \pmod{4}. \end{cases}$$

Also  $c(e) = 0$ ,  $c(\beta) = z^4$  and  $c(\alpha) = 2z^2$ . Complex conjugation acts on the coefficient ring  $K_*$  by  $\overline{z^j} = (-1)^j z^j$  for all integers  $j$ .

*Proof.* The results for  $r(z^i)$  and  $c(e)$  follow easily from Bott's sequence (2.1) with  $X = S^0$ . The first Chern class  $c_1(-): K^0(S^2) \rightarrow H^2(S^2)$  is an isomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}$  [20] and commutes with conjugation to show that  $\bar{z} = -z$ . The relations  $c(\beta) = z^4$  and  $c(\alpha) = 2z^2$  both follow from the fact that  $z^{2i} + \overline{z^{2i}}$  and  $2z^{2i}$  are equal.  $\square$

Real and complex  $K$ -theory are linked by Bott's exact sequence [12]

$$\dots \longrightarrow KO^{*+1}(X) \xrightarrow{\cdot e} KO^*(X) \xrightarrow{\chi} K^{*+2}(X) \xrightarrow{r} KO^{*+2}(X) \longrightarrow \dots \quad (2.1)$$

where the homomorphism  $\cdot e$  is multiplication by  $e$ , and  $\chi$  is complexification followed by multiplication by  $z^{-1}$ . We shall appeal to this sequence repeatedly.

An immediate consequence of exactness of (2.1) is that the product of any element of  $KO^*(X)$  with  $e$  is zero if and only if the element is in the image of the realification homomorphism. Whenever  $KO^*(X)$  is a free abelian group, (2.1) shows that  $r$  is an epimorphism, and  $c$  is a monomorphism.

### 2.0.1 Thom Algebras

The generalised diagonal map,  $\delta: T\theta \rightarrow X_+ \wedge T\theta$  [28, IV.5.36], sends a point  $x \in T\theta - \{x_0\} \simeq E(\theta)$  to  $p(x) \wedge x \in X_+ \wedge T\theta$ , and the basepoint  $x_0 \in T\theta$  to the basepoint. The generalised diagonal interacts with the diagonal of  $X$  according to the commutative diagram

$$\begin{array}{ccc} T\theta & \xrightarrow{\delta} & X_+ \wedge T\theta \\ \uparrow \iota & & \uparrow id \wedge \iota \\ X & \xrightarrow{\Delta} & X_+ \wedge X \end{array} \quad (2.2)$$

where  $\iota$  is the inclusion of  $X$  as the zero section, and  $\Delta$  the standard diagonal.

For any homotopy commutative ring spectrum  $D$ , there is a canonical external product

$$\wedge: D^*(X) \otimes_{D_*} D^*(Y) \rightarrow D^*(X \wedge Y) \quad (2.3)$$



of reduced cohomology  $D_*$ -algebras, for all CW-complexes  $X$  and  $Y$  [30, page 270]. We compose  $\wedge$  with the homomorphism induced by  $\delta$  to obtain the action

$$\delta^*: D^*(X_+) \otimes_{D_*} D^*(T\theta) \rightarrow D^*(T\theta) \quad (2.4)$$

exhibiting  $D^*(T\theta)$  as a graded commutative algebra over  $D^*(X_+)$ . We refer to this description of  $D^*(T\theta)$  as the *Thom algebra* of  $\theta$ . To emphasise the module structure, we will often write  $\delta^*(x \otimes y)$  as  $x \cdot y$  for  $x \in D^*(X_+)$  and  $y \in D^*(T\theta)$ , and may refer to  $D^*(T\theta)$  as a *Thom module*<sup>1</sup>.

**Lemma 2.0.4.** *As a  $D^*(X_+)$ -module, the image of the Thom algebra under restriction to the zero section is a sub-ideal of  $D^*(X)$ .*

*Proof.* Applying  $D^*$  to (2.2), we construct the following diagram.

$$\begin{array}{ccccc} D^*(X_+) \otimes D^*(T\theta) & \xrightarrow{\wedge} & D^*(X_+ \wedge T\theta) & \longrightarrow & D^*(T\theta) \\ \downarrow id \otimes \iota^* & & \downarrow (id \wedge \iota)^* & & \downarrow \iota^* \\ D^*(X_+) \otimes D^*(X) & \xrightarrow{\wedge} & D^*(X_+ \wedge X) & \xrightarrow{\Delta^*} & D^*(X). \end{array} \quad (2.5)$$

Commutativity follows from naturality of the product (2.3) (see [30, 13.52] for details), and the diagram implies that

$$\iota^*(x \cdot y) = (\Delta^* \circ \wedge \circ id \otimes \iota^*)(x \otimes y) = (\Delta^* \circ \wedge)(x \otimes \iota^*(y)) = x \iota^*(y)$$

in  $D^*(X)$ , for any  $x \in D^*(X_+)$  and  $y \in D^*(T\theta)$ . □

## 2.0.2 Euler, Chern and Thom Classes

The unit  $S^0 \rightarrow D$  represents an element  $s_n^D \in D^n(S^n)$ , which generates  $D^*(S^n)$  as a free  $D_*$ -module for any  $n \geq 0$ . An  $n$ -dimensional vector bundle  $\theta$  is  *$D$ -oriented* whenever it admits a Thom class  $t^D(\theta) \in D^n(T\theta)$ . Thom classes are characterised by the property that they restrict to  $s_n^D$  on the fibre [7].

Let  $\eta_n$  denote the canonical complex  $n$ -plane bundle over the classifying space  $BU(n)$  for every  $n \geq 1$ . For the canonical complex line bundle  $\eta_1$  over  $\mathbb{C}P^\infty$  we will

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<sup>1</sup>Disambiguation: Our use of ‘‘Thom modules’’ throughout the thesis should not be confused with that in [13], where a Thom module refers to a purely algebraic analogue of the cohomology of a Thom space.

usually just write  $\eta$ . For any complex oriented ring spectrum  $E$ , the orientation class  $v^E$  lies in  $E^2(\mathbb{C}P^\infty)$ , and determines an isomorphism

$$E^*(\mathbb{C}P^\infty) \cong E_*[[v^E]] \quad (2.6)$$

of  $E_*$ -algebras. The orientation leads to the construction of canonical Chern classes  $c_j^E \in E^{2j}(BU(n))$  for  $1 \leq j \leq n$ , and  $E^*(BU(n))$  is isomorphic to the formal power series algebra

$$E_*[[c_1^E, \dots, c_n^E]] \quad (2.7)$$

as  $E_*$ -algebras. For  $n = 1$ , the first Chern class  $c_1^E(\eta)$  coincides with  $v^E$ , and (2.7) reduces to (2.6). A good reference for this material is Switzer [30, Chapter 16].

**Notation 2.0.5.** Our notation for Thom classes  $t^D(\theta) \in D^{2n}(T\theta)$  is indicative of both the cohomology theory and the vector bundle. When clear by context, we may omit the information about the bundle and simply write  $t^D$ . For computations, we find it convenient to define the Thom class  $t(\theta) := z^n t^K(\theta) \in K^0(T\theta)$ , where  $z \in K^{-2}(S^0)$  is the periodicity element. Again, we may find it convenient to simply write  $t$  when context allows. Note that  $t(\theta)$  differs from  $t^K(\theta)$  only by the invertible element  $z^n \in K_*$ , and so  $t(\theta)$  is a Thom class.

Similarly for the Chern classes  $c_j^E(\theta) \in E^{2j}(X)$ , we may omit the superscript if the cohomology theory is clear by context.

**Example 2.0.6.** Define  $x := \eta - \mathbb{C} \in K^0(\mathbb{C}P^\infty)$  where  $\mathbb{C}$  is the trivial line bundle over  $\mathbb{C}P^\infty$ . Then there is an isomorphism of  $K_*$ -algebras [20]

$$K^*(\mathbb{C}P_+^\infty) \cong K_*[[z^{-1}x]],$$

identifying  $z^{-1}x \in K^2(\mathbb{C}P^\infty)$  as the first Chern class  $c_1 \in K^2(\mathbb{C}P^\infty)$ .

The element  $x \in K^0(\mathbb{C}P^\infty)$  performs an important role in several of our calculations, and we list some of its properties below.

For any complex line bundle  $\phi$  over  $X$ , the product  $\phi \otimes \bar{\phi}$  is isomorphic to the trivial line bundle  $\mathbb{C}$  over  $X$  [20]. In particular,  $\mathbb{C} = (x + \mathbb{C})(\overline{x + \mathbb{C}})$ , and we deduce

the equation

$$\bar{x} = \sum_{i=1}^n (-1)^i x^i. \quad (2.8)$$

It follows immediately that

$$-x\bar{x} = x + \bar{x}. \quad (2.9)$$

If  $\theta$  is  $D$ -oriented, then its *Euler class*  $\ell^D(\theta)$  is the element  $\iota^*(t^D(\theta)) \in D^{2n}(X)$ , and the image of the Thom algebra is isomorphic to the principal ideal  $(\ell^D(\theta))$  as a  $D^*(X_+)$ -module. Irrespective of orientability, we refer to  $\iota^*(D^*(T\theta))$  as the *Euler ideal of  $\theta$* , and describe its elements as *Eulerian*.

In several of our examples,  $\iota^*$  is a monomorphism, and defines an isomorphism from the Thom algebra to the Euler ideal. In these circumstances we rewrite the element  $(\iota^*)^{-1}(w) \in D^*(T\theta)$  as  $\langle w \rangle$  for each Eulerian  $w$ . Using Lemma 2.0.4, we may express the isomorphism using the formulae

$$u \cdot \langle w \rangle = \langle uw \rangle \quad \text{and} \quad \langle vw \rangle = \langle v \rangle \langle w \rangle \quad (2.10)$$

for every  $u \in D^*(X_+)$  and every Eulerian  $v, w \in D^*(X)$ .

The classical Thom isomorphism [30] may be restated as follows.

**Proposition 2.0.7.** *The bundle  $\theta$  is  $D$ -oriented if and only if  $D^*(T\theta)$  is a free  $D^*(X_+)$ -module on a single generator  $t^D$ . The Thom algebra is generated by  $t^D$ , with the single relation*

$$(t^D)^2 = \ell^D \cdot t^D$$

We note that Proposition 2.0.7 is a special case of (2.4). In general, the Thom algebra of  $\theta$  is more complicated, and  $D^*(T\theta)$  is a  $D^*(X_+)$ -module with many generators and relations.

All complex bundles are both  $H$ -oriented and  $K$ -oriented. The underlying real bundle is  $KO$ -orientable if and only if  $\theta$  is *Spin* [7]. A necessary and sufficient condition for a bundle  $\theta$  to admit a *Spin* structure is that both  $w_1(\theta)$  and  $w_2(\theta)$  are zero [12], where  $w_i(\theta) \in H^i(X; \mathbb{Z}_2)$  denotes the  $i$ th Stiefel-Whitney class of  $\theta$  [27].

For any line bundle  $\phi$ , the line bundle  $\phi^{2k}$  is a pullback of  $\eta^2 \rightarrow \mathbb{C}P^\infty$ , which is universal for line bundles with *Spin* structure. For any two bundles  $\theta, \theta'$  over  $X$ , the

Whitney sum  $\theta \oplus \theta' \rightarrow X$  has second Stiefel-Whitney class  $w_2(\theta) + w_2(\theta')$ , so  $2\theta$  is  $KO$ -orientable for any complex bundle  $\theta$ . So in some sense, such bundles are close to  $KO$ -orientability.

In the following examples, we use the fact that for any complex oriented cohomology theory  $E$ , the top Chern class  $c_n^E(\theta) \in E^{2n}(X)$  and the Euler class  $\ell^E(\theta) \in E^{2n}(X)$  coincide.

**Example 2.0.8.** For any complex bundle  $\theta$  over an even space  $X$ , the first Stiefel-Whitney class  $w_1(\theta) \in H^1(X; \mathbb{Z}_2) = 0$  is zero. This is a necessary and sufficient condition for  $H$ -orientability of  $\theta$  [25, IV 4.20], and  $H^*(T\theta)$  is a free  $H^*(X_+)$ -module on a single generator  $t^H$ . The Thom algebra is generated by  $t^H$ , with the single relation

$$(t^H)^2 = c_n^H(\theta) \cdot t^H.$$

**Example 2.0.9.** For any complex bundle  $\theta$  over an even space  $X$ , the second Stiefel-Whitney class  $w_2(\theta) \in H^2(X; \mathbb{Z}_2)$  is the mod 2 reduction of the first Chern class  $c_1(\theta) \in H^2(X)$ . This is a sufficient condition for  $K$ -orientability of  $\theta$  [29, page 293], and  $K^*(T\theta)$  is a free  $K^*(X_+)$ -module on a single generator  $t^K$ . The Thom algebra is generated by  $t^K$ , with the single relation

$$(t^K)^2 = c_n^K(\theta) \cdot t^K.$$

**Example 2.0.10.** Let  $\eta_n$  over  $BU(n)$  be the canonical complex  $n$ -plane bundle. Then  $K^*(MU(n))$  is a free  $K^*(BU(n)_+)$ -module on a single generator  $t^K$ . The Thom algebra is generated by  $t^K$ , with the single relation

$$(t^K)^2 = c_n^K \cdot t^K.$$

Conjugation interacts with the Thom class via the relation

$$\overline{t^K} = \kappa_n \cdot t^K.$$

where  $\kappa_n \in K^0(BU(n))$  is defined by the equation  $\kappa_n c_n = \overline{c_n}$ .

*Proof.* The first relation follows immediately from Example 2.0.9. For the second relation, the Thom isomorphism implies that there exists some element  $\kappa_n \in K^0(BU(n))$  such that  $\overline{t^K} = \kappa_n \cdot t^K$ . Then  $t^K \overline{t^K} = \kappa_n \cdot (t^K)^2 = \kappa_n c_n \cdot t^K$ . But  $t^K \overline{t^K} = \overline{t^K t^K} = \overline{\kappa_n c_n \kappa_n} \cdot t^K$ . As there are no multiplicative relations in  $K^*(BU(n))$  [30], the relation  $\kappa_n c_n \cdot t^K = \overline{\kappa_n c_n \kappa_n} \cdot t^K$  simplifies to  $c_n = \overline{\kappa_n c_n}$ .  $\square$

For computations, it is convenient to work with the element  $t(\theta) = z^n \cdot t^K(\theta) \in K^0(T\theta)$ . Since  $t(\theta)$  differs from  $t^K(\theta)$  only by the invertible element  $z^n \in K_*$ , it follows that  $t(\theta)$  is a Thom class.

**Corollary 2.0.11.** *If  $\theta \rightarrow X$  is a complex  $n$ -plane bundle, then  $K^*(T\theta)$  is a free  $K^*(X_+)$ -module on a single generator  $t(\theta) \in K^0(T\theta)$ . The Thom algebra is generated by  $t(\theta)$ , with the single relation*

$$(t(\theta))^2 = z^n c_n^K(\theta) \cdot t(\theta).$$

*Conjugation interacts with  $t(\theta)$  via the relation*

$$\overline{t(\theta)} = (-1)^n \kappa_n(\theta) \cdot t(\theta),$$

*where  $\kappa_n(\theta) \in K^0(X)$  denotes the pullback of  $\kappa_n \in K^0(BU(n))$  by the classifying map  $\theta: X \rightarrow BU(n)$ .*

*Proof.* If  $\theta$  is the canonical  $n$ -plane bundle  $\eta_n$  over  $BU(n)$ , then the corollary is an immediate consequence of Example 2.0.10. For a general  $n$ -plane bundle  $\theta$ , we simply pull back along the map of Thom spaces  $T\theta \rightarrow MU(n)$  corresponding to the classifying map  $\theta: X \rightarrow BU(n)$ .  $\square$

We give an explicit formula for  $\kappa_1 \in K^0(X)$  in terms of the first Chern class.

**Lemma 2.0.12.**

$$\kappa_1 = 1 - zc_1 + z^2c_1^2 - z^3c_1^3 + z^4c_1^4 - \dots$$

*Proof.* The equation  $\kappa_1 c_1 = \overline{c_1}$  is equivalent to  $\kappa_1 z^{-1}x = -z^{-1}\overline{x}$ . It follows that

$$\begin{aligned} \kappa_1 &= -\overline{x}/x \\ &= 1 - x + x^2 - x^3 + \dots && \text{by (2.8)} \\ &= 1 - zc_1 + z^2c_1^2 - z^3c_1^3 + \dots \end{aligned}$$

□

We apply Corollary 2.0.11 frequently throughout to explicitly describe the complex  $K$ -theory of each Thom space we consider.

### 2.0.3 Atiyah-Hirzebruch Spectral Sequence

We will carry out calculations using the Atiyah-Hirzebruch spectral sequence [22, Chapter 3].

**Proposition 2.0.13.** *Suppose  $X$  is a space with the homotopy type of a finite dimensional CW-complex. Then there is a half-plane spectral sequence with*

$$E_2^{p,q} \cong H^p(X; KO^q(S^0)),$$

*converging to  $KO^*(X)$ .*

**Definition 2.0.14.** We say that the  $E_r$ -term of a spectral sequence is *concentrated in even degrees* if  $E_r^{p,q} = 0$  when either  $p$  or  $q$  is odd.

If the  $E_r$ -term is concentrated in even degrees then an immediate consequence is that all subsequent differentials  $d_r, d_{r+1}, \dots$  are trivial, and the spectral sequence collapses at the  $r$ th term.

Since we only consider CW-complexes  $X$  which are even,  $H^*(X)$  is a free abelian group, so  $H^p(X; KO^q(S^0))$  and  $H^p(X) \otimes KO^q(S^0)$  are isomorphic. Since  $H^p(X)$  is zero whenever  $p$  is odd and  $KO^q(S^0)$  is isomorphic to  $\mathbb{Z}_2$  when  $q \equiv 7 \pmod{8}$  but is otherwise zero when  $q$  is odd, it follows that with the exception of the rows  $E_2^{p,-8t-1}$ , the  $E_2$ -term is concentrated in even degree. If there are any non-trivial differentials, either the source or target must lie in these rows.

In the  $E_2$ -term, the differentials are described by Fujii [17] as

$$\begin{aligned} d_2^{p,-8t} &= Sq^2: H^p(X; \mathbb{Z}) \rightarrow H^{p+2}(X; \mathbb{Z}_2), \\ d_2^{p,-8t-1} &= Sq^2: H^p(X; \mathbb{Z}_2) \rightarrow H^{p+2}(X; \mathbb{Z}_2). \end{aligned} \tag{2.11}$$

That is, the non-zero differentials may be described as  $Sq^2 \otimes \cdot e$ , where  $\cdot e$  denotes multiplication by  $e \in KO_1$  and  $Sq^2$  is the Steenrod square operation, precomposed with reduction modulo 2 if necessary.

We make use of Proposition 2.0.1 in resolving extension problems encountered whilst reconstructing  $KO^*(X)$  from the  $E_\infty$ -term. Define

$$F^p KO^i(X) := \ker(KO^i(X) \rightarrow KO^i(X^{(p-1)})),$$

the kernel of the restriction to the  $(p-1)$ -skeleton  $X^{(p-1)}$  of  $X$ . There is an isomorphism of graded groups [22, Theorem 3.6]

$$E_\infty^{p,i-p} \cong F^p KO^i(X) / F^{p+1} KO^i(X),$$

and we compute  $F^0 KO^i(X) \cong KO^i(X)$  by finding  $F^p KO^i(X)$  for decreasing  $p$  using short exact sequences

$$0 \rightarrow F^{p+1} KO^i(X) \rightarrow F^p KO^i(X) \rightarrow E_\infty^{p,i-p} \rightarrow 0.$$

Since the existence of a  $\mathbb{Z}_2$ -summand in  $F^p KO^i(X)$  necessitates the existence of a  $\mathbb{Z}_2$ -summand in  $F^{p-1} KO^i(X)$ , and hence in  $KO^i(X)$ , Proposition 2.0.1 is, in many cases, sufficient enable computation of  $KO^i(X)$  from the  $E_\infty$ -term.

The following group will be useful in stating many of our results:

**Definition 2.0.15.** The  $\mathbb{Z}$ -graded abelian group  $\mathbb{Z}(n)$  is isomorphic to  $\mathbb{Z}^n$  in even dimensions, and trivial in odd dimensions.

## 2.0.4 Inverse Limits

In several of our calculations, we wish to compute  $KO^*(X_+)$  where  $X$  is an infinite dimensional complex that may be described as the union of an increasing sequence of finite subcomplexes  $X_0 \subset X_1 \subset \dots$ . Applying a cohomology functor  $D^*$  to such a sequence of subcomplexes and inclusions, we obtain a sequence of  $D_*$ -algebras and homomorphisms, from which we derive information on  $D^*(X)$ .

**Definition 2.0.16.** Given a sequence of abelian groups  $\{G_j \mid j \in \mathbb{Z}\}$  and homomorphisms  $a_j: G_j \rightarrow G_{j-1}$

$$\dots \rightarrow G_2 \xrightarrow{a_2} G_1 \xrightarrow{a_1} G_0$$

the *inverse limit*  $\varprojlim G_j$  is defined to be the subgroup of  $\prod_j G_j$  (with coordinatewise addition) consisting of all sequences  $(g_j)$  with  $a_j(g_j) = g_{j-1}$  for  $j \geq 1$ .

Given  $\delta: \prod G_j \rightarrow \prod G_j$  where  $\delta(\dots, g_j, \dots) = (\dots, g_j - a_{j+1}(g_{j+1}), \dots)$ , we see that  $\varprojlim G_j$  is isomorphic to the kernel of  $\delta$ . We define  $\varprojlim^1 G_j := \text{coker } \delta$ .

**Theorem 2.0.17.** *For a CW-complex  $X$  which is the union of an increasing sequence of subcomplexes  $X_0 \subset X_1 \subset \dots$  there is an exact sequence*

$$0 \rightarrow \varprojlim^1 D^{i-1}(X_j) \rightarrow D^i(X) \rightarrow \varprojlim D^i(X_j) \rightarrow 0$$

for any (reduced or unreduced) cohomology theory  $D^*$ .

*Proof.* See Hatcher [19]. □

*Remark 2.0.18.* The following remarks are taken from [19]:

- $\varprojlim G_j = \varprojlim^1 G_j = 0$  if each  $a_j$  is zero.
- $\varprojlim^1 G_j = 0$  if each  $a_j$  is surjective.
- $\varprojlim G_j$  and  $\varprojlim^1 G_j$  are unchanged if we replace  $\dots \rightarrow G_2 \rightarrow G_1 \rightarrow G_0$  by a subsequence (with homomorphisms appropriate compositions of  $a_j$ s).

Now assume that for any  $i \in \mathbb{Z}$ , we are able to find some subsequence of

$$\dots \rightarrow D^i(X_j) \rightarrow D^i(X_{j-1}) \rightarrow \dots \rightarrow D^i(X_1) \rightarrow D^i(X_0)$$

such that each homomorphism is surjective. Then  $\varprojlim^1 D^i(X_j)$  is zero for all  $i \in \mathbb{Z}$ , and by Theorem 2.0.17 we have an isomorphism  $D^i(X) \cong \varprojlim D^i(X_j)$  for each  $i \in \mathbb{Z}$ .

Note that we have defined inverse limits for sequences of abelian groups and group homomorphisms. We wish to compute the  $D_*$ -algebra  $D^*(X_+)$ , and so require multiplication

$$\varprojlim D^i(X_j) \otimes \varprojlim D^{i'}(X_j) \longrightarrow \varprojlim D^{i+i'}(X_j)$$

for all  $i, i' \in \mathbb{Z}$ .

Given  $x_i \in \varprojlim D^i(X_j)$ ,  $x_{i'} \in \varprojlim D^{i'}(X_j)$ , we define  $m_n(x_i \otimes x_{i'}) \in D^{i+i'}(X_n)$  by  $m_n(x_i \otimes x_{i'}) = p_n(x_i)p_n(x_{i'})$  where  $p_n: \varprojlim D^i(X_j) \rightarrow D^i(X_n)$  is the projection given by restricting to the  $n$ th coordinate, for any  $n \geq 0$ . So for any  $n$ , we have multiplication

$$m_n: \varprojlim D^i(X_j) \otimes \varprojlim D^{i'}(X_j) \longrightarrow D^{i+i'}(X_n).$$



By the universal property of inverse limits (we are taking limits in the category of abelian groups),  $m_n$  lifts to a unique homomorphism

$$m: \varprojlim D^i(X_j) \otimes \varprojlim D^{i'}(X_j) \longrightarrow \varprojlim D^{i+i'}(X_j) \quad (2.12)$$

such that  $p_n m = m_n$  for all  $n$ . Note that  $p_n m(x_i \otimes x_{i'})$  coincides with the product of  $p_n(x_i)$  and  $p_n(x_{i'})$  in  $D^*(X_n)$  using the usual  $D_*$ -algebra multiplication. So by equipping the graded abelian group  $D^*(X) \cong \bigoplus_i D^i(X)$  with multiplication  $m$ , we have the limit of  $D^*(X_j)$  in the category of  $D_*$ -algebras.

**Example 2.0.19.**  $K^*(\mathbb{C}P_+^\infty) \cong K_*[[x]]$ .

$K^*(\mathbb{C}P_+^n) \cong K_*[x]/(x^{n+1})$ , so  $K^{-2i}(\mathbb{C}P^n)$  is isomorphic to  $\mathbb{Z}^n$  with a basis given by the monomials  $z^i x, \dots, z^i x^n$ , whereas  $K^{-2i+1}(\mathbb{C}P^n)$  is zero. The homomorphism induced by inclusion  $\mathbb{C}P^{n-1} \rightarrow \mathbb{C}P^n$  is the identity on  $x^j$  when  $1 \leq j \leq n-1$ , but annihilates  $x^n$ .

Since all homomorphism are surjective, the groups  $K^i(\mathbb{C}P^\infty)$  and  $\varprojlim K^i(\mathbb{C}P^n)$  are isomorphic for all  $i \in \mathbb{Z}$  (Theorem 2.0.17). Since  $\varprojlim K^0(\mathbb{C}P^n)$  consists of sequences

$$\left( a_1 x, a_1 x + a_2 x^2, \dots, \sum_{j=1}^n a_j x^j, \dots \right)$$

for some  $a_j \in \mathbb{Z}$ , we have an isomorphism of  $\varprojlim K^0(\mathbb{C}P^n)$  and the infinite direct product  $\prod \mathbb{Z}$  by identifying each sequence with its limit. We write any element of  $K^0(\mathbb{C}P^\infty)$  in the form  $\sum_{j=1}^\infty a_j x^j$ . Similarly, for any  $i \in \mathbb{Z}$ , there is an isomorphism (of groups)  $K^{-2i}(\mathbb{C}P_+^\infty) \cong z^i \mathbb{Z}[[x]]$ .

Multiplication (2.12) coincides with the obvious power series multiplication, so we have isomorphisms (of rings and  $K_*$ -algebras respectively)  $K^0(\mathbb{C}P_+^\infty) \cong \mathbb{Z}[[x]]$  and  $K^*(\mathbb{C}P_+^\infty) \cong K_*[[x]]$ .

# Chapter 3

## Complex Projective Space

In [17], Fujii computes  $KO^i(\mathbb{C}P_+^n)$ , describing bases for the  $\mathbb{Z}$ -modules  $KO^i(\mathbb{C}P_+^n)$ ,  $-7 \leq i \leq 0$ . Partial algebra structure is also calculated. Civan and Ray [14] extend Fujii's results to state  $KO^*(\mathbb{C}P_+^2)$  and  $KO^*(\mathbb{C}P_+^\infty)$  as  $KO_*$ -algebras, but do not provide a proof. Our first aim is therefore to fill this gap.

We recover Fujii's results, first calculating  $KO^*(\mathbb{C}P^n)$  as a graded abelian group for all  $n \in \mathbb{Z}$ , then giving bases for  $KO^i(\mathbb{C}P^{2n})$ . By computing multiplicative relations, we write  $KO^*(\mathbb{C}P_+^n)$  as a  $KO_*$ -algebra. Other than the case  $n = 2$ , stated in [14], we believe these  $KO_*$ -algebras are not in the literature. Using an inverse limit argument we deduce the  $KO_*$ -algebra  $KO^*(\mathbb{C}P_+^\infty)$ . The calculations of  $KO^*(\mathbb{C}P_+^n)$  as a  $KO_*$ -algebra (for  $n \in \mathbb{Z}^{\geq 0}$ ) have recently been made independently in [32].

Since  $\mathbb{C}P^\infty$  has the homotopy type of  $BU(1)$ , it is a classifying space for principal  $U(1)$  bundles or equivalently for complex line bundles. For an arbitrary complex line bundle  $\phi$  over  $X$ , we introduce characteristic classes by pulling back the generators of the  $KO_*$ -algebra  $KO^*(\mathbb{C}P_+^\infty)$  along the classifying map  $\phi: X \rightarrow \mathbb{C}P^\infty$ .

Using the homotopy equivalence  $\iota: \mathbb{C}P^\infty \rightarrow T\eta$ , we interpret the preceding results in the light of the action (2.4), giving generators and relations for the Thom algebra of  $\eta$  over  $\mathbb{C}P^\infty$ . We denote the generators as spurious Thom classes, and study their pullbacks to the line bundles  $\eta(n)$  over  $\mathbb{C}P^n$  to describe the Thom algebra of  $\eta(n)$  for all positive integers  $n$ .

Subsequent chapters of this thesis will broadly maintain the structure of this

chapter, in the hope of easier reading.

### 3.1 Integral Cohomology and Complex $K$ -theory

We begin by reviewing the well-known integral cohomology and complex  $K$ -theory of  $\mathbb{C}P^n$  (see [19] and [20] respectively, for example) before examining the real  $K$ -theory in detail. For  $K^*(\mathbb{C}P_+^\infty)$ , we repeat material from the preceding chapter for ease of reference (see Example 2.0.6).

Integral cohomology is a complex oriented cohomology theory, and we have the isomorphism (2.6) of  $H_*$ -algebras

$$H^*(\mathbb{C}P_+^\infty) \cong \mathbb{Z}[c_1^H]$$

where  $c_1^H \in H^2(\mathbb{C}P^\infty)$  is the first Chern class.

We restrict via the inclusion  $\eta(n): \mathbb{C}P^n \rightarrow \mathbb{C}P^\infty$  to obtain the following result. As  $H_*$ -algebras,  $H^*(\mathbb{C}P_+^n)$  and  $\mathbb{Z}[c_1]/(c_1^{n+1})$  are isomorphic (see, e.g., [19] or [30]).

Turning to the complex  $K$ -theory, we define the element  $x = \eta - \mathbb{C} \in K^0(\mathbb{C}P^\infty)$ , and have the isomorphism of  $K_*$ -algebras (Example 2.0.6)

$$K^*(\mathbb{C}P_+^\infty) \cong K_*[[x]]. \quad (3.1)$$

For the canonical line bundle  $\eta(n)$  over  $\mathbb{C}P^n$ , we pull back along the inclusion  $\eta(n): \mathbb{C}P^n \rightarrow \mathbb{C}P^\infty$ . We still write  $x$  for the restriction  $\eta(n)^*(x) = \eta(n) - \mathbb{C} \in KO^0(\mathbb{C}P^n)$ , and have the isomorphism of  $K_*$ -algebras [20]

$$K^*(\mathbb{C}P_+^n) \cong K_*[x]/(x^{n+1}). \quad (3.2)$$

Corollary 2.0.11 describes the Thom algebras. Firstly  $K^*(T\eta)$  is a free  $K^*(\mathbb{C}P_+^\infty)$ -module on one generator  $t$ . The Thom algebra is generated by  $t \in K^0(T\eta)$ , with the single relation

$$t^2 = x \cdot t. \quad (3.3)$$

Conjugation interacts with the Thom class via the relation

$$\bar{t} = -\kappa_1 \cdot t \quad (3.4)$$

where  $\kappa_1 \in K^0(BU(n))$  is (Lemma 2.0.12)

$$\kappa_1 = 1 - x + x^2 - x^3 + \dots$$

For the Thom algebra of  $K^*(T\eta(n))$ , we simply restrict the above using the inclusion  $T\eta(n): T\eta(n) \rightarrow MU(1)$ . As above,  $K^*(T\eta(n))$  is a free  $K^*(\mathbb{C}P_+^n)$ -module on one generator  $t$ . The Thom algebra is generated by  $t$ , with the single relation

$$t^2 = x \cdot t.$$

Conjugation interacts with the Thom class via the relation

$$\bar{t} = -\kappa_1(\eta(n)) \cdot t.$$

where  $\kappa_1(\eta(n)) \in K^0(\mathbb{C}P^n)$  is

$$\kappa_1(\eta(n)) = \sum_{j=0}^n (-1)^j x^j.$$

## 3.2 Real $K$ -theory

We now compute the  $KO_*$ -algebra  $KO^*(\mathbb{C}P_+^n)$  for all  $n \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$ , beginning with the group structure for finite  $n$ .

### 3.2.1 Group Structure

We first compute  $KO^*(\mathbb{C}P^n)$  as a graded abelian group using the Atiyah-Hirzebruch spectral sequence. The following proposition may be deduced from [17, Theorem 2].

**Proposition 3.2.1.** *There is an isomorphism of graded abelian groups*

$$KO^*(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z}(\frac{n}{2}) & n \text{ even,} \\ \mathbb{Z}(\frac{n-1}{2}) \oplus KO^*(S^{2n}) & n \text{ odd.} \end{cases}$$

*Proof.* We use the Atiyah-Hirzebruch spectral sequence to calculate  $KO^*(\mathbb{C}P^n)$ . The  $E_2$ -term is given by  $E_2^{p,q} \cong H^p(\mathbb{C}P^n) \otimes KO_{-q}$  (Proposition 2.0.13), and the differentials by  $d_2 = Sq^2 \otimes \cdot e$  (2.11). The  $E_2$ -term is concentrated in even degree, with the

exception of the rows  $E_2^{p,-8t-1}$ , so for dimensional reasons, the only non-zero differentials must be homomorphisms either into or out of these rows. It is straightforward to verify the following equations:  $d_2(c_1^{2k+1} \otimes 1) = c_1^{2k+2} \otimes e$ ,  $d_2(c_1^{2k+1} \otimes e) = c_1^{2k+2} \otimes e^2$  and  $d_2(c_1^{2k} \otimes 1) = d_2(c_1^{2k} \otimes e) = 0$ .

Let  $n$  be even. Then the kernel of  $d_2: E_2^{p,-1} \rightarrow E_2^{p+2,-2}$  consists of elements  $c_1^{2k} \otimes e$ ,  $1 \leq k \leq \frac{n}{2}$ . But  $d_2(c_1^{2k-1} \otimes 1) = c_1^{2k} \otimes e$ , so  $E_3^{p,-1}$  is zero for any integer  $p$ , and  $E_3^{p,-8t-1}$  is zero by periodicity. Then  $E_3^{*,*}$  is concentrated in even degrees, and the spectral sequence collapses at the  $E_3$ -term. Noting that the kernel of any homomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}_2$  is isomorphic to  $\mathbb{Z}$ , we see that for each copy of  $\mathbb{Z}$  in the  $E_2$ -term, a copy of  $\mathbb{Z}$  survives to the  $E_3$ -term and hence to the  $E_\infty$ -term. Because the  $E_\infty$ -term is concentrated in even degrees,  $KO^{2i+1}(\mathbb{C}P^n)$  is zero, so  $KO^{2i}(\mathbb{C}P^n)$  is free abelian (Proposition 2.0.1), implying that given any extension problem

$$0 \rightarrow F^{p+1}KO^{2i}(\mathbb{C}P^n) \rightarrow F^pKO^{2i}(\mathbb{C}P^n) \rightarrow E_\infty^{p,2i-p} \rightarrow 0$$

where  $E_\infty^{p,2i-p}$  and  $\mathbb{Z}_2$  are isomorphic,  $F^{p+1}$  and  $F^p$  are isomorphic. To determine  $KO^{2i}(\mathbb{C}P^n)$ , we need only count the number of copies of the integers in the diagonal  $E_\infty^{p,2i-p}$ , or equivalently the number of copies of the integers in the diagonal  $E_2^{p,2i-p}$ . This proves the proposition for even  $n$ .

Let  $n$  be odd. The final column,  $E_2^{2n,q}$ , is in the kernel of  $d_2$ , but neither  $c_1^n \otimes e$  nor  $c_1^n \otimes e^2$  is in the image of  $d_2$ , so all groups in the column survive to the  $E_3$ -term. As in the case where  $n$  is even, for every copy of  $\mathbb{Z}$  in the  $E_2$ -term, a copy of  $\mathbb{Z}$  survives to the  $E_3$ -term, and, aside from  $E_3^{2n,-1} \cong \mathbb{Z}_2$ , the row  $E_2^{p,-1}$  is zero. So with the exception of  $E_3^{2n,-1-8t}$ , the  $E_3$ -term is concentrated in even degrees. Clearly  $E_3^{2n,-2} \cong \mathbb{Z}_2$  cannot be killed by subsequent differentials, so  $KO^{2n-2}(\mathbb{C}P^n)$  contains a  $\mathbb{Z}_2$ -summand. It follows that  $KO^{2n-1}(\mathbb{C}P^n)$  contains a  $\mathbb{Z}_2$ -summand (Proposition 2.0.1) which is possible only if the group  $\mathbb{Z}_2 \cong E_3^{2n,-1}$  survives to the  $E_\infty$ -term, indicating that  $d_r = 0$  for  $r \geq 3$ . Then  $KO^{2n-1}(\mathbb{C}P^n)$  is isomorphic to  $\mathbb{Z}_2$ , and  $KO^{2n-2}(\mathbb{C}P^n)$  contains a single  $\mathbb{Z}_2$ -summand. As above, we see that  $KO^{2i}(\mathbb{C}P^n)$  is torsion free unless  $i \equiv n-1 \pmod{4}$ , implying that given any extension problem

$$0 \rightarrow F^{p+1}KO^{2i}(\mathbb{C}P^n) \rightarrow F^pKO^{2i}(\mathbb{C}P^n) \rightarrow E_\infty^{p,2i-p} \rightarrow 0$$

where  $p < 2n$  and  $E_\infty^{p,2i-p} \cong \mathbb{Z}_2$ , the groups  $F^{p+1}$  and  $F^p$  are isomorphic. This proves the proposition for odd  $n$ .  $\square$

### 3.2.2 Basis Elements

Following Fujii, we define elements  $u_i$  which generate the  $KO_*$ -algebra  $KO^*(\mathbb{C}P_+^n)$ . We then find bases for  $KO^{-2i}(\mathbb{C}P^{2n})$  in terms of these elements.

**Definition 3.2.2.** Define  $u_i \in KO^{-2i}(\mathbb{C}P^n)$  by  $u_i := r(z^i x)$  for all  $i \in \mathbb{Z}$ .

We make use of (2.8) to describe the complexification  $c(u_i)$ .

$$z^{-i}c(u_i) = x + (-1)^i \bar{x} = \begin{cases} x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n & i \text{ even,} \\ 2x - x^2 + x^3 - x^4 + \cdots + (-1)^{n+1} x^n & i \text{ odd.} \end{cases} \quad (3.5)$$

**Lemma 3.2.3.** *The following equations hold in  $KO^{-2i}(\mathbb{C}P^n)$  for any integer  $i$  and any  $2 \leq j \leq n$ :*

$$r(z^i x^j) = u_0(r(z^i x^{j-1}) + r(z^i x^{j-2})), \quad j \geq 3, \text{ and}$$

$$r(z^i x^2) = \begin{cases} u_i(u_0 + 2) & i \text{ even,} \\ u_i u_0 & i \text{ odd.} \end{cases}$$

*Proof.* Complexifying  $r(z^i x^j)$ , we have:

$$\begin{aligned} z^i x^j + \overline{z^i x^j} &= (x + \bar{x})(z^i x^{j-1} + \overline{z^i x^{j-1}}) - x\bar{x}(z^i x^{j-2} + \overline{z^i x^{j-2}}) \\ &= (x + \bar{x})(z^i x^{j-1} + \overline{z^i x^{j-1}} + z^i x^{j-2} + \overline{z^i x^{j-2}}) \quad \text{by (2.9)} \\ &= c(u_0(r(z^i x^{j-1}) + r(z^i x^{j-2}))). \end{aligned}$$

If  $j = 2$ , then  $z^i x^2 + \overline{z^i x^2} = (z^i x + \overline{z^i x})(x + \bar{x}) + (z^i + \bar{z}^i)(x + \bar{x})$ . Since  $z^i + \bar{z}^i$  is equal to  $2z^i$  if  $i$  is even, and equal to 0 if  $i$  is odd, the result follows by injectivity of  $c$ .  $\square$

We apply Lemma 3.2.3 to give a basis for  $KO^{-2i}(\mathbb{C}P^{2n})$ .

**Proposition 3.2.4.** *The elements  $u_i u_0^k$ ,  $0 \leq k \leq n - 1$  are a basis for the abelian group  $KO^{-2i}(\mathbb{C}P^{2n})$ .*

*Proof.* Since  $KO^{-2i-1}(\mathbb{C}P^{2n})$  is zero, realification is a surjection on  $K^{-2i}(\mathbb{C}P^{2n})$  and complexification  $c: KO^{-2i}(\mathbb{C}P^{2n}) \rightarrow K^{-2i}(\mathbb{C}P^{2n})$  is injective (2.1). Surjectivity of  $r$  implies that the elements  $\{r(z^i x^j) \mid 1 \leq j \leq 2n\}$  span  $KO^{-2i}(\mathbb{C}P^{2n})$ . Repeated application of Lemma 3.2.3 shows that  $r(z^i x^j)$  is equal to some linear combination of elements  $\{u_i u_0^k \mid k \geq 0\}$ . But we have the relation  $u_i u_0^n = 0$ . Indeed, if we complexify, it is clear from (3.5) that  $c(u_i u_0^n)$  has  $x^{2n+1} = 0$  as a factor for any  $i$ , and  $c$  is injective. It follows that the  $n$  elements  $\{u_i u_0^k \mid 0 \leq k \leq n-1\}$  span  $KO^{-2i}(\mathbb{C}P^{2n}) \cong \mathbb{Z}^n$ .  $\square$

### 3.2.3 Multiplicative Relations

We describe the interaction of  $u_i \in KO^{-2i}(\mathbb{C}P^{2n})$  with the coefficient ring, then the products  $u_i u_j \in KO^{-2(i+j)}(\mathbb{C}P^{2n})$  for all integers  $i, j$ . This furnishes us with a description of  $KO^*(\mathbb{C}P_+^{2n})$  as a  $KO_*$ -algebra. Our computations rely on the injectivity of  $c: KO^*(\mathbb{C}P^{2n}) \rightarrow K^*(\mathbb{C}P^{2n})$ , which follows from (2.1) and Proposition 3.2.1.

**Proposition 3.2.5.** *For any integer  $i$ , the following relations hold in  $KO^*(\mathbb{C}P_+^{2n})$ .*

$$eu_i = 0, \quad \alpha u_i = 2u_{i+2}, \quad \beta u_i = u_{i+4}.$$

*Proof.* Referring to (2.1),  $eu_i$  is zero because  $u_i$  is in the image of realification. The remaining relations follow easily by complexifying and referring to Lemma 2.0.3. For example,

$$c(\alpha u_i) = c(\alpha)c(u_i) = 2z^{i+2}(x + \bar{x}) = 2c(u_{i+2}).$$

Injectivity of  $c$  implies that  $\alpha u_i$  and  $2u_{i+2}$  are equal.  $\square$

**Lemma 3.2.6.** *In  $KO^*(\mathbb{C}P^{2n})$ , we have the following relations for any integers  $i, j$ . The relations lie in  $KO^{-4(i+j)}(\mathbb{C}P^{2n})$  and  $KO^{-2(i+j)}(\mathbb{C}P^{2n})$  respectively.*

$$u_{2i-1}u_{2j+1} = (4 + u_0)u_{2(i+j)}, \quad u_i u_j = u_{i+2}u_{j-2}.$$

*Proof.* For the first relation, we complexify  $u_{2i-1}u_{2j+1}$  and obtain

$$\begin{aligned} z^{2i-1}(x - \bar{x})z^{2j+1}(x - \bar{x}) &= z^{2(i+j)}(x^2 + \bar{x}^2 - 2x\bar{x}) \\ &= z^{2(i+j)}[(x + \bar{x})^2 + 4(x + \bar{x})] & (2.9) \\ &= z^{2(i+j)}(x + \bar{x})(4 + x + \bar{x}) \\ &= c(u_{2(i+j)}(4 + u_0)). \end{aligned}$$

The second relation is easily proved by complexifying.  $\square$

### 3.2.4 Finite Projective Spaces

We now examine  $KO^*(\mathbb{C}P_+^n)$  beginning with the case where  $n$  is even before examining the less straightforward case where  $n$  is odd. When  $n$  is odd,  $KO^*(\mathbb{C}P_+^n)$  has a more complicated structure due to fringe effects associated with the top cell.

**Proposition 3.2.7.** *There is an isomorphism of  $KO_*$ -algebras*

$$KO^*(\mathbb{C}P_+^{2n}) \cong KO_*[u_i \mid i \in \mathbb{Z}]/I(2n)$$

where  $I(2n)$  is the ideal

$$(eu_i, \alpha u_i - 2u_{i+2}, \beta u_i - u_{i+4}, u_i u_j - u_{i+2} u_{j-2}, u_{2i-1} u_{2j+1} - (4+u_0)u_{2(i+j)}, u_i u_0^n \mid \text{all } i, j).$$

*Proof.* The relation  $u_i u_0^n = 0$  is obvious if we complexify. Indeed, it is clear from (3.5) that  $c(u_i u_0^n)$  has  $x^{2n+1} = 0$  as a factor for any  $i$ , and  $c$  is injective. Proposition 3.2.4 states that  $\{u_i u_0^k \mid 0 \leq k \leq n-1\}$  is a basis for  $KO^{-2i}(\mathbb{C}P^{2n})$ , and we note that the relations are sufficient to rewrite a polynomial  $P \in KO_*[u_i \mid i \in \mathbb{Z}]$  in terms of the basis.  $\square$

Due to the periodicity relation  $\beta u_i = u_{i+4}$ , the  $KO_*$ -algebra requires only four generators  $u_0, u_1, u_2, u_3$ , for example, as in Fujii [17]. We retain the other  $u_i$  for notational convenience.

It follows from Proposition 3.2.7 that the composition of the restriction homomorphisms

$$KO^*(\mathbb{C}P^{2n+2}) \xrightarrow{j_{2n+1}^*} KO^*(\mathbb{C}P^{2n+1}) \xrightarrow{j_{2n}^*} KO^*(\mathbb{C}P^{2n}) \quad (3.6)$$

is the identity on  $u_i u_0^k$  for  $0 \leq k \leq n-1$ , and annihilates  $u_i u_0^n$ .

We now examine  $KO^*(\mathbb{C}P^{2n+1})$  using the long exact sequence

$$\dots \rightarrow KO^*(S^{2m}) \xrightarrow{p^*} KO^*(\mathbb{C}P^{m+1}) \xrightarrow{j_m^*} KO^*(\mathbb{C}P^m) \rightarrow \dots \quad (3.7)$$

induced by the cofibre sequence of the inclusion  $j_m: \mathbb{C}P^m \rightarrow \mathbb{C}P^{m+1}$ , where  $p$  is the projection onto the top cell. Since  $j_m^*$  is a homomorphism of  $KO_*$ -algebras, the relations of Lemmas 3.2.3, 3.2.6 and Proposition 3.2.5 apply to  $KO^*(\mathbb{C}P^{2n+1})$ .



**Definition 3.2.8.** We define  $\sigma_{-2n-1} \in KO^{4n+2}(\mathbb{C}P^{2n+1})$  as  $p^*(s_{4n+2}^{KO})$ , and note that  $(\sigma_{-2n-1})^2 = 0$ .

**Proposition 3.2.9.** *There is an isomorphism of  $KO_*$ -algebras*

$$KO^*(\mathbb{C}P_+^{2n+1}) \cong KO_*(\mathbb{C}P^{2n+2})[\sigma_{-2n-1}]/I(2n+1)$$

where  $I(2n+1)$  is the ideal

$$(\sigma_{-2n-1}^2, u_{-2n-1}u_0^n - 2\sigma_{-2n-1}, u_{-2n}u_0^n - e^2\sigma_{-2n-1}, u_{-2n+1}u_0^n - \alpha\sigma_{-2n-1}, u_{-2n+2}u_0^n).$$

*Proof.* Firstly we show that the split short exact sequence

$$0 \rightarrow KO^*(S^{4n+2}) \rightarrow KO^*(\mathbb{C}P^{2n+1}) \xrightarrow{j_{2n}^*} KO^*(\mathbb{C}P^{2n}) \rightarrow 0.$$

of abelian groups (Proposition 3.2.1) splits as a sequence of  $KO_*$ -modules.

The corresponding  $K$ -theory exact sequence

$$0 \rightarrow K^*(S^{4n+2}) \rightarrow K^*(\mathbb{C}P^{2n+1}) \xrightarrow{j_{2n}^*} K^*(\mathbb{C}P^{2n}) \rightarrow 0$$

splits as a sequence of  $K_*$ -modules, as the action of the coefficient ring is multiplication by the periodicity elements. When  $KO^i(\mathbb{C}P^{2n+1})$  is torsion free, complexification is injective on  $KO^i(\mathbb{C}P^{2n+1})$ , and the result follows by complexifying and referring to Lemma 2.0.3. But  $KO^i(S^{4n+2})$  and  $\mathbb{Z}_2$  are isomorphic when  $i \equiv 4n+1$  or  $4n \pmod{8}$ . Take an element  $u + u'$  of  $KO^{4n+2}(\mathbb{C}P^{2n+1})$ , where  $u$  and  $u'$  are elements of  $KO^{4n+2}(S^{4n+2})$  and  $KO^{4n+2}(\mathbb{C}P^{2n})$  respectively. Any non-zero  $e(u + u') \in KO^{4n+2}(\mathbb{C}P^{2n+1})$  restricts to zero in  $KO^{4n+1}(\mathbb{C}P^{2n}) \cong \{0\}$ , and so equals  $eu = eu + 0 = eu + eu'$  as required. It follows that  $e^2(u + u') \in KO^{4n}(\mathbb{C}P^{2n+1})$  equals  $e(e(u + 0)) = e^2u + e^2u'$ .

Complexification is a monomorphism on  $KO^{4n-4}(\mathbb{C}P^{2n+1})$ , resulting in the relation  $u_{-2n+2}u_0^n = 0$ . By Proposition 3.2.1, both  $KO^{4n+2}(\mathbb{C}P^{2n+1})$  and  $KO^{4n-2}(\mathbb{C}P^{2n+1})$  are torsion free, so  $c$  is monic on both. This gives us the relations  $2\sigma_{-2n-1} = u_{-2n-1}u_0^n$  and  $\alpha\sigma_{-2n-1} = u_{-2n+1}u_0^n$ . Applying  $KO^{4n}(-)$  to the cofibre sequence  $\mathbb{C}P^{2n+1} \xrightarrow{j_{2n+1}^*} \mathbb{C}P^{2n+2} \rightarrow S^{4n+4}$  shows that  $j_{2n+1}^*$  is an epimorphism. The restrictions (3.6) show that  $u_{-2n}u_0^n = e^2\sigma_{-2n-1}$ .  $\square$

Bases for the abelian groups  $KO^i(\mathbb{C}P^{2n+1})$  are as follows:

$$\begin{aligned}
& u_{-2n-1}, u_{-2n-1}u_0, \dots, u_{-2n-1}u_0^{n-1}, \sigma_{-2n-1}, & i = 4n + 2; \\
& e\sigma_{-2n-1}, & i = 4n + 1; \\
& u_{-2n}, u_{-2n}u_0, \dots, u_{-2n}u_0^{n-1}, e^2\sigma_{-2n-1}, & i = 4n; \\
& u_{-2n+1}, u_{-2n+1}u_0, \dots, u_{-2n+1}u_0^{n-1}, u_{-2n+1}u_0^n, & i = 4n - 2; \\
& u_{-2n+2}, u_{-2n+2}u_0, \dots, u_{-2n+2}u_0^{n-1}, & i = 4n - 4; \\
& 0, & i = 4n - 1, 4n - 3, 4n - 5.
\end{aligned}$$

where  $2\sigma_{-2n-1} = u_{-2n-1}u_0^n$ ,  $e^2\sigma_{-2n-1} = u_{-2n}u_0^n$  and  $\alpha\sigma_{-2n-1} = u_{-2n+1}u_0^n$ .

We have now confirmed the results of [17] on  $KO^*(\mathbb{C}P^n)$  and of [14] on  $KO^*(\mathbb{C}P^2)$ .

### 3.2.5 Infinite Projective Space

We now deduce the  $KO_*$ -algebra  $KO^*(\mathbb{C}P_+^\infty)$  from Proposition 3.2.7 using inverse limits. Noting that  $j_{2n}^* \circ j_{2n+1}^*: KO^{-2i}(\mathbb{C}P^{2n+2}) \rightarrow KO^{-2i}(\mathbb{C}P^{2n})$  is surjective (3.6) and applying Theorem 2.0.17, we see that  $KO^{-2i}(\mathbb{C}P^\infty) \cong \varprojlim KO^{-2i}(\mathbb{C}P^{2n})$  is isomorphic to  $\prod \mathbb{Z}$  and we write the elements in the form  $\sum_{n=0}^\infty a_n u_i u_0^n$ , where  $a_n$  are integers.

Equipping  $KO^*(\mathbb{C}P_+^\infty)$  with multiplication (2.12), we have isomorphisms (of rings and  $\mathbb{Z}[[u_0]]$ -modules respectively)  $KO^0(\mathbb{C}P_+^\infty) \cong \mathbb{Z}[[u_0]]$  and  $KO^{-2i}(\mathbb{C}P^\infty) \cong u_i \mathbb{Z}[[u_0]]$ . Compare with Example 2.0.19.

The above demonstrates that  $KO^*(\mathbb{C}P_+^\infty)$  is generated over  $KO_*$  as an algebra by the elements  $u_i$ ,  $i \in \mathbb{Z}$ . Certain multiplicative relations are carried over from  $KO^*(\mathbb{C}P_+^{2n})$ . Recall that, by the definition of inverse limits, an element in  $KO^i(\mathbb{C}P_+^\infty)$  equals the zero element if and only if it is annihilated by restriction to  $KO^i(\mathbb{C}P_+^{2n})$  for every  $n \geq 0$ . Thus, referring to Proposition 3.2.7, we have the following relations in  $KO^*(\mathbb{C}P_+^\infty)$ :  $eu_i = 0$ ,  $\alpha u_i - 2u_{i+2} = 0$ ,  $\beta u_i - u_{i+4} = 0$ ,  $u_i u_j - u_{i+2} u_{j-2} = 0$  and  $u_{2i-1} u_{2j+1} - (4 + u_0) u_{2(i+j)} = 0$  for any integers  $i, j$ .

Note that  $u_i u_0^n$  is non-zero  $KO^{-2i}(\mathbb{C}P^\infty)$  for any  $i$ , because  $u_i u_0^m$  is a non-zero element of  $KO^{-2i}(\mathbb{C}P_+^{2n})$  for every  $n \geq m + 1$ .

Summarising the above, we have the following result.

**Proposition 3.2.10.** *There are isomorphisms (of rings and  $\mathbb{Z}[[u_0]]$ -modules respectively)*

$$KO^0(\mathbb{C}P_+^\infty) \cong \mathbb{Z}[[u_0]]$$

and

$$KO^{-2i}(\mathbb{C}P^\infty) \cong u_i \mathbb{Z}[[u_0]].$$

For any  $i \in \mathbb{Z}$ ,  $KO^{-2i-1}(\mathbb{C}P^\infty)$  is isomorphic to the trivial group. The action of  $KO_*$  on  $KO^*(\mathbb{C}P_+^\infty)$  is given by the relations

$$eu_i = 0, \quad \alpha u_i = 2u_{i+2} \quad \text{and} \quad \beta u_i = u_{i+4} \quad \text{for all } i \in \mathbb{Z}.$$

The algebra relations are given by

$$u_i u_j = u_{i+2} u_{j-2} \quad \text{and} \quad u_{2i-1} u_{2j+1} = (4 + u_0) u_{2(i+j)} \quad \text{for all } i, j \in \mathbb{Z}.$$

### 3.2.6 Characteristic Classes

We call  $u_i \in KO^{-2i}(\mathbb{C}P^\infty)$  the  *$i$ th universal Fujii class*, and denote its pullback along the map classifying an arbitrary line bundle  $\phi: X \rightarrow \mathbb{C}P^\infty$  as  $u_i(\phi) \in KO^{-2i}(X)$ , for any integer  $i$ . We call  $u_i(\phi)$  the  *$i$ th Fujii class* of  $\phi$ . The Fujii classes of  $\phi$  generate a  $KO_*$ -subalgebra of  $KO^*(X)$ , which we call the *Fujii subalgebra* of  $\phi$  and write as  $FS(\phi)$ . Due to the periodicity relation,  $FS(\phi)$  is unaffected when we restrict to  $0 \leq i \leq 3$ . To measure the Fujii subalgebra we define the *Fujii quotient* as the quotient of  $KO_*$ -modules

$$FQ(\phi) = KO^*(X)/FS(\phi).$$

**Example 3.2.11.** It is immediate that  $FS(\eta)$ , the Fujii subalgebra of  $\eta$ , coincides with  $KO^*(\mathbb{C}P^\infty)$ . It follows that  $FQ(\eta)$  is isomorphic to the trivial group.

Using the inclusion  $\eta(m): \mathbb{C}P^m \rightarrow \mathbb{C}P^\infty$  classifying  $\eta(m)$ , we see that the Fujii classes  $u_i(\eta(m))$  are the elements  $u_i \in KO^{-2i}(\mathbb{C}P^m)$ .

**Example 3.2.12.** When  $m = 2n$  is even,  $u_0, u_1, u_2$  and  $u_3$  generate  $KO^*(\mathbb{C}P^m)$ . Then  $FS(\eta(m)) = KO^*(\mathbb{C}P^m)$ , and  $FQ(\eta(m)) = 0$ .

**Example 3.2.13.** When  $m = 2n + 1$  is odd, the Fujii subalgebra does not contain the elements  $\sigma_{-m}$  and  $e\sigma_{-m}$  in  $KO^*(\mathbb{C}P^m)$ . As a  $KO_*$ -module,  $FQ(\eta(m))$  is generated by  $\sigma_{-m}$ ; as an abelian group it is isomorphic to  $\mathbb{Z}_2$  in dimensions  $2m - 8k$  and  $2m - 8k - 1$ , and is zero otherwise.

### 3.3 Thom Algebra

We now discuss the Thom algebra of  $\eta$  over  $\mathbb{C}P^\infty$ . The notion of a Thom algebra is a new viewpoint, in which the Thom isomorphism is a special case. Since  $\eta$  is not *Spin*, the Thom algebra of  $\eta$  has more than one generator and several relations.

The zero section  $\iota: \mathbb{C}P^\infty \rightarrow T\eta$  is a homotopy equivalence, and so induces the isomorphism  $\iota^*: KO^*(T\eta) \rightarrow KO^*(\mathbb{C}P^\infty)$ . Then the Euler ideal is the subalgebra  $KO^*(\mathbb{C}P^\infty) \subset KO^*(\mathbb{C}P_+^\infty)$ , and every element  $u \in KO^*(\mathbb{C}P^\infty)$  is Eulerian and corresponds to a unique element  $(\iota^*)^{-1}(u) = \langle u \rangle \in KO^*(T\eta)$ . We have the relations

$$v \cdot \langle u \rangle = \langle vu \rangle \quad \text{and} \quad \langle uu' \rangle = \langle u \rangle \langle u' \rangle \quad (3.8)$$

for every  $u, u' \in KO^*(\mathbb{C}P^\infty)$  and  $v \in KO^*(\mathbb{C}P_+^\infty)$ .

We read off the structure of the Euler ideal directly from Proposition 3.2.10, and lift the results back to  $KO^*(T\eta)$ . Using the notation of (2.10), we have the following result.

**Theorem 3.3.1.** *The Thom module  $KO^*(T\eta)$  is generated over  $KO^*(\mathbb{C}P_+^\infty)$  by the elements  $\langle u_0 \rangle, \langle u_1 \rangle, \langle u_2 \rangle, \langle u_3 \rangle$ , with relations*

1.  $e \cdot \langle u_0 \rangle = e \cdot \langle u_1 \rangle = e \cdot \langle u_2 \rangle = e \cdot \langle u_3 \rangle = 0$ ,
2.  $\alpha \cdot \langle u_0 \rangle = 2\langle u_2 \rangle, \quad \alpha \cdot \langle u_1 \rangle = 2\langle u_3 \rangle, \quad \alpha \cdot \langle u_2 \rangle = 2\beta \cdot \langle u_0 \rangle, \quad \alpha \cdot \langle u_3 \rangle = 2\beta \cdot \langle u_1 \rangle$ ,
3.  $u_j \cdot \langle u_0 \rangle = u_{j-2} \cdot \langle u_2 \rangle, \quad u_j \cdot \langle u_1 \rangle = u_{j-2} \cdot \langle u_3 \rangle$ ,
4.  $u_{4j-1} \cdot \langle u_1 \rangle = (u_{4j} + 4\beta^j) \cdot \langle u_0 \rangle$ ,  
 $u_{4j+1} \cdot \langle u_1 \rangle = (u_{4j} + 4\beta^j) \cdot \langle u_2 \rangle$ .

The Thom algebra is generated by the same elements, with the additional relation

$$\langle u_i \rangle \langle u_j \rangle = u_j \cdot \langle u_i \rangle.$$

for all integers  $0 \leq i, j \leq 3$ .

Alternative choices of generators are of course possible, since the periodicity relation  $\beta^k \cdot \langle u_i \rangle = \langle u_{i+4k} \rangle$  holds for all integers  $i$  and  $k$ .

**Definition 3.3.2.** For any complex line bundle  $\phi: X \rightarrow \mathbb{C}P^\infty$ , we may pull back the generators  $\langle u_i \rangle \in KO^{-2i}(T\eta)$  back along the induced map of Thom complexes to elements  $\langle u_i \rangle(\phi) \in KO^{-2i}(T\phi)$ , where  $0 \leq i \leq 3$ . We call these the *spurious Thom classes* of  $\phi$ .

Note that the spurious Thom classes are linked to the Fujii classes by

$$i^*(\langle u_i \rangle(\phi)) = u_i(\phi) \text{ in } KO^{-2i}(X) \text{ for all } 0 \leq i \leq 3. \quad (3.9)$$

The spurious Thom classes may or may not generate the corresponding Thom module, but they obey the pullback relations, and generate the image of  $KO^*(T\eta)$  under  $T\phi^*$ .

We now apply the above to the complex line bundles  $\eta(n): \mathbb{C}P^n \rightarrow \mathbb{C}P^\infty$ , pulling back the generators  $\langle u_i \rangle \in KO^{-2i}(T\eta)$  along the induced map of Thom complexes to elements  $\langle u_i \rangle(\eta(n)) \in KO^{-2i}(T\eta(n))$ . To avoid an excess of notation we abbreviate  $\langle u_i \rangle(\eta(n))$  to  $\langle u_i \rangle$  in the following results.

**Theorem 3.3.3.** *The Thom module  $KO^*(T\eta(2n-1))$  over  $KO^*(\mathbb{C}P_+^{2n-1})$  is generated by the elements  $\langle u_0 \rangle, \langle u_1 \rangle, \langle u_2 \rangle$  and  $\langle u_3 \rangle$ , with relations*

1.  $e \cdot \langle u_0 \rangle = e \cdot \langle u_1 \rangle = e \cdot \langle u_2 \rangle = e \cdot \langle u_3 \rangle = 0$ ,
2.  $\alpha \cdot \langle u_0 \rangle = 2\langle u_2 \rangle, \quad \alpha \cdot \langle u_1 \rangle = 2\langle u_3 \rangle, \quad \alpha \cdot \langle u_2 \rangle = 2\beta \cdot \langle u_0 \rangle, \quad \alpha \cdot \langle u_3 \rangle = 2\beta \cdot \langle u_1 \rangle$ ,
3.  $u_j \cdot \langle u_0 \rangle = u_{j-2} \cdot \langle u_2 \rangle, \quad u_j \cdot \langle u_1 \rangle = u_{j-2} \cdot \langle u_3 \rangle$ ,
4.  $u_{4j-1} \cdot \langle u_1 \rangle = (u_{4j} + 4\beta^j) \cdot \langle u_0 \rangle$ ,  
 $u_{4j+1} \cdot \langle u_1 \rangle = (u_{4j} + 4\beta^j) \cdot \langle u_2 \rangle$ ,

$$\begin{aligned}
5. \quad & \sigma_{1-2n} \cdot \langle u_0 \rangle = \sigma_{1-2n} \cdot \langle u_2 \rangle = 0, \\
& \sigma_{1-2n} \cdot \langle u_1 \rangle = 2u_{2-2n}u_0^{n-2} \cdot \langle u_0 \rangle, \\
& \sigma_{1-2n} \cdot \langle u_3 \rangle = 2u_{4-2n}u_0^{n-2} \cdot \langle u_0 \rangle.
\end{aligned}$$

The Thom algebra is generated by the same elements, with the additional relation

$$\langle u_i \rangle \langle u_j \rangle = u_i \cdot \langle u_j \rangle.$$

for all integers  $0 \leq i, j \leq 3$ .

The reader should note that  $u_0^n \in KO^0(\mathbb{C}P_+^{2n-1})$  is equal to  $e^2\beta^{\frac{n}{2}}\sigma_{1-2n}$  if  $n$  is even, and 0 if  $n$  is odd (Proposition 3.2.9). The relation  $u_0^n \cdot \langle u_i \rangle = 0$  follows. This corresponds to the relation  $u_i u_0^n = 0$  in Proposition 3.2.7, identifying  $KO^*(\mathbb{C}P^{2n})$  as a truncation of  $KO^*(\mathbb{C}P^\infty)$ .

*Proof.* Pulling back the generators of  $KO^{-2i}(T\eta)$  preserves the relations of Theorem 3.3.1. It remains to prove the relations involving products  $\sigma_{1-2n} \cdot \langle u_i \rangle$ .

Complexification is injective on  $KO^*(T\eta(2n-1))$ , and  $c(\sigma_{1-2n})$  equals  $z^{1-2n}x^{2n-1}$  in  $K^{4n-2}(T\eta(2n-1))$ . Commutativity of  $c$  with homomorphisms induced by maps between spaces gives the formula  $c\langle u_i \rangle = \frac{c(u_i)}{x} \cdot t$  in  $K^{-2i}(T\eta(2n-1))$ . The relations are easily proved by referring to (3.5).  $\square$

**Theorem 3.3.4.** *The Thom module  $KO^*(T\eta(2n))$  over  $KO^*(\mathbb{C}P_+^{2n})$  is generated by the elements  $\langle u_0 \rangle, \langle u_1 \rangle, \langle u_2 \rangle, \langle u_3 \rangle$  and  $\sigma_{-2n-1}$ , with relations*

1.  $e \cdot \langle u_0 \rangle = e \cdot \langle u_1 \rangle = e \cdot \langle u_2 \rangle = e \cdot \langle u_3 \rangle = 0,$
2.  $\alpha \cdot \langle u_0 \rangle = 2\langle u_2 \rangle, \quad \alpha \cdot \langle u_1 \rangle = 2\langle u_3 \rangle, \quad \alpha \cdot \langle u_2 \rangle = 2\beta \cdot \langle u_0 \rangle, \quad \alpha \cdot \langle u_3 \rangle = 2\beta \cdot \langle u_1 \rangle,$
3.  $u_j \cdot \langle u_0 \rangle = u_{j-2} \cdot \langle u_2 \rangle, \quad u_j \cdot \langle u_1 \rangle = u_{j-2} \cdot \langle u_3 \rangle,$
4.  $u_{4j-1} \cdot \langle u_1 \rangle = (u_{4j} + 4\beta^j) \cdot \langle u_0 \rangle,$   
 $u_{4j+1} \cdot \langle u_1 \rangle = (u_{4j} + 4\beta^j) \cdot \langle u_2 \rangle,$
5.  $u_i \cdot \sigma_{-2n-1} = 0,$
6.  $2\beta^j \cdot \sigma_{-2n-1} = (u_{4j-2n-1}u_0^{n-1}) \cdot \langle u_0 \rangle,$

$$7. e^2 \beta^j \cdot \sigma_{-2n-1} = (u_{4j-2n} u_0^{n-1}) \cdot \langle u_0 \rangle,$$

$$8. \alpha \beta^j \cdot \sigma_{-2n-1} = (u_{4j-2n+1} u_0^{n-1}) \cdot \langle u_0 \rangle,$$

$$9. (u_{4j-2n+2} u_0^{n-1}) \cdot \langle u_0 \rangle = 0.$$

The Thom algebra is generated by the same elements, with the additional relations

$$\langle u_i \rangle \langle u_j \rangle = u_i \cdot \langle u_j \rangle \quad \text{and} \quad \langle u_i \rangle \sigma_{-2n-1} = 0 \pmod{e^2 \cdot \sigma_{-2n-1}}$$

for all integers  $0 \leq i, j \leq 3$ .

The reader should note that  $u_0^{n+1}$  equals zero in  $KO^*(\mathbb{C}P_+^{2n})$  (Proposition 3.2.7), implying the relation  $u_0^{n+1} \cdot \langle u_i \rangle = 0$  in  $KO^*(T\eta(2n))$  for any integer  $i$ . This corresponds to the relation  $u_i u_0^{n+1} = 0$  in  $KO^*(\mathbb{C}P^{2n+1})$  (Proposition 3.2.9).

*Proof.* Pulling back the generators of  $KO^{-2i}(T\eta)$  preserves the relations of Theorem 3.3.1. Relations 6 to 9, describing the interaction of  $\sigma_{-2n-1}$  with the coefficients  $KO_*$ , come from Proposition 3.2.9 via the homeomorphism  $T\eta(2n) \cong \mathbb{C}P^{2n+1}$ .

It remains to prove that  $u_i \sigma_{-2n-1}$  equals zero. We use complexification, which annihilates  $e^2 \cdot \sigma_{-2n-1} \in KO^{4n}(T\eta(2n))$ , but is injective on  $KO^*(T\eta(2n))/(e^2 \cdot \sigma_{-2n-1})$ .

□

# Chapter 4

## Stunted Projective Spaces

In this chapter, we compute the Thom algebra of the Whitney sum  $m\eta$  of canonical line bundles over  $\mathbb{C}P^n$  where  $n \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$ . The Thom space  $Tm\eta$  is homeomorphic to the quotient space  $\mathbb{C}P^{m+n}/\mathbb{C}P^{m-1}$  and is studied in the context of  $K$ -theory by Atiyah and Todd [8].

We divide the family of bundles  $m\eta$  in half. When  $m$  is even,  $m\eta$  is *Spin*, and the Thom algebra of  $m\eta$  is generated by  $t^{KO} \in KO^{2m}(Tm\eta)$  with a single relation. When  $m$  is odd,  $m\eta$  is not *Spin*, and the Thom algebra has several generators and a larger set of relations. The reader should note this contrast in complexity depending on the parity of  $m$ . Note that  $1\eta$  is simply the canonical line bundle  $\eta$  over  $\mathbb{C}P^n$ .

We begin by discussing the complex  $K$ -theory of  $Tm\eta$ , and the Thom algebra of  $Tm\eta$  where  $m$  is even. The bulk of our computations concern the case where  $m$  is odd. We compute the graded group  $KO^*(Tm\eta)$  for all  $n$ , then describe a basis for the  $KO^*(\mathbb{C}P_+^n)$ -module  $KO^{-2i}(Tm\eta)$ . By computing multiplicative relations, we determine the Thom algebra of  $m\eta$  over  $\mathbb{C}P^n$  for all finite  $n$ , then use inverse limits to study the infinite version of this result.

Other than Lemma 4.2.3, which is no doubt a well known linear algebra result, we believe that the material in this chapter concerning real  $K$ -theory is original. The complex  $K$ -theory of  $Tm\eta$  is well known.



## 4.1 Complex $K$ -theory

As we discussed the complex  $K$ -theory of the base space  $\mathbb{C}P^n$  in the preceding chapter (see (3.1) and (3.2)) we only comment on the complex  $K$ -theory of the Thom space  $Tm\eta(n)$  here.

Using the Whitney sum formula,  $c_k(\xi \oplus \theta) = \sum_{i+j=k} c_i(\xi)c_j(\theta)$  [30], it follows that  $c_m(m\eta) = c_1(\eta)^m = z^{-m}x^m$  (Example 2.0.6). Since there are no multiplicative relations in  $K^*(\mathbb{C}P_+^\infty)$ , the element  $\kappa_m(m\eta) \in K^0(\mathbb{C}P_+^\infty)$  is given by

$$\kappa_m(m\eta) = \overline{c_m(m\eta)}/c_m(m\eta) = \overline{z^{-m}x^m}/z^{-m}x^m = (1 + \bar{x})^m \quad \text{by (2.9).}$$

Then by Corollary 2.0.11 we have the following result.  $K^*(Tm\eta)$  is a free  $K^*(\mathbb{C}P_+^\infty)$ -module on one generator  $t \in K^0(Tm\eta)$ . The Thom algebra is generated by  $t$ , with the single relation

$$t^2 = x^m \cdot t. \tag{4.1}$$

Conjugation interacts with the Thom class via the relation

$$\begin{aligned} \bar{t} &= (-1 - \bar{x})^m \cdot t \\ &= (-\bar{\eta})^m \cdot t. \end{aligned} \tag{4.2}$$

For the Thom algebra of  $m\eta(n)$  over  $\mathbb{C}P^n$ , we restrict the above and observe that  $K^*(Tm\eta(n))$  is a free  $K^*(\mathbb{C}P_+^n)$ -module on one generator  $t \in K^0(Tm\eta(n))$ . The Thom class  $t$  satisfies both (4.1) and (4.2), but we have the relation  $x^{n+1} = 0$  in  $K^*(\mathbb{C}P^n)$ .

## 4.2 Real $K$ -theory

### 4.2.1 The $KO$ -orientable case

Let  $m = 2a$  be even. Using the Whitney sum formula, we see that the second Stiefel-Whitney class  $w_2(m\eta) = m \cdot w_2(\eta) \in H^2(\mathbb{C}P^\infty; \mathbb{Z}_2)$  is zero, and the bundle  $m\eta$  is  $KO$ -orientable.

**Lemma 4.2.1.** *Unless  $n \equiv 1 \pmod{4}$ , there is a unique element  $t^{KO} \in KO^{4a}(T2a\eta)$  with complexification  $c(t^{KO}) = \bar{\eta}^a \cdot t^K$ . It is a Thom class, with square  $(t^{KO})^2 = (u_{-2})^a \cdot t^{KO} \in KO^{8a}(T2a\eta)$ . When  $n \equiv 1 \pmod{4}$ ,  $t^{KO}$  is only well-defined up to a unique element of order 2 in  $KO^{4a}(T2a\eta)$ .*

*Proof.* The bundle  $2a\eta$  is *Spin* and we have the isomorphism of abelian groups  $KO^i(T2a\eta) \cong KO^{i-4a}(\mathbb{C}P_+^n)$ . Unless  $n \equiv 1 \pmod{4}$ , we have the following exact sequence:

$$0 \rightarrow KO^{4a}(T2a\eta) \xrightarrow{\chi} K^{4a+2}(T2a\eta) \xrightarrow{r} KO^{4a+2}(T2a\eta) \rightarrow 0$$

and  $t^{KO}$  is uniquely defined, if it exists.

$$\text{Using } \bar{t}^K = \bar{\eta}^{2a} \cdot t^K,$$

$$cr(z^{-1}\bar{\eta}^a \cdot t^K) = z^{-1}(\bar{\eta}^a \cdot t^K - \eta^a \cdot \bar{t}^K) = z^{-1}((\bar{\eta}^a - \eta^a\bar{\eta}^{2a}) \cdot t^K) = 0$$

so  $r(z^{-1}\bar{\eta}^a \cdot t^K) = 0$ , and  $t^{KO}$  exists. It is a Thom class because  $\bar{\eta}^a \cdot t^K$  is a Thom class, and  $c$  a map of ring spectra. Using  $(t^K)^2 = z^{-2a}x^{2a} \cdot t^K = z^{-2a}(\eta - 1)^{2a} \cdot t^K$ , we have the following:

$$\begin{aligned} c((t^{KO})^2) &= z^{-2a}(\eta - 1)^{2a}\bar{\eta}^{2a} \cdot t^K = z^{-2a}(1 - \bar{\eta})^a(\eta - 1)^a\bar{\eta}^a \cdot t^K \\ &= z^{-2a}(x + \bar{x})^a \cdot c(t^{KO}) = c(u_{-2}^a \cdot t^{KO}) \end{aligned}$$

So  $(t^{KO})^2 = (u_{-2})^a \cdot t^{KO}$  in  $KO^{8a}(T2a\eta)$ .

If  $n \equiv 1 \pmod{4}$ , then we have the exact sequence

$$0 \rightarrow \mathbb{Z}_2 \xrightarrow{e} KO^{4a}(T2a\eta) \xrightarrow{\chi} K^{4a+2}(T2a\eta) \xrightarrow{r} KO^{4a+2}(T2a\eta) \xrightarrow{e} \mathbb{Z}_2 \rightarrow 0 \quad (4.3)$$

and  $t^{KO}$  is well-defined up to the unique element of order 2 in  $KO^{4a}(T2a\eta)$ . Since any element of order 2 in  $KO^{4a}(T2a\eta)$  is of the form  $e^2\tau$  for some  $\tau$  in  $KO^{4a+2}(T2a\eta)$  (see (4.3)) and  $(t^{KO} + e^2\tau)^2 = (t^{KO})^2 + 2e^2\tau t^{KO} + e^4\tau^2 = (t^{KO})^2$ , the square of  $t^{KO}$  is unaffected by the choice of Thom class.  $\square$

Then  $KO^*(Tm\eta)$  is a  $KO^*(\mathbb{C}P_+^n)$ -module on one generator  $t^{KO} \in KO^{2m}(Tm\eta)$ . The Thom algebra is generated by  $t^{KO}$  with the single relation  $(t^{KO})^2 = (u_{-2})^a \cdot t^{KO}$ .

### 4.2.2 Group Structure

We now turn to the more complicated case: computing the Thom algebra  $KO^*(Tm\eta)$  over  $KO^*(\mathbb{C}P_+^n)$  for all  $n$  where  $m$  is odd, and consequently  $m\eta$  fails to be *Spin*. Note that for the case  $m = 1$ , we replicate the results of the preceding chapter. For the remainder of the chapter,  $m$  will be an odd, positive integer. There is a homeomorphism of spaces  $Tm\eta \cong \mathbb{C}P^{m+n}/\mathbb{C}P^{m-1} := \mathbb{C}P_m^{m+n}$ . [19]

**Proposition 4.2.2.** *Let  $\eta$  over  $\mathbb{C}P^n$  be the canonical line bundle. Then there is an isomorphism of graded abelian groups*

$$KO^*(Tm\eta) \cong \begin{cases} \mathbb{Z}(\frac{n+1}{2}) & n \text{ odd,} \\ \mathbb{Z}(\frac{n}{2}) \oplus KO^*(S^{2(m+n)}) & n \text{ even.} \end{cases}$$

*Proof.* We study the real  $K$ -theory long exact sequence induced by the cofibre sequence

$$\mathbb{C}P^{m-1} \hookrightarrow \mathbb{C}P^{m+n} \rightarrow Tm\eta.$$

If  $n$  is odd, we have isomorphisms  $KO^{2i}(\mathbb{C}P^{m+n}) \cong \mathbb{Z}^{\frac{m+n}{2}}$  and  $KO^{2i}(\mathbb{C}P^{m-1}) \cong \mathbb{Z}^{\frac{m-1}{2}}$  and both  $KO^{2i+1}(\mathbb{C}P^{m+n})$  and  $KO^{2i+1}(\mathbb{C}P^{m-1})$  are zero for all  $i \in \mathbb{Z}$  (Proposition 3.2.1). So the long exact sequence becomes

$$0 \rightarrow KO^{2i}(Tm\eta) \rightarrow KO^{2i}(\mathbb{C}P^{m+n}) \rightarrow KO^{2i}(\mathbb{C}P^{m-1}) \rightarrow KO^{2i+1}(Tm\eta) \rightarrow 0.$$

The monomorphism  $KO^{2i}(Tm\eta) \rightarrow KO^{2i}(\mathbb{C}P^{m+n})$  shows that  $KO^{2i}(Tm\eta)$  is torsion free, and it follows that  $KO^{2i+1}(Tm\eta)$  is zero (Proposition 2.0.1). Because  $KO^{2i}(\mathbb{C}P^{m-1})$  is free abelian, the sequence splits, proving the proposition for odd  $n$ .

Let  $n$  be even. Then  $KO^{2i+1}(\mathbb{C}P^{m+n})$  is isomorphic to  $\mathbb{Z}_2$  if  $i \equiv m+n-1 \pmod{4}$ , and is zero otherwise (Proposition 3.2.1). We examine these two cases in turn.

Suppose  $i \equiv m+n-1 \pmod{4}$ . Then we have the following exact sequence:

$$0 \rightarrow KO^{2i}(Tm\eta) \rightarrow KO^{2i}(\mathbb{C}P^{m+n}) \rightarrow KO^{2i}(\mathbb{C}P^{m-1}) \rightarrow KO^{2i+1}(Tm\eta) \rightarrow \mathbb{Z}_2 \rightarrow 0.$$

The monomorphism  $KO^{2i}(Tm\eta) \rightarrow KO^{2i}(\mathbb{C}P^{m+n}) \cong \mathbb{Z}^{\frac{m+n-1}{2}} \oplus \mathbb{Z}_2$  indicates that  $KO^{2i}(Tm\eta)$  has at most a single  $\mathbb{Z}_2$ -summand. Then by 2.0.1,  $KO^{2i+1}(Tm\eta)$  is

isomorphic to either  $\mathbb{Z}_2$  or the trivial group. But  $KO^{2i+1}(Tm\eta) \rightarrow \mathbb{Z}_2$  is an epimorphism, so  $KO^{2i+1}(Tm\eta)$  is non-zero, and the epimorphism is an isomorphism. The remainder of the sequence is a (split) short exact sequence

$$0 \rightarrow KO^{2i}(Tm\eta) \rightarrow KO^{2i}(\mathbb{C}P^{m+n}) \rightarrow KO^{2i}(\mathbb{C}P^{m-1}) \rightarrow 0.$$

Suppose  $i \not\equiv m+n-1 \pmod{4}$ . Then we have the following exact sequence:

$$0 \rightarrow KO^{2i}(Tm\eta) \rightarrow KO^{2i}(\mathbb{C}P^{m+n}) \rightarrow KO^{2i}(\mathbb{C}P^{m-1}) \rightarrow KO^{2i+1}(Tm\eta) \rightarrow 0.$$

Recall that  $KO^{2i}(\mathbb{C}P^{m+n})$  is free abelian. Since  $KO^{2i}(Tm\eta) \rightarrow KO^{2i}(\mathbb{C}P^{m+n})$  is a monomorphism,  $KO^{2i}(Tm\eta)$  is torsion free, and  $KO^{2i+1}(Tm\eta)$  must be zero. Again we are left with the (split) short exact sequence

$$0 \rightarrow KO^{2i}(Tm\eta) \rightarrow KO^{2i}(\mathbb{C}P^{m+n}) \rightarrow KO^{2i}(\mathbb{C}P^{m-1}) \rightarrow 0.$$

Then for any  $i \in \mathbb{Z}$ , we have the following isomorphisms of abelian groups.

$$KO^{2i}(\mathbb{C}P^{m+n}) \cong KO^{2i}(Tm\eta) \oplus KO^{2i}(\mathbb{C}P^{m-1})$$

and

$$KO^{2i+1}(Tm\eta) \cong \begin{cases} \mathbb{Z}_2 & i \equiv m+n-1 \pmod{4}, \\ 0 & \text{otherwise,} \end{cases}$$

proving the proposition for even  $n$ . □

### 4.2.3 Basis Elements

We now consider bases for the graded groups  $KO^*(Tm\eta)$ . First we need the following lemma.

**Lemma 4.2.3.** *Let  $x_1, \dots, x_n$  be a set of non-zero, linearly independent elements of  $\mathbb{Z}^n$  with the property that if the sum  $a_1x_1 + \dots + a_nx_n$  is divisible by  $k \in \mathbb{Z}$ , then  $a_i$  is divisible by  $k$  for each  $i$ . Then  $x_1, \dots, x_n$  is a basis of  $\mathbb{Z}^n$ .*

*Proof.* Let  $[e_1, \dots, e_n]^T$  be the column vector of standard generators of  $\mathbb{Z}^n$ . Then  $[x_1, \dots, x_n]^T = A[e_1, \dots, e_n]^T$  for some square matrix  $A$ . There exists a matrix

$[b_{ij}] = B$  such that  $BA = mI_n$  for some non-zero integer  $m$ . Then  $B[x_1, \dots, x_n]^T = m[e_1, \dots, e_n]^T$ . For each  $e_i$ , we can write

$$me_i = \sum_j b_{ij}x_j$$

so  $m$  divides  $\sum_j b_{ij}x_j$ . But then  $m$  divides  $b_{ij}$  for all  $j$ , and we may write

$$e_i = \sum_j \frac{b_{ij}}{m}x_j.$$

So the set  $x_1, \dots, x_n$  is a spanning set, and therefore is a basis.  $\square$

We define elements  $g_i$  which generate the Thom module of  $m\eta$  over  $\mathbb{C}P^n$ .

**Definition 4.2.4.** Define  $g_i \in KO^{-2i}(Tm\eta)$  by  $g_i := r(z^i\bar{\eta}^{\frac{m-1}{2}} \cdot t)$ .

**Proposition 4.2.5.** Let  $\eta$  be the canonical bundle over  $\mathbb{C}P^{2n+1}$ . The elements  $u_0^k \cdot g_i$ ,  $0 \leq k \leq n$ , form a basis for the abelian group  $KO^{-2i}(Tm\eta)$ .

*Proof.* Since  $m$  is odd,  $\bar{t} = \overline{z^m \cdot t^K} = -\bar{\eta}^m \cdot t$ . Then

$$\begin{aligned} c(u_0^k \cdot g_{2i}) &= z^{2i}(x + \bar{x})^k (\bar{\eta}^{\frac{m-1}{2}} \cdot t + \eta^{\frac{m-1}{2}} \cdot \bar{t}) \\ &= z^{2i}(x + \bar{x})^k (\bar{\eta}^{\frac{m-1}{2}} - \eta^{\frac{m+1}{2}}) \cdot t \\ &= z^{2i}((1 - x + x^2 - \dots)^{\frac{m-1}{2}} - (1 - x + x^2 - \dots)^{\frac{m+1}{2}}) \\ &\quad (x^2 - x^3 + x^4 - \dots)^k \cdot t \\ &= z^{2i}((1 - (\frac{m-1}{2})x + \dots) - (1 - (\frac{m+1}{2})x + \dots)) \\ &\quad (x^{2k} - kx^{2k+1} + \dots) \cdot t \\ &= z^{2i}(x^{2k+1} - \dots) \cdot t \end{aligned}$$

and

$$\begin{aligned} c(u_0^k \cdot g_{2i+1}) &= z^{2i+1}(\bar{\eta}^{\frac{m-1}{2}} \cdot t - \eta^{\frac{m-1}{2}} \cdot \bar{t})(x + \bar{x})^k \\ &= z^{2i}((1 - (\frac{m-1}{2})x + \dots) + (1 - (\frac{m+1}{2})x + \dots)) \\ &\quad (x^{2k} - kx^{2k+1} + \dots) \cdot t \\ &= z^{2i+1}(2x^{2k} - (m+2k)x^{2k+1} + \dots) \cdot t. \end{aligned}$$

Choose any  $i \in \mathbb{Z}$ . Then, noting that  $K^{-2i}(Tm\eta)$  has a basis  $z^i x^j \cdot t$  where  $0 \leq j \leq 2n+1$ , the above equations show that the images of  $u_0^k \cdot g_i$  under complexification are linearly independent, non-zero elements of  $K^{-2i}(Tm\eta)$  for  $k \leq n$ . A linear combination

$$a_0 c(g_i) + a_1 c(u_0 \cdot g_i) + \dots + a_n c(u_0^n \cdot g_i)$$

is divisible by an integer  $l$  precisely when every  $a_i$  is divisible by  $l$  (noting that  $m+2k$  and 2 are coprime when  $m$  is odd). Because complexification is injective,  $u_0^k \cdot g_i$  fulfil the conditions of Lemma 4.2.3, and so are a basis for  $KO^{-2i}(Tm\eta) \cong \mathbb{Z}^{n+1}$ .  $\square$

#### 4.2.4 Multiplicative Relations

Let  $\eta$  be the canonical bundle over  $\mathbb{C}P^{2n+1}$ . We now describe the interaction of  $g_i \in KO^{-2i}(Tm\eta)$  with the coefficient ring  $KO_*$ , then the products  $u_i \cdot g_j$  and  $g_i g_j \in KO^{-2(i+j)}(Tm\eta)$ . This furnishes us with a description of the Thom algebra of  $m\eta$  over  $\mathbb{C}P^{2n+1}$ . The proofs rely on injectivity of complexification, which holds by (2.1) since  $KO^{-2i+1}(Tm\eta)$  is zero.

**Lemma 4.2.6.** *The following relations hold in the  $KO_*$ -module  $KO^*(Tm\eta)$ .*

$$e \cdot g_i = 0, \quad \alpha \cdot g_i = 2g_{i+2}, \quad \beta \cdot g_i = g_{i+4}$$

*Proof.* This is similar to the proof of Proposition 3.2.5.  $\square$

**Lemma 4.2.7.** *In  $KO^{-2(i+j)}(Tm\eta)$ , we have the following relation for any integers  $i, j$ .*

$$u_j \cdot g_i = u_{j-2} \cdot g_{i+2}.$$

*Proof.* Immediate by complexification.  $\square$

**Lemma 4.2.8.** *In  $KO^{-4i-4j-2}(Tm\eta)$ , we have the following relation for any integers  $i, j$ .*

$$u_{2j+1} \cdot g_{2i} = u_0 \cdot g_{2i+2j+1}.$$

*Proof.*

$$\begin{aligned} c(u_{2j+1} \cdot g_{2i}) &= z^{2i+2j+1} \left( \bar{\eta}^{\frac{m-1}{2}} \cdot t + \eta^{\frac{m-1}{2}} \cdot \bar{t} \right) (x - \bar{x}) \\ &= z^{2i+2j+1} \left( \bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}} \right) (\eta - \bar{\eta}) \cdot t \\ &= z^{2i+2j+1} \left( \bar{\eta}^{\frac{m-3}{2}} - \bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}} + \bar{\eta}^{\frac{m+3}{2}} \right) \cdot t \\ &= z^{2i+2j+1} \left( \bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}} \right) (\eta + \bar{\eta} - 2) \cdot t \\ &= \left( z^{2i+2j+1} \bar{\eta}^{\frac{m-1}{2}} \cdot t + \bar{z}^{2i+2j+1} \eta^{\frac{m-1}{2}} \cdot \bar{t} \right) (x + \bar{x}) \\ &= c(u_0 \cdot g_{2i+2j+1}) \end{aligned}$$

$\square$

**Lemma 4.2.9.** *In  $KO^{-4(i+j)}(Tm\eta)$ , we have the following relation for any integers  $i, j$ .*

$$u_{2j-1} \cdot g_{2i+1} = (u_0 + 4) \cdot g_{2(i+j)}.$$

*Proof.*

$$\begin{aligned} c(u_{2j-1} \cdot g_{2i+1}) &= z^{2i+2j} (\bar{\eta}^{\frac{m-1}{2}} \cdot t - \eta^{\frac{m-1}{2}} \cdot \bar{t}) (x - \bar{x}) \\ &= z^{2i+2j} (\bar{\eta}^{\frac{m-1}{2}} + \bar{\eta}^{\frac{m+1}{2}}) (\eta - \bar{\eta}) \cdot t \\ &= z^{2i+2j} (\bar{\eta}^{\frac{m-3}{2}} + \bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}} - \bar{\eta}^{\frac{m+3}{2}}) \cdot t \\ &= z^{2i+2j} (\bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}}) (\eta + \bar{\eta} + 2) \cdot t \\ &= (z^{2i+2j} \bar{\eta}^{\frac{m-1}{2}} \cdot t + \bar{z}^{2i+2j} \eta^{\frac{m-1}{2}} \cdot \bar{t}) (x + \bar{x} + 4) \\ &= c((u_0 + 4) \cdot g_{2(i+j)}) \end{aligned}$$

□

**Lemma 4.2.10.** *In  $KO^{-4i-2j}(Tm\eta)$ , we have the following relation for any integers  $i, j$ .*

$$g_{2i} g_j = u_0^{\frac{m+1}{2}} \cdot g_{2i+j}.$$

*Proof.* We need the following relation:

$$\begin{aligned} (\bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}}) x^m &= (1 - \bar{\eta}) x \bar{\eta}^{\frac{m-1}{2}} x^{m-1} \\ &= (-x\bar{x}) \left(-\frac{\bar{x}}{x}\right)^{\frac{m-1}{2}} x^{m-1} \\ &= (-1)^{\frac{m-1}{2}} (x + \bar{x}) (-1)^{\frac{m-1}{2}} (x + \bar{x})^{\frac{m-1}{2}} \\ &= (x + \bar{x})^{\frac{m+1}{2}} \\ &= c(u_0^{\frac{m+1}{2}}). \end{aligned}$$

Then

$$\begin{aligned} c(g_{2i} g_j) &= z^{2i} (\bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}}) (z^j \bar{\eta}^{\frac{m-1}{2}} - \bar{z}^j \eta^{\frac{m+1}{2}}) \cdot t^2 \\ &= (\bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}}) x^m (z^{2i+j} \bar{\eta}^{\frac{m-1}{2}} - \bar{z}^{2i+j} \eta^{\frac{m+1}{2}}) \cdot t \\ &= c(u_0^{\frac{m+1}{2}} \cdot g_{2i+j}). \end{aligned}$$

□

**Lemma 4.2.11.** *In  $KO^{-4(i+j)}(Tm\eta)$ , we have the following relation for any integers  $i, j$ .*

$$g_{2i+1} g_{2j-1} = u_0^{\frac{m-1}{2}} (u_0 + 4) \cdot g_{2(i+j)}.$$

*Proof.*

$$\begin{aligned}
\bar{\eta}^{\frac{m-3}{2}}(1 + \bar{\eta}^2)x^{m-1} &= \left(-\frac{\bar{x}}{x}\right)^{\frac{m-3}{2}}(1 + \bar{\eta})^2x^{m-1} \\
&= \left(-\frac{\bar{x}}{x}\right)^{\frac{m-3}{2}}x^{m-3}(1 + \bar{\eta})^2x^2 \\
&= \left(-x\bar{x}\right)^{\frac{m-3}{2}}(x + x\bar{\eta})^2 \\
&= \left(-x\bar{x}\right)^{\frac{m-3}{2}}(x - \bar{x})^2 \\
&= c(u_0^{\frac{m-3}{2}}(u_0^2 + 4u_0)) \\
&= c(u_0^{\frac{m-1}{2}}(u_0 + 4))
\end{aligned}$$

We have used the fact that  $(x - \bar{x})^2 = c(u_{-1}u_1) = c(u_0(u_0 + 4))$  from Proposition 3.2.10. Then

$$\begin{aligned}
c(g_{2i+1}g_{2j-1}) &= z^{2(i+j)}\left(\bar{\eta}^{\frac{m-1}{2}} + \bar{\eta}^{\frac{m+1}{2}}\right)^2 \cdot t^2 \\
&= z^{2(i+j)}\left(\bar{\eta}^{\frac{m-1}{2}} + \bar{\eta}^{\frac{m+1}{2}}\right)^2 x^m \cdot t \\
&= z^{2(i+j)}\bar{\eta}^{m-1}(1 + \bar{\eta})^2 x^m \cdot t \\
&= z^{2(i+j)}x\bar{\eta}\bar{\eta}^{\frac{m-1}{2}} \cdot t\bar{\eta}^{\frac{m-3}{2}}(1 + \bar{\eta})^2 x^{m-1} \\
&= z^{2(i+j)}(1 - \bar{\eta})\bar{\eta}^{\frac{m-1}{2}} \cdot tc(u_0^{\frac{m-1}{2}}(u_0 + 4)) \\
&= z^{2(i+j)}\left(\bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}}\right) \cdot tc(u_0^{\frac{m-1}{2}}(u_0 + 4)) \\
&= c(u_0^{\frac{m-1}{2}}(u_0 + 4)) \cdot c(g_{2(i+j)}).
\end{aligned}$$

□

### 4.3 Thom Algebra

Using the above, we now discuss the Thom modules  $KO^*(Tm\eta)$  for  $\eta$  over  $\mathbb{C}P^n$ .

Where necessary, we write  $\eta(n)$  to emphasis the base space.

**Theorem 4.3.1.** *Let  $\eta$  be the canonical complex line bundle over  $\mathbb{C}P^{2n+1}$ . Then the Thom module  $KO^*(Tm\eta)$  is generated over  $KO^*(\mathbb{C}P_+^{2n+1})$  by the elements  $g_0, g_1, g_2$  and  $g_3$ , with relations*

1.  $e \cdot g_0 = e \cdot g_1 = e \cdot g_2 = e \cdot g_3 = 0$ ,
2.  $\alpha \cdot g_0 = 2g_2, \alpha \cdot g_1 = 2g_3, \alpha \cdot g_2 = 2\beta \cdot g_0, \alpha \cdot g_3 = 2\beta \cdot g_1$ ,
3.  $u_j \cdot g_0 = u_{j-2} \cdot g_2, u_j \cdot g_1 = u_{j-2} \cdot g_3$ ,



$$4. \quad u_{4j+1} \cdot g_0 = u_{4j} \cdot g_1, \quad u_{4j+3} \cdot g_0 = u_{4j} \cdot g_3,$$

$$5. \quad u_{4j-1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_0,$$

$$u_{4j+1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_2,$$

$$6. \quad \sigma_{-2n-1} \cdot g_0 = \sigma_{-2n-1} \cdot g_2 = 0,$$

$$\sigma_{-2n-1} \cdot g_1 = \begin{cases} 2u_{-2n}u_0^{n-1} \cdot g_0, & n \text{ even,} \\ 2u_{-2n-2}u_0^{n-1} \cdot g_2, & n \text{ odd,} \end{cases}$$

$$\sigma_{-2n-1} \cdot g_3 = \begin{cases} 2u_{-2n}u_0^{n-1} \cdot g_2, & n \text{ even,} \\ 2u_{-2n+2}u_0^{n-1} \cdot g_0, & n \text{ odd.} \end{cases}$$

The Thom algebra is generated by the same elements, with the additional relation

$$g_i g_j = \begin{cases} u_0^{\frac{m-1}{2}} (u_0 + 4) \cdot g_{i+j}, & i \text{ and } j \text{ odd,} \\ u_0^{\frac{m+1}{2}} \cdot g_{i+j}, & \text{else} \end{cases}$$

for all integers  $0 \leq i, j \leq 3$ .

Since  $u_0^{n+1}$  is equal to  $e^2 \beta^{\frac{n}{2}} \sigma_{-2n-1}$  if  $n$  is even and equal to 0 if  $n$  is odd (Proposition 3.2.9) it follows that  $u_0^{n+1} \cdot g_i = 0$  for any  $i$ . This implies that the composition of the restriction homomorphisms

$$KO^*(Tm\eta(2n+1)) \xrightarrow{Tj_{2n}^*} KO^*(Tm\eta(2n)) \xrightarrow{Tj_{2n-1}^*} KO^*(Tm\eta(2n-1)) \quad (4.4)$$

is the identity on  $u_0^k \cdot g_i$  for  $0 \leq k \leq n-1$ , and annihilates  $u_0^n \cdot g_i$ .

*Proof.* Recall relations of Lemmas 4.2.6 to 4.2.11. It remains to prove the relations involving products  $\sigma_{-2n-1} \cdot g_i$ .

Multiplying  $\sigma_{-2n-1} \cdot g_i$  by two, we have  $u_0^n u_{-2n-1} \cdot g_i$ . This is equal to  $u_0^{n+1} \cdot g_{i-2n-1}$  if  $i$  is even and to  $(u_0 + 4)u_0^n \cdot g_{i-2n-1}$  if  $i$  is odd (Lemmas 4.2.8 and 4.2.9 respectively). Since  $u_0^{n+1} \in KO^0(\mathbb{C}P^{2n+1})$  is equal to either zero or  $\beta^{\frac{n}{2}} e^2 \sigma_{-2n-1}$  (Proposition 3.2.9),  $2\sigma_{-2n-1} \cdot g_i$  is equal to zero if  $i$  is even and to  $4u_0^n \cdot g_{i-2n-1}$  if  $i$  is odd. Since  $KO^*(Tm\eta)$  is free from 2-torsion, this implies the required relations.

We obtain the basis  $\{\beta^{\frac{i-i'}{4}} u_0^k \cdot g_{i'} \mid i \in \mathbb{Z}, 0 \leq k \leq n\}$  for  $KO^{-2i}(Tm\eta)$  (where  $i'$  is the reduction of  $i$  modulo 4) by applying the periodicity relation  $\beta \cdot g_i = g_{i+4}$  to Proposition 4.2.5. Recalling (see comment above (4.4)) that the relations imply

that  $u_0^{n+1} \cdot g_i$  equals 0 for all  $i$ , the relations allow us to rewrite any polynomial  $P \in KO^*(\mathbb{C}P_+^{2n+1})[g_i \mid i \in \mathbb{Z}]$  in terms of the basis  $\{\beta^{\frac{i-i'}{4}} u_0^k \cdot g_{i'} \mid i \in \mathbb{Z}, 0 \leq k \leq n\}$  for  $KO^*(Tm\eta)$ , so the relations are sufficient.  $\square$

The inclusion  $j_{2n-1}: \mathbb{C}P^{2n-1} \rightarrow \mathbb{C}P^{2n}$  induces an inclusion of Thom spaces. We apply  $KO^*(-)$  to the cofibre sequence

$$\mathbb{C}P_m^{m+2n-1} \rightarrow \mathbb{C}P_m^{m+2n} \xrightarrow{p} S^{2m+4n}$$

where  $p$  is the projection onto the top cell. Similarly to the proof of Proposition 3.2.9, we obtain a split short exact sequence of  $KO_*$ -modules

$$0 \rightarrow KO^*(S^{2m+4n}) \xrightarrow{p^*} KO^*(\mathbb{C}P_m^{m+2n}) \rightarrow KO^*(\mathbb{C}P_m^{m+2n-1}) \rightarrow 0,$$

and we define  $\nu_{-m-2n} \in KO^{2m+4n}(\mathbb{C}P_m^{m+2n})$  by  $\nu_{-m-2n} := p^*(s_{2m+4n}^{KO})$ .

**Proposition 4.3.2.** *Let  $\eta$  be the canonical bundle over  $\mathbb{C}P^{2n}$ . Then bases for the abelian groups  $KO^i(Tm\eta)$  are as follows:*

$$\begin{aligned} g_{-m-2n}, u_0 \cdot g_{-m-2n}, \dots, u_0^{n-1} \cdot g_{-m-2n}, \nu_{-m-2n}, & i = 2m + 4n; \\ e \cdot \nu_{-m-2n}, & i = 2m + 4n - 1; \\ g_{-m-2n+1}, u_0 \cdot g_{-m-2n+1}, \dots, u_0^{n-1} \cdot g_{-m-2n+1}, e^2 \cdot \nu_{-m-2n}, & i = 2m + 4n - 2; \\ g_{-m-2n+2}, u_0 \cdot g_{-m-2n+2}, \dots, u_0^{n-1} \cdot g_{-m-2n+2}, u_0^n \cdot g_{-m-2n+2} & i = 2m + 4n - 4; \\ g_{-m-2n+3}, u_0 \cdot g_{-m-2n+3}, \dots, u_0^{n-1} \cdot g_{-m-2n+3}, & i = 2m + 4n - 6; \end{aligned}$$

where  $2\nu_{-m-2n} = u_0^n \cdot g_{-m-2n}$ ,  $e^2 \cdot \nu_{-m-2n} = u_0^n \cdot g_{-m-2n+1}$  and  $\alpha \cdot \nu_{-m-2n} = u_0^n \cdot g_{-m-2n+2}$ . For the remaining groups,  $KO^i(Tm\eta)$  is isomorphic to the trivial group if  $i$  is equal to  $2m + 4n - 3$ ,  $2m + 4n - 5$  or  $2m + 4n - 7$ .

*Proof.* Complexification is a monomorphism on  $KO^{2m+4n-6}(Tm\eta)$ , resulting in the relation  $u_0^n \cdot g_{-m-2n+3} = 0$ . Since both  $KO^{2m+4n}(Tm\eta)$  and  $KO^{2m+4n-4}(Tm\eta)$  are torsion free (By Proposition 4.2.2),  $c$  is monic on both. This gives us the relations  $2\nu_{-m-2n} = u_0^n \cdot g_{-m-2n}$  and  $\alpha \cdot \nu_{-m-2n} = u_0^n \cdot g_{-m-2n+2}$ . Applying  $KO^{2m+4n-2}(-)$  to the cofibre sequence  $\mathbb{C}P_m^{m+2n} \xrightarrow{Tj_{2n}} \mathbb{C}P_m^{m+2n+1} \xrightarrow{p} S^{2m+4n}$  shows that  $Tj_{2n}^*$  is an epimorphism. The restrictions (4.4) show that  $u_0^n \cdot g_{-m-2n} = e^2 \cdot \nu_{-m-2n}$ .  $\square$

**Theorem 4.3.3.** *Let  $\eta$  be the canonical complex line bundle over  $\mathbb{C}P^{2n}$ . Then the Thom module  $KO^*(Tm\eta)$  is generated over  $KO^*(\mathbb{C}P_+^{2n})$  by the elements  $g_0, g_1, g_2, g_3$  and  $\nu_{-m-2n}$ , with relations*

1.  $e \cdot g_0 = e \cdot g_1 = e \cdot g_2 = e \cdot g_3 = 0$ ,
2.  $\alpha \cdot g_0 = 2g_2, \alpha \cdot g_1 = 2g_3, \alpha \cdot g_2 = 2\beta \cdot g_0, \alpha \cdot g_3 = 2\beta \cdot g_1$ ,
3.  $u_j \cdot g_0 = u_{j-2} \cdot g_2, u_j \cdot g_1 = u_{j-2} \cdot g_3$ ,
4.  $u_{4j+1} \cdot g_0 = u_{4j} \cdot g_1, u_{4j+3} \cdot g_0 = u_{4j} \cdot g_3$ ,
5.  $u_{4j-1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_0$ ,  
 $u_{4j+1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_2$ ,
6.  $u_{-m-2n-1}u_0^{n-1} \cdot g_1 = 2\nu_{-m-2n}$ ,  
 $u_{-m-2n-1}u_0^{n-1} \cdot g_0 = e^2 \cdot \nu_{-m-2n}$ ,  
 $u_{-m-2n-3}u_0^{n-1} \cdot g_1 = \alpha \cdot \nu_{-m-2n}$ ,  
 $u_{-m-2n-3}u_0^{n-1} \cdot g_0 = 0$ .

The Thom algebra is generated by the same elements, with the additional relations

$$g_i g_j = \begin{cases} u_0^{\frac{m-1}{2}} (u_0 + 4) \cdot g_{i+j}, & i \text{ and } j \text{ odd,} \\ u_0^{\frac{m+1}{2}} \cdot g_{i+j}, & \text{else} \end{cases}$$

and

$$\nu_{-m-2n} g_i = 0 \pmod{e^2 \cdot \nu_{-m-2n}}$$

for all integers  $0 \leq i, j \leq 3$ .

Since  $u_i u_0^n \in KO^0(\mathbb{C}P^{2n})$  is equal to 0 (Proposition 3.2.7) it follows that  $u_i u_0^n \cdot g_j$  equals 0 for any integers  $i, j$ .

*Proof.* The inclusion  $j_{2n}: \mathbb{C}P^{2n} \rightarrow \mathbb{C}P^{2n+1}$  induces an inclusion of Thom spaces. Pulling back from  $KO^*(Tm\eta(2n+1))$  preserves the relations of Lemmas 4.2.6 to 4.2.11. Relation 6. is Proposition 4.3.2, and it remains to prove that  $\nu_{-m-2n} g_i$  equals

zero. We use complexification, which annihilates  $e^2 \cdot \nu_{-m-2n} \in KO^{2m+4n-2}(Tm\eta)$ , but is injective on  $KO^*(Tm\eta)/(e^2 \cdot \nu_{-m-2n})$ .

The relations allow us to rewrite any polynomial  $P \in KO^*(\mathbb{C}P_+^{2n})[g_i \mid i \in \mathbb{Z}]$  in terms of the basis given in Proposition 4.3.2. Hence the relations are sufficient.  $\square$

Let  $\eta$  be the canonical line bundle over  $\mathbb{C}P^\infty$ . We now deduce the Thom algebra  $KO^*(Tm\eta)$  from Proposition 4.3.1 using inverse limits. First note that the homomorphism  $Tj_{2n}^* \circ Tj_{2n+1}^* : KO^{-2i}(Tm\eta(2n+1)) \rightarrow KO^{-2i}(Tm\eta(2n-1))$  is surjective (4.4) and applying Theorem 2.0.17, we see that  $KO^{-2i}(Tm\eta) \cong \varprojlim_{n=0} KO^{-2i}(Tm\eta(2n+1))$  is isomorphic to  $\prod \mathbb{Z}$ . We write the elements in the form  $\sum_{n=0}^\infty a_n u_0^n \cdot g_i$ , where  $a_n$  are integers.

Equipping  $KO^*(Tm\eta)$  with multiplication (2.12), we have an isomorphism of  $\mathbb{Z}[[u_0]]$ -modules  $KO^{-2i}(Tm\eta) \cong \mathbb{Z}[[u_0]] \cdot g_i$  for each integer  $i$ .

The above demonstrates that  $KO^*(Tm\eta)$  is generated over  $KO^*(\mathbb{C}P_+^\infty)$  by the elements  $g_i$ ,  $i \in \mathbb{Z}$ . We retain some multiplicative relations from  $KO^*(Tm\eta(2n+1))$ . Recall that, by the definition of inverse limits, an element in  $KO^i(Tm\eta)$  equals the zero element if and only if it is annihilated by restriction to  $KO^i(Tm\eta(2n+1))$  for every  $n \geq 0$ . Thus, referring to Proposition 4.3.1, we retain those relations which do not depend on  $n$ , i.e. those proved in Lemmas 4.2.6 to 4.2.11.

Note that  $u_0^n \cdot g_i$  is non-zero in  $KO^{-2i}(Tm\eta)$  for any  $i$ , because  $u_0^k \cdot g_i$  is a non-zero element of  $KO^{-2i}(Tm\eta(2n+1))$  for every  $n \geq k$ .

Summarising the above, we have the following result.

**Theorem 4.3.4.** *The Thom module  $KO^*(Tm\eta)$  is generated over  $KO^*(\mathbb{C}P_+^\infty)$  by the elements  $g_0, g_1, g_2$  and  $g_3$ , with relations*

1.  $e \cdot g_0 = e \cdot g_1 = e \cdot g_2 = e \cdot g_3 = 0$ ,

2.  $\alpha \cdot g_0 = 2g_2, \alpha \cdot g_1 = 2g_3, \alpha \cdot g_2 = 2\beta \cdot g_0, \alpha \cdot g_3 = 2\beta \cdot g_1$ ,

3.  $u_j \cdot g_0 = u_{j-2} \cdot g_2, u_j \cdot g_1 = u_{j-2} \cdot g_3$ ,

4.  $u_{4j+1} \cdot g_0 = u_{4j} \cdot g_1, u_{4j+3} \cdot g_0 = u_{4j} \cdot g_3$ ,

$$5. \quad u_{4j-1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_0,$$

$$u_{4j+1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_2.$$

The Thom algebra is generated by the same elements, with the additional relation

$$g_i g_j = \begin{cases} u_0^{\frac{m-1}{2}} (u_0 + 4) \cdot g_{i+j}, & i \text{ and } j \text{ odd,} \\ u_0^{\frac{m+1}{2}} \cdot g_{i+j}, & \text{else} \end{cases}$$

for all integers  $0 \leq i, j \leq 3$ .

Other choices of generators are of course possible since the periodicity relation  $\beta^k \cdot g_i = g_{i+4k}$  holds for all integers  $i$  and  $k$ .

# Chapter 5

## Line Bundles over Complex

## Projective Space

As in the previous chapter, we study a family of vector bundles over  $\mathbb{C}P^n$  which are derived from a number of copies of the canonical line bundle  $\eta \rightarrow \mathbb{C}P^n$ . The  $m$ -fold tensor product bundle  $\eta^m \rightarrow \mathbb{C}P^\infty$  is the complex line bundle classified by the composite map

$$\mathbb{C}P^\infty \xrightarrow{\Delta} \underbrace{\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty}_m \xrightarrow{m} \mathbb{C}P^\infty$$

where  $\Delta$  is the diagonal map and  $m$  is  $m$ -fold multiplication. We compute the  $KO$ -theory Thom algebra of the line bundles  $\eta^m$  over  $\mathbb{C}P^n$  for all  $n \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$ .

When  $m$  is even,  $\eta^m$  is *Spin*, and the Thom algebra is generated by one element  $t^{KO} \in KO^2(T\eta^m)$  with a single relation. When  $m$  is odd,  $\eta^m$  is not *Spin*, and the Thom algebra has several generators and a larger set of relations. Note that  $\eta^1$  is simply the canonical line bundle  $\eta$ .

We begin by discussing the complex  $K$ -theory of  $T\eta^m$ , and the Thom algebra of  $T\eta^m$  where  $m$  is even. The bulk of our computations concern the case where  $m$  is odd. We compute the graded group  $KO^*(T\eta^m)$  for all  $n$ , then describe a basis for the  $KO^*(\mathbb{C}P_+^n)$ -module  $KO^{-2i}(T\eta^m)$ . By computing multiplicative relations, we determine the Thom algebra of  $\eta^m$  over  $\mathbb{C}P^n$  for all finite  $n$ , then use inverse limits to study the infinite version of this result.

The real  $KO$ -theory algebras for  $\eta^2$  over  $\mathbb{C}P^\infty$  and  $\eta(1)^m$  over  $\mathbb{C}P^1$  for all  $m$  are determined in [14]. We confirm these results in Proposition 5.2.1 and Theorem 5.3.1 respectively, where we prove more general results.

Other than where we repeat Civan and Ray's results, we believe that the material in this chapter concerning real  $K$ -theory is original. The complex  $K$ -theory of  $T\eta^m$  is well known.

## 5.1 Integral Cohomology and Complex $K$ -theory

We begin by reviewing the integral cohomology and complex  $K$ -theory of  $T\eta^m$ .

Since  $c_1(\eta^m)$  and  $mc_1(\eta)$  are equal, the first Chern class  $c_1$  provides an isomorphism between the multiplicative group of complex line bundles over a CW-complex  $X$  and  $H^2(X; \mathbb{Z})$  [14].

The first Chern class  $c_1(\eta^m) \in H^2(\mathbb{C}P^n)$  is equal to  $mc_1(\eta)$ , and by Example 2.0.8, the Thom module  $H^*(T\eta^m)$  is generated over  $H^*(\mathbb{C}P_+^n)$  by a single generator  $t^H \in H^2(T\eta^m)$ . The Thom algebra is generated by  $t^H$ , with the relation

$$(t^H)^2 = mc_1(\eta) \cdot t^H. \tag{5.1}$$

We refer to (3.1) and (3.2) for the complex  $K$ -theory of the base space  $\mathbb{C}P^n$  and we only comment on the complex  $K$ -theory of  $T\eta(n)^m$  here.

Pulling back the universal first Chern class  $c_1 = z^{-1}(\eta - 1) \in K^2(\mathbb{C}P^\infty)$  along the classifying map  $\eta^m: \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$ , we obtain  $c_1(\eta^m) = z^{-1}(\eta^m - 1)$ . Since there are no multiplicative relations in  $K^*(\mathbb{C}P^\infty)$ , the element  $\kappa_1(\eta^m) \in K^0(\mathbb{C}P_+^\infty)$  is given by

$$\kappa_1(\eta^m) = \frac{\overline{c_1(\eta^m)}}{c_1(\eta^m)} = \frac{\overline{z^{-1}(\eta^m - 1)}}{z^{-1}(\eta^m - 1)} = \frac{1 - \overline{\eta}^m}{\eta^m - 1} = \frac{\overline{\eta}^m(\eta^m - 1)}{\eta^m - 1} = \overline{\eta}^m.$$

Then by Corollary 2.0.11 we have the following result.  $K^*(T\eta^m)$  is a free  $K^*(\mathbb{C}P_+^\infty)$ -module on one generator  $t \in K^0(T\eta^m)$ . The Thom algebra is generated by  $t$ , with the single relation

$$\begin{aligned} t^2 &= (\eta^m - 1) \cdot t \\ &= ((x + 1)^m - 1) \cdot t. \end{aligned} \tag{5.2}$$

Conjugation interacts with the Thom class via the relation

$$\begin{aligned} \bar{t} &= -\bar{\eta}^m \cdot t \\ &= -(1 + \bar{x})^m \cdot t. \end{aligned} \tag{5.3}$$

For the Thom algebra of  $\eta(n)^m$  over  $\mathbb{C}P^n$ , we restrict the above, and observe that  $K^*(T\eta(n)^m)$  is a free  $K^*(\mathbb{C}P_+^n)$ -module on one generator  $t \in K^0(T\eta^m)$ . The Thom class  $t$  satisfies both (5.2) and (5.3), but we have the relation  $x^{n+1} = 0$  in  $K^*(\mathbb{C}P^n)$ .

Note that both  $t(\eta^m) \in KO^0(T\eta^m)$  and  $t(m\eta) \in KO^0(Tm\eta)$  satisfy similar relations under conjugation (compare (4.2) and (5.3) when  $m$  is odd). Many of the results in Chapter 4 relied solely on the formula for  $\overline{t(m\eta)}$ , and we exploit this to prove similar results in this chapter.

## 5.2 Real $K$ -theory

### 5.2.1 The $KO$ -orientable case

Let  $m = 2b$  be even. Taking the mod 2 reduction of the first Chern class, the second Stiefel-Whitney class  $w_2(\eta^m) = mw_2(\eta) \in H^2(\mathbb{C}P^\infty; \mathbb{Z}_2)$  is zero, and  $\eta^m$  is  $KO$ -orientable.

**Proposition 5.2.1.** *There is a unique element  $t^{KO} \in KO^2(T\eta^{2b})$  with complexification  $c(t^{KO}) = \bar{\eta}^b \cdot t^K$ . It is a Thom class, with square  $(t^{KO})^2 = u_{-1}(\eta^b) \cdot t^{KO}$  in  $KO^4(T\eta^{2b})$ .*

*Proof.* The Thom isomorphism and Proposition 3.2.1 imply that the abelian group  $KO^3(T\eta^{2b}) \cong KO^1(\mathbb{C}P_+^\infty)$  is isomorphic to the trivial group. Referring to (2.1) with  $X = T\eta^{2b}$ , this implies that  $c$  is monic on  $KO^2(T\eta^{2b})$  and  $t^{KO}$  is unique, if it exists.

This result is proved for  $b = 1$  by Civan and Ray [14, Lemma 4.4] with the Thom class  $t_\square \in KO^2(T\eta^2)$ , where  $t_\square^2 = u_{-1} \cdot t_\square$ . The bundle  $\eta^{2b}$  is the pullback of  $\eta^2$  along the map  $\mathbb{C}P^n \rightarrow \mathbb{C}P^\infty$  of degree  $b$ . Pulling both  $t_\square$  and  $u_{-1}$  back along the induced  $KO$ -theory homomorphisms gives us  $(t^{KO})^2 = u_{-1}(\eta^b) \cdot t^{KO}$ . Pulling back along the induced  $K$ -theory homomorphisms gives us  $c(t^{KO}) = \bar{\eta}^b \cdot t^K$ .  $\square$



Then  $KO^*(T\eta^m)$  is a  $KO^*(\mathbb{C}P_+^n)$ -module on one generator  $t^{KO} \in KO^2(T\eta^m)$ . The Thom algebra is generated by  $t^{KO}$  with the single relation  $(t^{KO})^2 = u_{-1}(\eta^b) \cdot t^{KO}$ .

Although we may write  $u_{-1}(\eta^b) = r(z^{-1}((x+1)^b - 1))$  in the form  $\sum_{i=0}^n a_i u_{-1} u_0^i$  by repeated application of Lemma 3.2.3, we do not have a closed formula. The first few cases are as follows.

$$u_{-1}(\eta^b) = \begin{cases} u_{-1}, & b = 1; \\ u_{-1}(u_0 + 2), & b = 2; \\ u_{-1}(u_0^2 + 4u_0 + 3), & b = 3; \\ u_{-1}(u_0^3 + 6u_0^2 + 7u_0 + 4), & b = 4. \end{cases}$$

### 5.2.2 Characteristic Classes

Let  $m \in \mathbb{Z}$  be any integer, and let  $\psi^m$  be the Adams operation [1].

Since  $\psi^m(\eta - 1) = \psi^m(\eta) - \psi^m(1) = \eta^m - 1$  [12], we may rewrite the  $i$ th Fujii class of  $\eta^m$  in the form  $u_i(\eta^m) = r(z^i(\eta^m - 1)) = r(z^i\psi^m(x))$ .

As above, we may write  $u_i(\eta^m) = r(z^i((x+1)^m - 1))$  in the form  $\sum_{j=0}^n a_j u_i u_0^j$  using Lemma 3.2.3, but we do not have a closed formula. A partial solution to this problem is given in [3], where it is shown that  $r(\psi^m(x)) = m^2 u_0$  (modulo higher powers of  $u_0$ ). It follows easily that  $u_{2i}(\eta^m) = m^2 u_{2i}$  in  $KO^{-4i}(\mathbb{C}P^\infty)$  (modulo  $u_{2i} u_0^k$  where  $k > 0$ ).

### 5.2.3 Group Structure

We now turn to the more complicated case: computing the Thom algebra  $KO^*(T\eta^m)$  over  $KO^*(\mathbb{C}P_+^n)$  for all  $n$  where  $m$  is odd, and consequently  $\eta^m$  fails to be *Spin*. Note that for the case  $m = 1$ , we replicate the results of the Chapter 3. For the remainder of the chapter,  $m$  will be an odd, positive integer.

**Proposition 5.2.2.** *Let  $\eta^m$  over  $\mathbb{C}P^n$  be the  $m$ -fold tensor bundle. Then there is an isomorphism of graded abelian groups*

$$KO^*(T\eta^m) \cong \begin{cases} \mathbb{Z}(\frac{n+1}{2}) & n \text{ odd,} \\ \mathbb{Z}(\frac{n}{2}) \oplus KO^*(S^{2n+2}) & n \text{ even.} \end{cases}$$

*Proof.* This reduces to the proof of Proposition 3.2.1. We use the Atiyah-Hirzebruch spectral sequence. The  $E_2$ -term is given by  $E_2^{p,q} \cong H^p(T\eta^m) \otimes KO^q(S^0)$ , and the differentials by  $d_2 = Sq^2 \otimes \cdot e$  (2.11). The Thom algebra  $H^*(T\eta^m)$  is generated over  $H^*(\mathbb{C}P_+^n)$  by  $t^H \in H^2(T\eta^m)$  where  $(t^H)^2 = mc_1 \cdot t^H$  (5.1). Then  $d_2(t^H \otimes 1) = mc_1 \cdot t^H \otimes e = c_1 \cdot t^H \otimes e$ . So  $d_2$  does not depend on  $m$ , and we set  $m = 1$ . But  $T\eta^1$  and  $\mathbb{C}P^{n+1}$  are homeomorphic, and we may refer to the proof of Proposition 3.2.1.  $\square$

### 5.2.4 Basis Elements

Let  $\eta$  be the canonical bundle over  $\mathbb{C}P^n$ . We now describe a basis for the graded group  $KO^*(T\eta^m)$ . We begin by defining elements  $g_i$  which generate the Thom module of  $\eta^m$  over  $\mathbb{C}P^n$ .

**Definition 5.2.3.** Define  $g_i \in KO^{-2i}(Tm\eta)$  by  $g_i := r(z^i \bar{\eta}^{\frac{m-1}{2}} \cdot t)$ .

Note that  $g_i$  is well-defined for negative  $m$ , since  $\bar{\eta}^{-1}$  is equal to  $\eta$ . Compare Definition 4.2.4. As we discussed above (see Section 5.1),  $t(\eta^m) \in K^0(T\eta^m)$  and  $t(m\eta) \in K^0(Tm\eta)$  satisfy similar relations under conjugation, allowing us to prove similar results to those in Chapter 4.

Group structure and bases for  $KO^*(T\eta^m)$  are similar to those of  $KO^*(Tm\eta)$ , but there is a significant increase in complexity when we examine the  $KO^*(T\eta^m)$  as an algebra.

**Proposition 5.2.4.** *Let  $\eta$  be the canonical bundle over  $\mathbb{C}P^{2n+1}$ . The elements  $g_i u_0^k$ ,  $0 \leq k \leq n$ , are a basis for the abelian group  $KO^{-2i}(T\eta^m)$ .*

*Proof.* Similar to the proof of Proposition 4.2.5.  $\square$

### 5.2.5 Multiplicative Relations

Let  $\eta$  be the canonical line bundle over  $\mathbb{C}P^{2n+1}$ . We now describe the interaction of  $g_i \in KO^{-2i}(T\eta^m)$  with the coefficient ring  $KO_*$ , then the products  $u_i \cdot g_j$  and  $g_i g_j \in KO^{-2(i+j)}(T\eta^m)$ . This furnishes us with a description of the Thom algebra of

$\eta^m$  over  $\mathbb{C}P^{2n+1}$ . The proofs rely on injectivity of complexification, which holds by (2.1) since  $KO^{-2i+1}(T\eta^m)$  is zero.

**Lemma 5.2.5.** *The following relations hold in the  $KO_*$ -module  $KO^*(T\eta^m)$ .*

$$e \cdot g_i = 0, \quad \alpha \cdot g_i = 2g_{i+2}, \quad \beta \cdot g_i = g_{i+4}.$$

*Proof.* Similar to the proof of Lemma 4.2.6. □

**Lemma 5.2.6.** *In  $KO^*(T\eta^m)$  we have the following relations.*

- $u_j \cdot g_i = u_{j-2} \cdot g_{i+2}$
- $u_{2j+1} \cdot g_{2i} = u_0 \cdot g_{2i+2j+1}$
- $u_{2j-1} \cdot g_{2i+1} = (u_0 + 4) \cdot g_{2(i+j)}$

*Proof.* Similar to the proofs of Lemmas 4.2.7 to 4.2.9. □

The above results rely on the formula (5.3) for  $\bar{t}$ , and so the above lemmas are proved in a similar manner to the corresponding results concerning the Thom module of  $Tm\eta$ . The Thom algebra of  $T\eta^m$  also involves the formula (5.2) for  $t^2$ . Consequently, in contrast to the preceding chapter, the problem of expressing the product  $g_i g_j \in KO^{-2(i+j)}(T\eta^m)$  is more complicated.

**Lemma 5.2.7.** *In  $KO^{-4i-2j}(T\eta^m)$  we have the following relation for any integers  $i, j$ .*

$$g_{2i}g_j = \begin{cases} r(x(x+1)^{\frac{m-1}{2}}) \cdot g_{2i+j}, & m > 0; \\ r(x(\bar{x}+1)^{\frac{m-1}{2}}) \cdot g_{2i+j}, & m < 0. \end{cases}$$

*Proof.* Firstly, we note that  $c(g_i) = (z^i \bar{\eta}^{\frac{m-1}{2}} \cdot t + \bar{z}^i \eta^{\frac{m-1}{2}} \cdot \bar{t}) = (z^i \bar{\eta}^{\frac{m-1}{2}} - \bar{z}^i \eta^{\frac{m-1}{2}}) \cdot t$ .

Then

$$\begin{aligned} c(g_{2i}g_j) &= (z^{2i} \bar{\eta}^{\frac{m-1}{2}} - \bar{z}^{2i} \eta^{\frac{m+1}{2}})(z^j \bar{\eta}^{\frac{m-1}{2}} - \bar{z}^j \eta^{\frac{m+1}{2}}) \cdot t^2 \\ &= (\bar{\eta}^{\frac{m-1}{2}} - \eta^{\frac{m+1}{2}})(\eta^m - 1)(z^{2i+j} \bar{\eta}^{\frac{m-1}{2}} - \bar{z}^{2i+j} \eta^{\frac{m+1}{2}}) \cdot t \\ &= (\bar{\eta}^{\frac{m-1}{2}} - \eta^{\frac{m+1}{2}})(\eta^m - 1) \cdot c(g_{2i+j}). \end{aligned}$$

But

$$\begin{aligned} (\bar{\eta}^{\frac{m-1}{2}} - \eta^{\frac{m+1}{2}})(\eta^m - 1) &= \eta^{\frac{m+1}{2}} + \bar{\eta}^{\frac{m+1}{2}} - \eta^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m-1}{2}} \\ &= cr(\eta^{\frac{m+1}{2}} - \eta^{\frac{m-1}{2}}), \end{aligned}$$

so  $c(g_{2i}g_j) = c(r((\eta - 1)\eta^{\frac{m-1}{2}}) \cdot g_{2i+j})$ . □

**Lemma 5.2.8.** *In  $KO^{-4(i+j)}(T\eta^m)$  we have the following relation for any integers  $i, j$ .*

$$g_{2i+1}g_{2j-1} = \begin{cases} \left\{ r(x(x+1)^{\frac{m-1}{2}}) + 4 + 4r(\eta + \cdots + \eta^{\frac{m-1}{2}}) \right\} \cdot g_{2(i+j)}, & m > 0; \\ \left\{ r(x(\bar{x}+1)^{\frac{m-1}{2}}) + 4 + 4r(\eta + \cdots + \eta^{\frac{m-1}{2}}) \right\} \cdot g_{2(i+j)}, & m < 0. \end{cases}$$

*Proof.* Firstly, we note that  $(\eta^k + \eta^{k-1} + \cdots + \bar{\eta} + \bar{\eta}^2 + \cdots + \bar{\eta}^k)(\bar{\eta}^k - \bar{\eta}^{k+1})$  is equal to  $1 + \bar{\eta} + \cdots + \bar{\eta}^{2k} - (\bar{\eta} + \bar{\eta}^2 + \cdots + \bar{\eta}^{2k+1})$ , which simplifies to  $1 - \bar{\eta}^{2k+1}$ . Then

$$\begin{aligned} c(g_{2i+1}g_{2j-1}) &= z^{2(i+j)}(\bar{\eta}^{\frac{m-1}{2}} + \bar{\eta}^{\frac{m+1}{2}})^2(\eta^m - 1) \cdot t \\ &= z^{2(i+j)} \left\{ (\bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}})^2 + 4\bar{\eta}^m \right\} (\eta^m - 1) \cdot t \\ &= z^{2(i+j)} \left\{ (\bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}})^2 (\eta^m - 1) + 4(1 - \bar{\eta}^m) \right\} \cdot t \\ &= z^{2(i+j)} (\bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}}) \left\{ (\bar{\eta}^{\frac{m-1}{2}} - \bar{\eta}^{\frac{m+1}{2}}) (\eta^m - 1) \right. \\ &\quad \left. + 4(\eta^{\frac{m-1}{2}} + \eta^{\frac{m-3}{2}} + \cdots + 1 + \bar{\eta} + \cdots + \bar{\eta}^{\frac{m-1}{2}}) \right\} \cdot t \\ &= c \left( \left\{ r(x(x+1)^{\frac{m-1}{2}}) + 4 + 4r(\eta + \cdots + \eta^{\frac{m-1}{2}}) \right\} \cdot g_{2(i+j)} \right). \end{aligned}$$

□

Again, we do not have a closed formula for  $r(x(x+1)^{\frac{m-1}{2}})$  or  $r(\sum_{i=1}^{\frac{m-1}{2}} (x+1)^i)$  in terms of our preferred basis for  $KO^0(\mathbb{C}P^{2n+1})$ .

### 5.3 Thom Algebra

Using the above, we now discuss the Thom modules  $KO^*(T\eta^m)$  for  $\eta$  over  $\mathbb{C}P^n$ . Where necessary, we write  $\eta(n)$  to emphasise the base space.

**Theorem 5.3.1.** *Let  $\eta$  be the canonical complex line bundle over  $\mathbb{C}P^{2n+1}$ . The Thom module  $KO^*(T\eta^m)$  is generated over  $KO^*(\mathbb{C}P_+^{2n+1})$  by the elements  $g_0, g_1, g_2, g_3$ , with relations*

1.  $e \cdot g_0 = e \cdot g_1 = e \cdot g_2 = e \cdot g_3 = 0$ ,
2.  $\alpha \cdot g_0 = 2g_2, \alpha \cdot g_1 = 2g_3, \alpha \cdot g_2 = 2\beta \cdot g_0, \alpha \cdot g_3 = 2\beta \cdot g_1$ ,
3.  $u_j \cdot g_0 = u_{j-2} \cdot g_2, u_j \cdot g_1 = u_{j-2} \cdot g_3$ ,

$$4. \quad u_{4j+1} \cdot g_0 = u_{4j} \cdot g_1, \quad u_{4j+3} \cdot g_0 = u_{4j} \cdot g_3,$$

$$5. \quad u_{4j-1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_0,$$

$$u_{4j+1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_2,$$

$$6. \quad \sigma_{-2n-1} \cdot g_0 = \sigma_{-2n-1} \cdot g_2 = 0,$$

$$\sigma_{-2n-1} \cdot g_1 = \begin{cases} 2u_{-2n}u_0^{n-1} \cdot g_0, & n \text{ even}, \\ 2u_{-2n-2}u_0^{n-1} \cdot g_2, & n \text{ odd}, \end{cases}$$

$$\sigma_{-2n-1} \cdot g_3 = \begin{cases} 2u_{-2n}u_0^{n-1} \cdot g_2, & n \text{ even}, \\ 2u_{-2n+2}u_0^{n-1} \cdot g_0, & n \text{ odd}. \end{cases}$$

The Thom algebra is generated by the same elements, with the additional relations

$$g_i g_j = \begin{cases} r(x(x+1)^{\frac{m-1}{2}}) \cdot g_{i+j}, & m > 0, \\ r(x(\bar{x}+1)^{\frac{m-1}{2}}) \cdot g_{i+j}, & m < 0, \end{cases} \quad \text{if either } i \text{ or } j \text{ even,}$$

and

$$g_i g_j = \begin{cases} \left\{ r(x(x+1)^{\frac{m-1}{2}}) + 4 + 4r(\eta + \cdots + \eta^{\frac{m-1}{2}}) \right\} \cdot g_{i+j}, & m > 0, \\ \left\{ r(x(\bar{x}+1)^{\frac{m-1}{2}}) + 4 + 4r(\eta + \cdots + \eta^{\frac{m-1}{2}}) \right\} \cdot g_{i+j}, & m < 0. \end{cases} \quad i \text{ and } j \text{ odd.}$$

for all integers  $0 \leq i, j \leq 3$ .

Since  $u_0^{n+1}$  is equal to  $e^2 \beta^{\frac{n}{2}} \sigma_{-2n-1}$  if  $n$  is even and equal to 0 if  $n$  is odd (Proposition 3.2.9) it follows that  $u_0^{n+1} \cdot g_i = 0$  for any  $i$ . This implies that the composition of the restriction homomorphisms

$$KO^*(T\eta(2n+1)^m) \xrightarrow{Tj_{2n}^*} KO^*(T\eta(2n)^m) \xrightarrow{Tj_{2n-1}^*} KO^*(T\eta(2n-1)^m) \quad (5.4)$$

is the identity on  $u_0^k \cdot g_i$  for  $0 \leq k \leq n-1$ , and annihilates  $u_0^n \cdot g_i$ .

*Proof.* Recall the relations of Lemmas 5.2.5 to 5.2.8. It remains to prove the relations involving products  $\sigma_{-2n-1} \cdot g_i$ .

Multiplying  $\sigma_{-2n-1} \cdot g_i$  by two, we have  $u_0^n u_{-2n-1} \cdot g_i$ . This is equal to  $u_0^{n+1} \cdot g_{i-2n-1}$  if  $i$  is even and to  $(u_0 + 4)u_0^n \cdot g_{i-2n-1}$  if  $i$  is odd (Lemma 5.2.6). Since  $u_0^{n+1}$  in  $KO^0(\mathbb{C}P^{2n+1})$  is equal to either zero or  $\beta^{\frac{n}{2}} e^2 \sigma_{-2n-1}$  (Proposition 3.2.9),  $2\sigma_{-2n-1} \cdot g_i$  is equal to zero if  $i$  is even and to  $4u_0^n \cdot g_{i-2n-1}$  if  $i$  is odd. Since  $KO^*(T\eta^m)$  is free from 2-torsion, this implies the required relations.

We obtain the basis  $\{\beta^{\frac{i-i'}{4}} u_0^k \cdot g_{i'} \mid i \in \mathbb{Z}, 0 \leq k \leq n\}$  for  $KO^{-2i}(T\eta^m)$  (where  $i'$  is the reduction of  $i$  modulo 4) by applying the periodicity relation  $\beta \cdot g_i = g_{i+4}$  to Proposition 5.2.4. Recalling (see comment above (5.4)) that the relations imply that  $u_0^{n+1} \cdot g_i$  equals zero for all  $i$ , the relations allow us to rewrite any polynomial  $P \in KO^*(\mathbb{C}P_+^{2n+1})[g_i \mid i \in \mathbb{Z}]$  in terms of the basis  $\{\beta^{\frac{i-i'}{4}} u_0^k \cdot g_{i'} \mid i \in \mathbb{Z}, 0 \leq k \leq n\}$  for  $KO^*(T\eta^m)$ , so the relations are sufficient.  $\square$

The inclusion  $j_{2n-1}: \mathbb{C}P^{2n-1} \rightarrow \mathbb{C}P^{2n}$  induces an inclusion of Thom spaces. We apply  $KO^*(-)$  to the cofibre sequence

$$T\eta(2n-1)^m \rightarrow T\eta(2n)^m \xrightarrow{p} S^{4n+2}$$

where  $p$  is the projection onto the top cell. Similarly to the proof of Proposition 3.2.9, we obtain a split short exact sequence of  $KO_*$ -modules

$$0 \rightarrow KO^*(S^{2n+1}) \xrightarrow{p^*} KO^*(T\eta(2n)^m) \rightarrow KO^*(T\eta(2n-1)^m) \rightarrow 0,$$

and we define  $\nu_{-2n-1} \in KO^{4n+2}(T\eta(2n)^m)$  by  $\nu_{-2n-1} := p^*(s_{4n+2}^{KO})$ .

**Proposition 5.3.2.** *Let  $\eta$  be the canonical bundle over  $\mathbb{C}P^{2n}$ . Then additive bases for  $KO^i(T\eta^m)$  are as follows:*

$$\begin{aligned} g_{-2n-1}, \quad u_0 \cdot g_{-2n-1}, \dots, \quad u_0^{n-1} \cdot g_{-2n-1}, \quad \nu_{-2n-1}, & \quad i = 4n + 2; \\ e \cdot \nu_{-2n-1}, & \quad i = 4n + 1; \\ g_{-2n}, \quad u_0 \cdot g_{-2n}, \dots, \quad u_0^{n-1} \cdot g_{-2n}, \quad e^2 \cdot \nu_{-2n-1}, & \quad i = 4n; \\ g_{-2n+1}, \quad u_0 \cdot g_{-2n+1}, \dots, \quad u_0^{n-1} \cdot g_{-2n+1}, \quad u_0^n \cdot g_{-2n+1} & \quad i = 4n - 2; \\ g_{-2n+2}, \quad u_0 \cdot g_{-2n+2}, \dots, \quad u_0^{n-1} \cdot g_{-2n+2}, & \quad i = 4n - 4; \end{aligned}$$

where  $2\nu_{-2n-1} = u_0^n \cdot g_{-2n-1}$ ,  $e^2 \cdot \nu_{-2n-1} = u_0^n \cdot g_{-2n}$  and  $\alpha \cdot \nu_{-2n-1} = u_0^n \cdot g_{-2n+1}$ .

*Proof.* We apply  $KO^*(-)$  to the cofibre sequence  $T\eta(2n-1) \xrightarrow{j} T\eta(2n) \rightarrow S^{4n+2}$ .

The proof proceeds similarly to that of Proposition 4.3.2.  $\square$

**Theorem 5.3.3.** *Let  $\eta$  be the canonical complex line bundle over  $\mathbb{C}P^{2n}$ . The Thom module  $KO^*(T\eta^m)$  is generated over  $KO^*(\mathbb{C}P_+^{2n})$  by the elements  $g_0, g_1, g_2, g_3$  and  $\nu_{-2n-1}$ , with relations*

1.  $e \cdot g_0 = e \cdot g_1 = e \cdot g_2 = e \cdot g_3 = 0$ ,
2.  $\alpha \cdot g_0 = 2g_2$ ,  $\alpha \cdot g_1 = 2g_3$ ,  $\alpha \cdot g_2 = 2\beta \cdot g_0$ ,  $\alpha \cdot g_3 = 2\beta \cdot g_1$ ,
3.  $u_j \cdot g_0 = u_{j-2} \cdot g_2$ ,  $u_j \cdot g_1 = u_{j-2} \cdot g_3$ ,
4.  $u_{4j+1} \cdot g_0 = u_{4j} \cdot g_1$ ,  $u_{4j+3} \cdot g_0 = u_{4j} \cdot g_3$ ,
5.  $u_{4j-1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_0$ ,  
 $u_{4j+1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_2$ ,
6.  $u_{-2n-2}u_0^{n-1} \cdot g_1 = 2\nu_{-2n-1}$ ,  
 $u_{-2n}u_0^{n-1} \cdot g_0 = e^2 \cdot \nu_{-2n-1}$ ,  
 $u_{-2n}u_0^{n-1} \cdot g_1 = \alpha \cdot \nu_{-2n-1}$ ,  
 $u_{-2n+2}u_0^{n-1} \cdot g_0 = 0$ .

The Thom algebra is generated by the same elements, with the additional relations

$$g_i g_j = \begin{cases} r(x(x+1)^{\frac{m-1}{2}}) \cdot g_{i+j}, & m > 0, \\ r(x(\bar{x}+1)^{\frac{m-1}{2}}) \cdot g_{i+j}, & m < 0, \end{cases} \text{ if either } i \text{ or } j \text{ even,}$$

$$g_i g_j = \begin{cases} \left\{ r(x(x+1)^{\frac{m-1}{2}}) + 4 + 4r(\eta + \cdots + \eta^{\frac{m-1}{2}}) \right\} \cdot g_{i+j}, & m > 0, \\ \left\{ r(x(\bar{x}+1)^{\frac{m-1}{2}}) + 4 + 4r(\eta + \cdots + \eta^{\frac{m-1}{2}}) \right\} \cdot g_{i+j}, & m < 0. \end{cases} \text{ } i \text{ and } j \text{ odd,}$$

and

$$\nu_{-2n-1} g_i = 0 \pmod{e^2 \cdot \nu_{-2n-1}}$$

for all integers  $0 \leq i, j \leq 3$ .

Since  $u_i u_0^n \in KO^0(\mathbb{C}P^{2n})$  is equal to 0 (Proposition 3.2.7) it follows that  $u_i u_0^n \cdot g_j$  equals 0 for any integers  $i, j$ .

*Proof.* The inclusion  $j_{2n}: \mathbb{C}P^{2n} \rightarrow \mathbb{C}P^{2n+1}$  induces an inclusion of Thom spaces. Pulling back from  $KO^*(T\eta(2n+1)^m)$  preserves the relations of Lemmas 5.2.5 to 5.2.8. Relation 6. is Proposition 5.3.2, and it remains to prove that  $\nu_{-2n-1} \cdot g_i$  equals zero. We use complexification, which annihilates  $e^2 \cdot \nu_{-2n-1} \in KO^{4n}(T\eta^m)$ , but is injective on  $KO^*(T\eta^m)/(e^2 \cdot \nu_{-2n-1})$ .

The relations allow us to rewrite any polynomial  $P \in KO^*(\mathbb{C}P_+^{2n})[g_i \mid i \in \mathbb{Z}]$  in terms of the basis given in Proposition 5.3.2. Hence the relations are sufficient.  $\square$

Let  $\eta$  be the canonical line bundle over  $\mathbb{C}P^\infty$ . We now deduce the Thom algebra  $KO^*(T\eta^m)$  from Proposition 5.3.1 using inverse limits. First note that the homomorphism  $Tj_{2n}^* \circ Tj_{2n+1}^*: KO^{-2i}(T\eta(2n+1)^m) \rightarrow KO^{-2i}(T\eta(2n-1)^m)$  is surjective (5.4) and applying Theorem 2.0.17, we see that  $KO^{-2i}(T\eta^m) \cong \varprojlim KO^{-2i}(T\eta(2n+1)^m)$  is isomorphic to  $\prod \mathbb{Z}$ . We write the elements in the form  $\sum_{n=0}^\infty a_n u_0^n \cdot g_i$ , where  $a_n$  are integers.

Equipping  $KO^*(T\eta^m)$  with multiplication (2.12), we have an isomorphism of  $\mathbb{Z}[[u_0]]$ -modules  $KO^{-2i}(T\eta^m) \cong \mathbb{Z}[[u_0]] \cdot g_i$  for each integer  $i$ .

The above demonstrates that  $KO^*(T\eta^m)$  is generated over  $KO^*(\mathbb{C}P_+^\infty)$  by the elements  $g_i$ ,  $i \in \mathbb{Z}$ . We retain some multiplicative relations from  $KO^*(T\eta(2n+1)^m)$ . Recall that, by the definition of inverse limits, an element in  $KO^i(T\eta^m)$  equals the zero element if and only if it is annihilated by restriction to  $KO^i(T\eta(2n+1)^m)$  for every  $n \geq 0$ . Thus, referring to Proposition 5.3.1, we retain those relations which do not depend on  $n$ , i.e. those proved in Lemmas 5.2.5 to 5.2.8.

Note that  $u_0^n \cdot g_i$  is non-zero in  $KO^{-2i}(T\eta^m)$  for any  $i$ , because  $u_0^k \cdot g_i$  is a non-zero element of  $KO^{-2i}(T\eta(2n+1)^m)$  for every  $n \geq k$ .

Summarising the above, we have the following result.

**Theorem 5.3.4.** *Let  $\eta$  be the canonical complex line bundle over  $\mathbb{C}P^\infty$ . The Thom module  $KO^*(T\eta^m)$  is generated over  $KO^*(\mathbb{C}P_+^\infty)$  by the elements  $g_0, g_1, g_2, g_3$ , with relations*

1.  $e \cdot g_0 = e \cdot g_1 = e \cdot g_2 = e \cdot g_3 = 0$ ,
2.  $\alpha \cdot g_0 = 2g_2, \alpha \cdot g_1 = 2g_3, \alpha \cdot g_2 = 2\beta \cdot g_0, \alpha \cdot g_3 = 2\beta \cdot g_1$ ,
3.  $u_j \cdot g_0 = u_{j-2} \cdot g_2, u_j \cdot g_1 = u_{j-2} \cdot g_3$ ,
4.  $u_{4j+1} \cdot g_0 = u_{4j} \cdot g_1, u_{4j+3} \cdot g_0 = u_{4j} \cdot g_3$ ,
5.  $u_{4j-1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_0$ ,  
 $u_{4j+1} \cdot g_1 = (u_{4j} + 4\beta^j) \cdot g_2$ .



The Thom algebra is generated by the same elements, with the additional relations

$$g_i g_j = \begin{cases} r(x(x+1)^{\frac{m-1}{2}}) \cdot g_{i+j}, & m > 0, \\ r(x(\bar{x}+1)^{\frac{m-1}{2}}) \cdot g_{i+j}, & m < 0, \end{cases} \text{ if either } i \text{ or } j \text{ even,}$$

and

$$g_i g_j = \begin{cases} \left\{ r(x(x+1)^{\frac{m-1}{2}}) + 4 + 4r(\eta + \cdots + \eta^{\frac{m-1}{2}}) \right\} \cdot g_{i+j}, & m > 0, \\ \left\{ r(x(\bar{x}+1)^{\frac{m-1}{2}}) + 4 + 4r(\eta + \cdots + \eta^{\frac{m-1}{2}}) \right\} \cdot g_{i+j}, & m < 0, \end{cases} \text{ } i \text{ and } j \text{ odd,}$$

for all integers  $0 \leq i, j \leq 3$ .

Other choices of generators are of course possible since the periodicity relation  $\beta^k \cdot g_i = g_{i+4k}$  holds for all integers  $i$  and  $k$ .

# Chapter 6

## Wedges of 2-spheres

In this chapter we study an arbitrary complex line bundle  $\phi$  over a 2 dimensional space  $X$ . Since  $X$  is a CW-complex with cells in only even dimensions, it has the homotopy type of the wedge of 2-spheres  $W_n := S_1^2 \vee \dots \vee S_n^2$  for some  $n \geq 0$ . If  $n = 0$ , then  $X = W_0$  has the homotopy type of a point, and any  $n$ -dimensional complex bundle  $\theta$  over  $X$  is isomorphic to the trivial bundle. Then  $T\theta$  has the homotopy type of a  $2n$ -sphere, and the Thom module  $KO^*(T\theta)$  is generated over  $KO^*(S^0)$  by  $t^{KO} \in KO^{2n}(T\theta)$  with the single relation  $(t^{KO})^2 = 0$ .

For the remainder of the chapter we assume that  $X$  has the homotopy type of  $W_n$  where  $n \geq 1$  and restrict attention to line bundles  $\phi$  over  $X$ . The collection of complex line bundles over  $S^2$  is a multiplicative group under the tensor product operation generated by the canonical line bundle  $\eta(1)$  over  $S^2$  [20]. A complex line bundle over  $W_n$  is of the form  $\phi := \eta_1^{a_1} \otimes \dots \otimes \eta_n^{a_n}$  for a list of integers  $(a_1, \dots, a_n)$ .

We begin by discussing the integral cohomology of  $X$  and  $T\phi$ , then the Thom algebra of  $T\phi$  when  $\phi$  is *Spin*, before addressing the more involved case, where  $\phi$  is not orientable. We compute  $KO^*(T\phi)$  as a graded abelian group using the Atiyah-Hirzebruch spectral sequence, then define elements of  $KO^*(T\phi)$  using the realification homomorphism. We prove that these elements form a basis for the graded group, and compute multiplicative relations between the elements. We can then state  $KO^*(T\phi)$  as a  $KO_*$ -algebra, and deduce the Thom algebra of  $\phi$  over  $X$ .

We repeat some material from Chapter 4, specifically the description of the Thom

algebra of line bundles  $\phi$  over  $\mathbb{C}P^1 = S^2$ . This is also computed in [14, Proposition 4.5]. We believe that the computations concerning line bundles over  $W_n$  where  $n \geq 2$  is original.

Recall that the element  $s^D \in D^2(S^2)$  generates  $D^*(S^2)$  as a free  $D_*$ -module for any spectrum  $D$ . Note that since we are concerned only with 2-spheres in this chapter, we discard the subscript. Using the projections  $\pi_i: X \rightarrow S^2$  of  $X = S^2 \vee \cdots \vee S^2$  onto the  $i$ th copy of  $S^2$ , we define  $s_i^D \in D^2(X)$  by  $s_i^D := \pi_i^*(s^D)$ . Then  $D^*(X) \cong \bigoplus_{i=1}^n D^*(S^2)$  is a  $D_*$ -module generated by  $s_1^D, \dots, s_n^D$ , with relations  $s_i^D s_{i'}^D = 0$  for any integers  $1 \leq i, i' \leq n$ .

## 6.1 Integral Cohomology and Complex $K$ -Theory

We begin by reviewing the integral cohomology and complex  $K$ -theory of  $T\phi$ . The first Chern class  $c_1(\phi) \in H^2(X)$  is equal to  $a_1 s_1^H + \cdots + a_n s_n^H$ , and by Example 2.0.8, the Thom module  $H^*(T\phi)$  is generated over  $H^*(X_+)$  by one generator  $t^H \in H^2(T\phi)$ . The Thom algebra is generated by  $t^H$ , with the single relation

$$(t^H)^2 = (a_1 s_1^H + \cdots + a_n s_n^H) \cdot t^H. \quad (6.1)$$

Turning to the complex  $K$ -theory of  $X$ , it is convenient to work with elements in  $K^0(X)$ , so we define  $x_i \in K^0(X)$  as  $x_i = z s_i^K$ ,  $1 \leq i \leq n$ . Then  $K^*(X)$  is a  $K_*$ -module generated by  $x_1, \dots, x_n$ , with relations  $x_i x_{i'} = 0$  for any integers  $1 \leq i, i' \leq n$ . Restriction of  $K^*(X)$  to  $K^*(S^2)$  coincides with  $K^*(\mathbb{C}P^1)$ , and we refer to (2.8) to observe that conjugation interacts with  $x_i$  via the relation

$$\bar{x}_i = -x_i \quad (6.2)$$

for each  $1 \leq i \leq n$ . Pulling back  $c_1^K = z^{-1}(\eta - 1) \in K^2(\mathbb{C}P^\infty)$ , we obtain

$$z^{-1}(\phi - 1) = z^{-1} \left( \prod_i (1 + x_i)^{a_i} - 1 \right),$$

so, since  $x_i^2 = 0$ , the first Chern class  $c_1(\phi) \in K^2(X)$  equals  $z^{-1}(a_1 x_1 + \cdots + a_n x_n)$ .

Pulling  $\bar{t} \in K^0(T\phi)$  back along the inclusion  $j_i: S^2 \rightarrow X$  of the  $i$ th copy of  $S^2$ , we obtain  $j_i^*(\bar{t}(\phi)) = (a_i x - 1) \cdot t(\eta^{a_i}) \in K^0(T\eta(1)^{a_i})$  (5.3). The only element in  $K^0(T\phi) \cong K^0(X) \cdot t(\phi)$  satisfying this relation for each  $i$  is  $(a_1 x_1 + \dots + a_n x_n - 1) \cdot t$ .

Then by Corollary 2.0.11 we have the following result.  $K^*(T\phi)$  is a free  $K^*(X_+)$ -module on one generator  $t \in K^0(T\phi)$ . The Thom algebra is generated by  $t$ , with the single relation

$$t^2 = (a_1 x_1 + \dots + a_n x_n) \cdot t. \quad (6.3)$$

Conjugation interacts with the Thom class via the relation

$$\bar{t} = (a_1 x_1 + \dots + a_n x_n - 1) \cdot t. \quad (6.4)$$

## 6.2 Real $K$ -theory

### 6.2.1 The $KO$ -orientable case

Let  $a_j = 2b_j$  be even for each  $1 \leq j \leq n$ . Taking the mod 2 reduction of the first Chern class, the second Stiefel-Whitney class  $w_2(\phi) = a_1 s_1^H + \dots + a_n s_n^H \in H^2(X; \mathbb{Z}_2)$  is zero, and  $\phi$  is  $KO$ -orientable.

Since  $\phi = \eta_1^{a_1} \otimes \dots \otimes \eta_n^{a_n}$  is the pullback of  $\eta^2$  along the map  $X \rightarrow \mathbb{C}P^\infty$  classifying  $\phi' = \eta_1^{a_1/2} \otimes \dots \otimes \eta_n^{a_n/2}$ , we pull back the Thom algebra of  $\eta^2$  over  $\mathbb{C}P^\infty$ .

The Thom algebra of  $\eta^2$  is described in [14, Lemma 4.4] with the Thom class  $t_\square \in KO^2(T\eta^2)$ , where  $t_\square^2 = u_{-1} \cdot t_\square$ . Pulling both  $t_\square$  and  $u_{-1}$  back along the induced  $KO$ -theory homomorphisms gives us  $(t^{KO})^2 = u_{-1}(\phi') \cdot t^{KO}$ . Since

$$\begin{aligned} u_{-1}(\phi') &= r(z^{-1}(\phi' - 1)) \\ &= r(z^{-1}(\eta_1^{b_1} - 1)) + \dots + r(z^{-1}(\eta_n^{b_n} - 1)) \\ &= b_1 r(z^{-1}(\eta_1 - 1)) + \dots + b_n r(z^{-1}(\eta_n - 1)) \\ &= a_1 s_1 + \dots + a_n s_n, \end{aligned}$$

we deduce the following result. The Thom module  $KO^*(T\phi)$  is generated over  $KO^*(X_+)$  by one generator  $t^{KO} \in KO^2(T\phi)$ . The Thom algebra is generated by  $t^{KO}$  with the single relation

$$(t^{KO})^2 = (a_1 s_1 + \dots + a_n s_n) \cdot t^{KO}.$$

### 6.2.2 Group Structure

We now address the more involved case, where  $\phi$  is not orientable. We begin by computing  $KO^*(T\phi)$  as a graded abelian group using the Atiyah-Hirzebruch spectral sequence. We then define elements of  $KO^*(T\phi)$  using the realification homomorphism. We prove that these elements form a basis for the graded group, and compute multiplicative relations between the elements. We can then state  $KO^*(T\phi)$  as a  $KO_*$ -algebra, and deduce the structure of  $KO^*(T\phi)$  as a  $KO^*(X_+)$ -module.

Since  $\phi = \eta_1^{a_1} \otimes \cdots \otimes \eta_n^{a_n}$  is not *Spin*, at least one  $a_i$  must be odd.

**Proposition 6.2.1.** *Given a complex line bundle  $\phi$  over  $W_n$  such that  $\theta$  is not *Spin*, the graded abelian groups  $KO^*(T\phi)$  and  $\mathbb{Z}(1) \bigoplus_{n-1} KO^*(S^4)$  are isomorphic.*

*Proof.* We use the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} \cong H^p(T\phi; KO^q(S^0)) \Rightarrow KO^*(T\phi).$$

Part of the  $E_2$ -term is shown below. We only examine this portion, as the remainder follows by periodicity. Of the integral cohomology groups, only  $H^2(T\phi)$  and  $H^4(T\phi)$  are non-zero, and there are only two non-zero columns in the  $E_2$ -term. For dimensional reasons the only non-zero differential is  $d_2 = Sq^2 \otimes \cdot e$  (2.11).

$q \setminus p$	0	1	2	3	4	5	...
0			$\mathbb{Z}$		$\mathbb{Z}^n$		
-1			$\mathbb{Z}_2$		$\mathbb{Z}_2^n$		
-2			$\mathbb{Z}_2$		$\mathbb{Z}_2^n$		
-3							
-4			$\mathbb{Z}$		$\mathbb{Z}^n$		
-5							
-6							
-7							
-8			$\mathbb{Z}$		$\mathbb{Z}^n$		

Since  $(t^H)^2 = (a_1 s_1^H + \cdots + a_n s_n^H) \cdot t^H$  (6.1), the only non-zero differentials are  $d_2^{2,0}(t^H \otimes 1) = (a_1 s_1^H + \cdots + a_k s_k^H) \cdot t^H \otimes e$  and  $d_2^{2,-1}(t^H \otimes e) = (a_1 s_1^H + \cdots + a_k s_k^H) \cdot t^H \otimes e^2$ .

This leaves us with the  $E_3$ -term shown below, beyond which the spectral sequence collapses.

$q \backslash p$	0	1	2	3	4	5	$\dots$
0			$\mathbb{Z}$		$\mathbb{Z}^n$		
-1					$\mathbb{Z}_2^{n-1}$		
-2			$\mathbb{Z}_2$		$\mathbb{Z}_2^{n-1}$		
-3							
-4			$\mathbb{Z}$		$\mathbb{Z}^n$		
-5							
-6							
-7							
-8			$\mathbb{Z}$		$\mathbb{Z}^n$		

Our lone non-trivial extension problem

$$0 \rightarrow F^3 KO^0(T\phi) \rightarrow F^2 KO^0(T\phi) \rightarrow \mathbb{Z}_2 \rightarrow 0$$

is to establish whether  $F^2 KO^0(T\phi) \cong KO^0(T\phi)$  is isomorphic to  $\mathbb{Z}^n$  or to  $\mathbb{Z}^n \oplus \mathbb{Z}_2$ . But since  $KO^1(T\phi)$  is zero, Bott's sequence shows that  $\chi$  is a monomorphism from  $KO^0(T\phi)$  to the torsion free group  $K^2(T\phi)$ . So  $F^2 KO^0(T\phi)$  and  $\mathbb{Z}^n$  are isomorphic.  $\square$

### 6.2.3 Basis elements

We define elements  $v_i$  and  $o_j$  which generate the Thom algebra of  $\phi$  over  $X$ .

Let  $b_i = \lfloor \frac{a_i}{2} \rfloor$ , and let  $a'_i = a_i - 2b_i$  be the mod 2 reduction of  $a_i$ .

We make use of Bott's sequence (2.1). Realification  $r: K^{2i}(T\phi) \rightarrow KO^{2i}(T\phi)$  is surjective unless  $i \equiv 2 \pmod{4}$ . Complexification  $c: KO^{2i}(T\phi) \rightarrow K^{2i}(T\phi)$  is injective unless  $i \equiv 1 \pmod{4}$ . So  $r: K^4(T\phi) \rightarrow KO^4(T\phi)$  is not surjective. In fact, the image of  $r$  in  $KO^4(T\phi)$  is isomorphic to the subgroup  $\mathbb{Z} \oplus 2\mathbb{Z}^{n-1} \subset \mathbb{Z}^n$ . The image of  $r$  is spanned by the elements  $r(z^{-2}x_1 \cdot t), \dots, r(z^{-2}x_n \cdot t)$  and  $r(z^{-2} \cdot t) = \langle u_{-2} \rangle(\phi)$  (see Definition 3.3.2). We complexify, and, using (6.4) obtain the relations

$$cr(z^{-2}x_j \cdot t) = 2z^{-2}x_j \cdot t \quad \text{and} \quad c(\langle u_{-2} \rangle(\phi)) = z^{-2}(x_1 + \dots + x_n) \cdot t.$$

It follows that the image of  $r$  has a basis consisting of the  $n$  elements  $r(z^{-2}x_2 \cdot t^K)$ ,  $\dots$ ,  $r(z^{-2}x_n \cdot t^K)$  and  $v_{-2} := \langle u_{-2} \rangle(\phi) - \sum_j b_j r(z^{-2}x_j \cdot t)$ , with the relation

$$r(z^{-2}x_1 \cdot t) = 2v_{-2} - r(z^{-2}x_2 \cdot t) - \dots - r(z^{-2}x_n \cdot t), \quad (6.5)$$

We may immediately deduce that  $v_{-2}$  and  $\frac{1}{2}r(z^{-2}x_2 t)$ ,  $\dots$ ,  $\frac{1}{2}r(z^{-2}x_n t)$  are a linearly independent spanning set in  $KO^4(T\phi) \cong \mathbb{Z}^n$  and thus are a basis. We should note that we could have removed any  $r(z^{-2}x_j \cdot t)$  from the  $n + 1$  elements spanning the image of  $r$  in  $KO^4(T\phi)$ , and so  $o_j := \frac{1}{2}r(z^{-2}x_j \cdot t)$  is defined for  $1 \leq j \leq n$ .

**Definition 6.2.2.** We define the elements  $v_i \in KO^{-2i}(T\phi)$  and  $o_j \in KO^4(T\phi)$  as follows.

$$v_i := \begin{cases} r(z^i \cdot t), & i \text{ odd,} \\ r(z^i(1 - b_1x_1 - \dots - b_nx_n) \cdot t), & i \text{ even.} \end{cases}$$

$$o_j := \frac{1}{2}r(z^{-2}x_j \cdot t), \quad 1 \leq j \leq n.$$

Note that  $v_i \in KO^{-2i}(T\phi)$  coincides with  $\langle u_i \rangle(\phi)$  when  $i$  is odd.

The complexifications of  $v_i$  and  $o_j$  are as follows.

$$c(o_j) = z^{-2}x_j \cdot t, \quad (6.6)$$

$$c(v_i) = \begin{cases} z^i(a'_1x_1 + \dots + a'_nx_n) \cdot t, & i \text{ even} \\ z^i(2 - a_1x_1 - \dots - a_nx_n) \cdot t, & i \text{ odd.} \end{cases} \quad (6.7)$$

**Proposition 6.2.3.** *Bases for  $KO^i(T\phi)$  are as follows:*

$$\begin{aligned} v_{-2}, o_2, \dots, o_n, & \quad i = 4, \\ e \cdot o_2, \dots, e \cdot o_n, & \quad i = 3, \\ v_{-1}, e^2 \cdot o_2, \dots, e^2 \cdot o_n, & \quad i = 2, \\ v_0, \alpha \cdot o_2, \dots, \alpha \cdot o_n, & \quad i = 0, \\ v_1, & \quad i = -2, \\ 0, & \quad i = \pm 1 \text{ or } 3. \end{aligned}$$

*Proof.* Since  $o_j$  is not in the image of  $r$ , it follows that  $e \cdot o_j$  is a non-trivial element of  $KO^3(T\phi)$ , and we have the relation

$$e \cdot o_1 + \dots + e \cdot o_n = 0$$

(divide (6.5) by 2 and multiply by  $e$ ). So  $e \cdot o_2, \dots, e \cdot o_n$  are a basis for  $KO^3(T\phi) \cong \mathbb{Z}_2^{n-1}$ . Similarly  $e^2 \cdot o_2, \dots, e^2 \cdot o_n$  are a basis for the summand  $\mathbb{Z}_2^{n-1} \subset KO^2(T\phi)$ .

Realification  $r: K^2(T\phi) \rightarrow KO^2(T\phi) \cong \mathbb{Z} \oplus \mathbb{Z}_2^{n-1}$  is surjective, but complexification is not injective. It is the zero homomorphism on the summand  $\mathbb{Z}_2^{n-1}$ , and sends the generator of the  $\mathbb{Z}$ -summand to a basis element of  $K^2(T\phi)$ . For any  $j$ ,  $cr(z^{-1}x_j \cdot t)$  equals zero, but  $cr(z^{-1} \cdot t) = z^{-1}(2 - a_1x_1 + \dots + a_nx_n) \cdot t$ , so  $r(z^{-1} \cdot t) = v_{-1}$  must generate the  $\mathbb{Z}$ -summand. The generators of the remaining  $\mathbb{Z}_2$ -summands are  $e^2 \cdot o_2, \dots, e^2 \cdot o_n$ . Each  $e^2 \cdot o_j$  is in the image of  $r$ . Restricting  $\phi$  to a bundle over  $S^2$ , equality of  $e^2 \cdot o_j$  and  $r(z^{-1}x_j \cdot t)$  becomes obvious.

For the remaining non-zero groups,  $KO^0(T\phi)$  and  $KO^{-2}(T\phi)$ , life becomes easier, since realification is surjective, and complexification is injective. Realification indicates that  $KO^0(T\phi)$  is spanned by  $r(t), r(x_1 \cdot t), \dots, r(x_n \cdot t)$ , and complexification provides the relations

$$2v_0 = r(x_1 \cdot t) + \dots + r(x_n \cdot t) \quad \text{and} \quad r(x_j \cdot t) = \alpha \cdot o_j$$

so the elements  $v_0, \alpha \cdot o_2, \dots, \alpha \cdot o_n$  span  $KO^0(T\phi) \cong \mathbb{Z}^n$  and thus are a basis.

Similarly, for  $KO^{-2}(T\phi) \cong \mathbb{Z}$ , we see that the elements  $r(z \cdot t), r(zx_1 \cdot t), \dots, r(zx_n \cdot t)$  are a spanning set, but complexification shows that only  $r(z \cdot t)$  is non-zero and is therefore a generator.  $\square$

## 6.2.4 Multiplicative Relations

Our basis for  $KO^*(T\phi)$  shows that each  $o_j \in K^4(T\phi)$  is free over the coefficients. We first describe the interaction of  $v_i \in KO^{-2i}(T\phi)$  with the coefficient ring  $KO_*$ , then the products  $v_i v_j, v_i o_j \in KO^{-2(i+j)}(T\phi)$  and  $KO^{4-2i}(T\phi)$  respectively. This furnishes us with a description of the Thom algebra of  $\phi$  over  $X$ .

**Proposition 6.2.4.** *The following relations hold in  $KO^*(T\phi)$ .*

$$e \cdot v_i = 0, \quad \alpha \cdot v_i = 2 \cdot v_{i+2} \pmod{e^2 \cdot o_j}, \quad \beta \cdot v_i = v_{i+4}.$$

*Proof.* Referring to (2.1),  $e \cdot v_i$  is zero because  $v_i$  is in the image of realification. The remaining relations follow easily by complexifying and referring to Lemma 2.0.3,



except for  $\alpha \cdot v_i = 2 \cdot v_{i+2}$  when  $i \equiv 1 \pmod{4}$ , as complexification  $c: KO^{4-2i}(T\phi) \rightarrow K^{4-2i}(T\phi)$  annihilates  $\beta^{\frac{i-1}{4}} e^2 \cdot o_j$  for each  $1 \leq j \leq n$ .  $\square$

Recall that  $KO^*(X)$  is a free  $KO_*$ -module, generated by  $s_j^{KO} \in KO^2(X)$  where  $1 \leq j \leq n$ . We omit the superscript, and work with  $s_j := s_j^{KO}$ .

**Proposition 6.2.5.** *The following relations hold in  $KO^*(T\phi)$ .*

$$s_j \cdot t_{j'} = 0, \quad s_j \cdot v_0 = 0 \pmod{e^2 \cdot o_j}, \quad s_j \cdot v_2 = 0, \quad s_j \cdot v_1 = \alpha \cdot t_j, \quad s_j \cdot v_3 = 2\beta \cdot t_j.$$

*Proof.* Using Bott's sequence, we see that the complexification of  $s_j$  is equal to  $z^{-1}x_j$  for each  $j$ . Using this fact, the relations  $s_j \cdot o_{j'} = s_j \cdot v_2 = 0$ ,  $s_j \cdot v_1 = \alpha \cdot o_j$ ,  $s_j \cdot v_2 = 0$  and  $s_j \cdot v_3 = 2\beta \cdot o_j$  follow easily by complexification. Complexification  $c: KO^{4-2i}(T\phi) \rightarrow K^{4-2i}(T\phi)$  annihilates  $\beta^{\frac{i-1}{4}} e^2 \cdot o_j$  for each  $1 \leq j \leq n$ , so complexification shows that  $s_j \cdot v_0$  equals zero modulo  $e^2 \cdot o_j$ .  $\square$

**Proposition 6.2.6.** *The following relations hold in  $KO^*(T\phi)$ .*

$$o_j o_{j'} = 0$$

and

$$v_{2i+1} v_{2i'-1} = 4a_1 v_{2(i+i')} + 4r(z^{2(i+i')}) \sum_{j=2}^n (b_j - b_1) o_j$$

*Proof.* The relations  $o_j o_{j'} = 0$  and  $v_{2i+1} v_{2i'-1} = 4a_1 v_{2(i+i')} + 4r(z^{2(i+i')}) \sum_{j=2}^n (b_j - b_1) o_j$  are proved using complexification.  $\square$

**Proposition 6.2.7.** *The following relations hold in  $KO^*(T\phi)$  modulo  $e^2 \cdot o_j$ .*

$$o_j v_i = 0 \quad \text{and} \quad v_i v_{2i'} = 0.$$

*Proof.* The relations  $o_j v_i = v_i v_{2i'} = 0$  are easily proved by complexifying when  $c$  is injective. However,  $c: KO^2(T\phi) \rightarrow K^2(T\phi)$  fails to be injective, annihilating  $e^2 \cdot o_j$  for each  $1 \leq j \leq n$ . So  $o_j v_1$  and  $v_{2i} v_{2i'+1}$  both equal zero modulo  $e^2 \cdot o_j$ .  $\square$

### 6.3 Thom Algebra

Summarising the above, we have the following result. The relations are true modulo  $e^2 \cdot o_j \in KO^2(T\phi)$ .

**Proposition 6.3.1.** *Suppose  $\phi$  over  $X$  is a line bundle which is not Spin. Then the Thom module  $KO^*(T\phi)$  is generated over  $KO^*(X_+)$  by the elements  $v_0, v_1, v_2, v_3, o_2, \dots, o_n$ , with relations*

1.  $e \cdot v_0 = e \cdot v_1 = e \cdot v_2 = e \cdot v_3 = 0$ ,
2.  $\alpha \cdot v_0 = 2v_2, \alpha \cdot v_1 = 2v_3, \alpha \cdot v_2 = 2\beta \cdot v_0, \alpha \cdot v_3 = 2\beta \cdot v_1$ ,
3.  $s_j \cdot o_{j'} = 0$ ,
4.  $s_j \cdot v_0 = s_j \cdot v_2 = 0$ ,  
 $s_j \cdot v_1 = \alpha \cdot o_j$ ,  
 $s_j \cdot v_3 = 2\beta \cdot o_j$ .

The Thom algebra is generated by the same elements, with the additional relations

$$o_j o_{j'} = 0, \quad o_j v_i = 0, \quad v_{2i} v_{i'} = 0,$$

and

$$v_{2i+1} v_{2i'-1} = 4a_1 \cdot v_{2(i+i')} + 4r(z^{2(i+i')}) \sum_{j=2}^n (b_j - b_1) \cdot o_j$$

where  $4r(z^{2(i+i')})$  equals  $4\beta^{\frac{i+i'}{2}}$  when  $i+i'$  is even, and  $2\alpha\beta^{\frac{i+i'-1}{2}}$  when  $i+i'$  is odd.

Alternative choices of generators are of course possible, since the periodicity relation  $\beta^k v_i = v_{i+4k}$  holds for all integers  $i$  and  $k$ .

# Chapter 7

## Complex 2-plane Grassmannians

Previously, we have examined line bundles and bundles derived from line bundles via constructions such as the tensor product and direct sum operations. In this chapter we progress to the study of the real  $K$ -theory of 2-plane complex Grassmannians  $G_2(\mathbb{C}^n)$ , and the Thom algebra of  $\eta_2$ , the universal 2-plane bundle over  $BU(2)$ . As in Chapter 3, this case is of interest due to being a classifying space: 2-plane complex bundles over a space  $X$  are classified up to bundle isomorphism via homotopy classes of maps  $X \rightarrow BU(2)$ .

Hoggar partially computes  $KO^*(G_k(\mathbb{C}^n))$  as a graded group, stating the rank of the free part of  $KO^{2i}(G_k(\mathbb{C}^n))$  [23, Theorem B] and observing that  $KO^{2i+1}(G_k(\mathbb{C}^n))$  is either zero or a sum of copies of  $\mathbb{Z}_2$  (Proposition 2.0.1).

In the case  $k = 2$ , the graded group is fully calculated [23, Theorem C1], and a basis for  $KO^0(G_k(\mathbb{C}^n))$  as a  $\mathbb{Z}$ -module is given [23, Theorem C2]. The generators are described using the image under realification of the canonical 2-plane bundle  $\eta_2$  over  $G_2(\mathbb{C}^n)$  and its conjugate, acted on by the exterior power and  $\gamma$ -operations [5].

Many of our computations use techniques similar to those employed by Hoggar; notably Bott's sequence plays an important role in [23].

Kono and Hara [26] use the Atiyah-Hirzebruch spectral sequence to compute the torsion parts of the abelian groups  $KO^i(G_k(\mathbb{C}^n))$ . Along with Hoggar's results, this gives a complete description of  $KO^*(G_k(\mathbb{C}^n))$  as a graded group.

In [18], Hara offers a description of  $KO^i(BU(n))$  in terms of groups derived from

maps  $1 \pm \tau$  on  $K^0(BU(n))$ . Hara's description of  $KO^*(BU(n))$  appears to be unsuitable for our purposes, as there seems to be no obvious way of deducing the Thom algebra of  $\eta_2$  over  $BU(2)$ .

We begin by reviewing the complex  $K$ -theory and prove a series of lemmas describing the action of conjugation on the generators of the  $K_*$ -algebra  $K^*(BU(2)_+)$ . Turning to the real  $K$ -theory, we define elements that generate the  $KO_*$ -algebras  $KO^*(G_2(\mathbb{C}^{2n})_+)$  and  $KO^*(BU(2)_+)$ , then examine their properties using the complexification homomorphism. We restate Hoggar's results regarding  $KO^i(G_2(\mathbb{C}^n))$  as an abelian group, and find a basis for  $KO^i(G_2(\mathbb{C}^{2n})_+)$  as a  $\mathbb{Z}$ -module. We prove multiplicative relations between the generators, allowing us to express  $KO^*(BU(2)_+)$  as a  $KO_*$ -algebra.

Since  $BU(2)$  is a classifying space for principal  $U(2)$  bundles or equivalently for complex 2-plane bundles, we introduce characteristic classes by pulling back the generators of the  $KO_*$ -algebra  $KO^*(BU(2)_+)$  along the classifying map  $\theta: X \rightarrow BU(2)$  for an arbitrary complex 2-plane bundle  $\theta$  over  $X$ .

The zero section  $\iota: BU(2) \rightarrow MU(2)$  identifies  $KO^*(MU(2))$  as an ideal in  $KO^*(BU(2))$  (via Proposition 7.2.10), and we pull back to deduce generators and relations for the Thom algebra of  $\eta_2$  over  $BU(2)$ .

We begin with some geometric results linking 2-plane Grassmannians, complex projective spaces and the Thom complex of canonical 2-plane bundles. Let  $G_2(\mathbb{C}^n)$  be the space of 2-planes in  $\mathbb{C}^n$ . We have an inclusion  $G_1(\mathbb{C}^{n-1}) \hookrightarrow G_2(\mathbb{C}^n)$  given by adding a fixed complex line.

**Lemma 7.0.2.** *The quotient space  $G_2(\mathbb{C}^n)/G_1(\mathbb{C}^{n-1})$  is homeomorphic to the Thom complex  $T\eta_2$  of the canonical 2-plane bundle  $\eta_2 \rightarrow G_2(\mathbb{C}^{n-1})$ .*

*Proof.* We adapt the proof of [23, Lemma 3.2]. We define the projection  $p: G_2(\mathbb{C}^n) - G_1(\mathbb{C}^{n-1}) \rightarrow G_2(\mathbb{C}^{n-1})$  as the restriction to the first  $n - 1$  coordinates. Note that for any plane  $W \in G_2(\mathbb{C}^n) - G_1(\mathbb{C}^{n-1})$ , its image  $p(W) \in \mathbb{C}^{n-1}$  is a 2-plane, since the image of  $G_1(\mathbb{C}^{n-1})$  in  $G_2(\mathbb{C}^n)$  is precisely those 2-planes which project to a line.

If  $p$  is the identity on a plane  $W$ , i.e.  $W$  lies in  $G_2(\mathbb{C}^{n-1})$ , then we identify  $W$  with the pair  $(p(W), 0)$ . For the remainder of the proof we assume that  $W$  intersects  $p(W) \in G_2(\mathbb{C}^{n-1})$  along a line, and let  $e_n \in \mathbb{C}^n$  be the standard  $n$ th basis vector.

Given a plane  $V \in G_2(\mathbb{C}^{n-1})$  and a point  $0 \neq v \in V$ , we determine a plane  $W$  in  $G_2(\mathbb{C}^n) - G_1(\mathbb{C}^{n-1})$  as follows: Define  $W$  to be the plane which intersects  $V$  along the line through the origin perpendicular to  $v$ , and passes through the point  $\frac{v}{\|v\|} + \|v\|e_n \in \mathbb{C}^{n-1} \oplus \mathbb{C}$ . If  $v = 0 \in V$ , then  $W = V$ . Note that as  $v \rightarrow 0$ , the distance (parallel to  $e_n$ ) from the unit  $\frac{v}{\|v\|} \in V$  to  $W$  tends to zero, and so  $W$  tends towards  $V$ .

Given a plane  $W \in G_2(\mathbb{C}^n) - G_1(\mathbb{C}^{n-1})$ , the planes  $W$  and  $p(W)$  intersect in a line  $L$  passing through the origin. We define  $L^\perp$  to be the line through the origin in  $p(W)$  perpendicular to  $L$ , and take a unit vector  $l \in L^\perp$ . For any such  $l$ , there is a unique point  $w \in W$  such that  $p(w) = l$ , and  $w$  is of the form  $l + \|w - l\|ze_n$  for some unit  $z \in \mathbb{C}$ . Multiplying through by  $z^{-1}$  we have a unit vector  $l' = z^{-1}l \in L^\perp$  and a point  $w' = z^{-1}w = l' + \|w' - l'\|e_n \in W$ . Both  $l'$  and  $w'$  are uniquely defined. So given any plane  $W \in G_2(\mathbb{C}^n) - G_1(\mathbb{C}^{n-1})$ , we have a pair consisting of the 2-plane  $p(W) \in G_2(\mathbb{C}^{n-1})$  and the point  $\|w' - l'\|l' \in p(W)$ .

The above gives us a continuous bijection between planes  $W \in G_2(\mathbb{C}^n) - G_1(\mathbb{C}^{n-1})$  and pairs  $\{V, \text{point in } V\}$  where  $V$  is a 2-plane in  $G_2(\mathbb{C}^{n-1})$ , and hence an identification of  $G_2(\mathbb{C}^n) - G_1(\mathbb{C}^{n-1}) \xrightarrow{p} G_2(\mathbb{C}^{n-1})$  with the canonical 2-plane bundle  $\eta_2$  over  $G_2(\mathbb{C}^{n-1})$ . The spaces  $T\eta_2$  and  $G_2(\mathbb{C}^n)/G_1(\mathbb{C}^{n-1})$  are the one point compactifications of the total space of  $\eta_2$  and  $G_2(\mathbb{C}^n) - G_1(\mathbb{C}^{n-1})$  respectively.  $\square$

**Corollary 7.0.3.** *Let  $\eta_2 \rightarrow G_2(\mathbb{C}^{n-1})$  be the canonical 2-plane bundle. Then the following is a cofibre sequence.*

$$\mathbb{C}P^{n-2} \rightarrow G_2(\mathbb{C}^n) \rightarrow T\eta_2$$

where the first map is the inclusion of  $G_1(\mathbb{C}^{n-1})$  into  $G_2(\mathbb{C}^n)$  given by adding a fixed vector in the  $n$ th coordinate.

**Definition 7.0.4.** We define  $BU(2)$  to be the limit of the sequence of inclusions

$$G_2(\mathbb{C}^n) \subset G_2(\mathbb{C}^{n+1}) \subset G_2(\mathbb{C}^{n+2}) \dots$$

and define  $MU(2)$  as the Thom space of the canonical 2-plane bundle  $\eta_2 \rightarrow BU(2)$ .

**Lemma 7.0.5.** *The following sequence of spaces is a cofibre sequence.*

$$\mathbb{C}P^\infty \rightarrow BU(2) \xrightarrow{\iota} MU(2).$$

*Proof.* Up to homotopy type,  $BU(n-1) \rightarrow BU(n)$  is the sphere bundle of  $\eta_n \rightarrow BU(n)$  [30, proof of Theorem 16.10], so as a sequence of spaces, we have  $S(\eta_2) \rightarrow D(\eta_2) \rightarrow MU(2)$   $\square$

## 7.1 Complex $K$ -theory

**Definition 7.1.1.** Let  $c_i^\perp \in K^{2i}(G_2(\mathbb{C}^n))$  be the  $i$ th  $K$ -theoretic Chern class of  $\eta^\perp$ , the orthogonal complement of the canonical 2-plane bundle  $\eta_2$ .

**Proposition 7.1.2.** *There is an isomorphism of  $K_*$ -algebras*

$$K^*(G_2(\mathbb{C}^n)_+) \cong K_*[c_1, c_2, c_1^\perp, \dots, c_{n-2}^\perp]/I$$

where  $I$  is the ideal generated by the homogeneous parts of

$$(1 + c_1 + c_2)(1 + c_1^\perp + \dots + c_{n-2}^\perp).$$

*Proof.* See Karoubi [25, Chapter IV, Corollary 3.19]. We need the case  $p = 2$ .  $\square$

**Definition 7.1.3.** The elements  $x, y \in K^0(BU(2))$  are defined by  $x = zc_1$  and  $y = z^2c_2$ .

As a virtual bundle,  $y$  is  $\eta_2 - 2\mathbb{C}$ .

*Remark 7.1.4.* Although the elements  $x$  and  $y$  lie in  $K^0(BU(2))$ , we assign them degrees 2 and 4 respectively, corresponding to the Chern classes  $c_1 \in K^2(BU(2))$  and  $c_2 \in K^4(BU(2))$ .

**Lemma 7.1.5.** *A basis of  $K^{-2i}(G_2(\mathbb{C}^n))$  is given by all monomials  $z^i x^j y^k$  such that  $0 \leq j + k \leq n - 2$*

*Proof.* The split short exact sequence

$$0 \rightarrow K^*(T\eta_2) \rightarrow K^*(G_2(\mathbb{C}^n)) \rightarrow K^*(\mathbb{C}P^{n-2}) \rightarrow 0.$$

splits because  $K^*(\mathbb{C}P^{n-2})$  is free abelian for all  $n \geq 2$ , and, since the image of  $t \in K^0(T\eta_2)$  in  $K^0(G_2(\mathbb{C}^n))$  is  $y$ , we obtain the isomorphism

$$K^*(G_2(\mathbb{C}^n)) \cong K^*(\mathbb{C}P^{n-2}) \oplus yK^*(G_2(\mathbb{C}^{n-2})_+).$$

The lemma can be proved inductively via this isomorphism, or alternatively using the relations of Proposition 7.1.2.  $\square$

We may describe  $BU(2)$  as the union of the strictly increasing sequence of sub-complexes

$$G_2(\mathbb{C}^n) \rightarrow G_2(\mathbb{C}^{n+1}) \rightarrow G_2(\mathbb{C}^{n+2}) \rightarrow \dots$$

and apply Theorem 2.0.17 to examine  $KO^i(BU(2))$ . We are interested in the following special case of (2.7).

**Proposition 7.1.6.** *There is an isomorphism of  $K_*$ -algebras*

$$K^*(BU(2)_+) \cong K_*[[x, y]]$$

Corollary 2.0.11 states that  $K^*(MU(2))$  is a free  $K^*(BU(2)_+)$ -module on a single generator  $t \in K^0(MU(2))$ . Similarly, the Thom algebra is generated by  $t$ , with the single relation

$$t^2 = y \cdot t. \tag{7.1}$$

Conjugation interacts with the Thom class via the relation

$$\bar{t} = \kappa_2 \cdot t = \bar{y}/y \cdot t. \tag{7.2}$$

Note that  $y\kappa_2 = \bar{y} = y - xy - x^2y - y^2$  (modulo terms of degree 10).

Similarly, for the Thom algebra of  $\eta_2$  over  $G_2(\mathbb{C}^n)$ , we find that  $K^*(T\eta_2)$  is a free  $K^*(G_2(\mathbb{C}^n)_+)$ -module on a single generator  $t \in K^0(T\eta_2)$ , where  $t$  satisfies the relations (7.1) and (7.2).

### 7.1.1 Preliminary Relations

After discussing properties of  $K^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  and the injective homomorphism  $i_U^*: K^*(BU(2)) \rightarrow K^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ , we prove a series of lemmas establishing relations between  $x$ ,  $y$  and their conjugates.

Using the Künneth formula [4], we have the following isomorphisms of  $K_*$ -algebras.

$$\begin{aligned} K^*((\mathbb{C}P^m \times \mathbb{C}P^n)_+) &\cong K^*(\mathbb{C}P_+^m) \otimes K^*(\mathbb{C}P_+^n) \\ &\cong K_*[x_1, x_2]/(x_1^{m+1}, x_2^{n+1}). \end{aligned} \quad (7.3)$$

The elements  $x_1, x_2 \in K^0(\mathbb{C}P^m \times \mathbb{C}P^n)$  are defined by  $x_1 := x \otimes 1$  and  $x_2 := 1 \otimes x$ .

So  $K^0((\mathbb{C}P^m \times \mathbb{C}P^n)_+)$  and the ring  $\mathbb{Z}[x_1, x_2]/(x_1^{m+1}, x_2^{n+1})$  are isomorphic. Taking inverse limits, we deduce that  $K^0((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$  and the power series ring  $\mathbb{Z}[[x_1, x_2]]$  are isomorphic. By periodicity, we have the isomorphism of  $K_*$ -algebras

$$K^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+) \cong K_*[[x_1, x_2]]. \quad (7.4)$$

Note that  $x_1 = z(c_1 \otimes 1)$  and  $x_2 = z(1 \otimes c_1) \in K^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  are the products of  $z$  with the first Chern class in the first and second factor respectively.

We require the following well-known result. Although we only need Proposition 7.1.7 for complex  $K$ -theory, equivalent results hold for any complex oriented cohomology theory.

**Proposition 7.1.7.** *The homomorphism  $i_U^*: K^*(BU(2)) \rightarrow K^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  is injective. The Chern classes  $c_1 \in K^2(BU(2))$  and  $c_2 \in K^4(BU(2))$  have images  $1 \otimes c_1 + c_1 \otimes 1 \in K^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  and  $c_1 \otimes c_1 \in K^4(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  respectively.*

*Proof.* The homomorphism  $i_U^*$  is injective in ordinary integral cohomology [30], and the Chern character  $ch: (K^0(X) \oplus K^1(X)) \otimes \mathbb{Q} \rightarrow H^*(X; \mathbb{Q})$  is an isomorphism [20].

We deduce from the following commutative diagram

$$\begin{array}{ccc} K^0(BU(2)) \otimes \mathbb{Q} & \xrightarrow{i_U^*} & K^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \otimes \mathbb{Q} \\ \downarrow ch & & \downarrow ch \\ H^*(BU(2); \mathbb{Q}) & \xrightarrow{i_U^*} & H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; \mathbb{Q}) \end{array}$$

that  $i_U^*: K^0(BU(2)) \otimes \mathbb{Q} \rightarrow K^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \otimes \mathbb{Q}$  is injective. Injectivity of  $i_U^*: K^0(BU(2)) \rightarrow K^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  now follows by noting that both  $K^0(BU(2))$  and



$K^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  are torsion free. Since  $\eta \times \eta = i_U^*(\eta_2)$  is the direct sum  $\pi_1^*(\eta) \oplus \pi_2^*(\eta)$  (where  $\pi_i$  is the projection of  $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$  onto the  $i$ th factor) we see that the total Chern classes  $c(\eta \times \eta)$  and  $c(\eta)c(\eta)$  are equal. This implies the formulae for  $i_U^*(c_i)$ .  $\square$

Multiplying by appropriate powers of  $z$  we immediately obtain the following:

**Corollary 7.1.8.** *The images of  $x$  and  $y$  in  $K^0(BU(2))$  under  $i_U^*$  are as follows.*

$$i_U^*(x) = x_1 + x_2, \quad i_U^*(y) = x_1x_2$$

in  $K^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ .

We now apply  $i_U^*$  to prove several useful lemmas. We also need the fact that  $x_i\bar{x}_i = -(x_i + \bar{x}_i)$  in  $K^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  for  $i \in \{1, 2\}$  (2.9).

**Lemma 7.1.9.** *The following equation holds in  $K^0(BU(2))$ .*

$$(y + \bar{y})^2 = y\bar{y}(x + \bar{x} + y + \bar{y} + 2).$$

*Proof.* We prove that  $y^2 + \bar{y}^2 = y\bar{y}(x + \bar{x} + y + \bar{y})$ . Define  $a := x_1 + x_2 + x_1x_2 + 1$ , and  $b := x_1x_2 + \bar{x}_1\bar{x}_2$ . Then  $i_U^*(y\bar{y})$  equals  $x_1x_2\bar{x}_1\bar{x}_2 = x_1x_2 + \bar{x}_1x_2 + x_1\bar{x}_2 + \bar{x}_1x_2 = b + \bar{b}$ , and  $i_U^*(x + \bar{x} + y + \bar{y} + 2)$  equals  $a + \bar{a}$ . Since  $i_U^*(y^2 + \bar{y}^2)$  equals  $x_1^2x_2^2 + \bar{x}_1^2\bar{x}_2^2$ , we need to show that  $(a + \bar{a})(b + \bar{b})$  and  $x_1^2x_2^2 + \bar{x}_1^2\bar{x}_2^2$  are equal.

$$\begin{aligned} ab &= x_1^2x_2 - x_1x_2 - \bar{x}_1x_2 + x_1x_2^2 + \bar{x}_1x_2^2 \\ &\quad + x_1^2x_2^2 - x_1x_2^2 - \bar{x}_1x_2^2 + x_1x_2 + \bar{x}_1x_2 \\ &= x_1^2x_2 + x_1^2x_2^2, \end{aligned}$$

and

$$\begin{aligned} a\bar{b} &= (x_1 + x_2 + x_1x_2 + 1)(\bar{x}_1\bar{x}_2 + x_1\bar{x}_2) \\ &= -x_1\bar{x}_2 - \bar{x}_1\bar{x}_2 + x_1^2\bar{x}_2 - \bar{x}_1x_2 - \bar{x}_1\bar{x}_2 - x_1x_2 - x_1\bar{x}_2 \\ &\quad + x_1x_2 + \bar{x}_1x_2 + x_1\bar{x}_2 + \bar{x}_1\bar{x}_2 - x_1^2x_2 - x_1^2\bar{x}_2 + \bar{x}_1x_2 + x_1\bar{x}_2 \\ &= -x_1^2x_2. \end{aligned}$$

So  $ab + a\bar{b} = x_1^2x_2^2$ , and  $(a + \bar{a})(b + \bar{b}) = x_1^2x_2^2 + \bar{x}_1^2\bar{x}_2^2$ .  $\square$

**Lemma 7.1.10.** *The following equation holds in  $K^0(BU(2))$ .*

$$x\bar{x} = y\bar{y} - (x + \bar{x} + y + \bar{y}).$$

*Proof.* Since,  $i_U^*(x\bar{x})$  equals

$$(x_1 + x_2)(\bar{x}_1 + \bar{x}_2) = -(x_1 + x_2 + \bar{x}_1 + \bar{x}_2) + x_1\bar{x}_2 + \bar{x}_1x_2,$$

and  $i_U^*(y\bar{y} - (x + \bar{x} + y + \bar{y}))$  equals

$$\begin{aligned} x_1x_2\bar{x}_1\bar{x}_2 - (x_1 + x_2 + \bar{x}_1 + \bar{x}_2 + x_1x_2 + \bar{x}_1\bar{x}_2) \\ = x_1x_2 + \bar{x}_1x_2 + x_1\bar{x}_2 + \bar{x}_1x_2 - x_1 - x_2 - \bar{x}_1 - \bar{x}_2 - x_1x_2 - \bar{x}_1\bar{x}_2 \\ = \bar{x}_1x_2 + x_1\bar{x}_2 - x_1 - x_2 - \bar{x}_1 - \bar{x}_2, \end{aligned}$$

the lemma follows by injectivity of  $i_U^*$ .  $\square$

**Lemma 7.1.11.** *The following equation holds in  $K^0(BU(2))$ .*

$$x\bar{y} + \bar{x}y + 2y\bar{y} = 0.$$

*Proof.* Firstly,  $i_U^*(x\bar{y})$  equals  $(x_1 + x_2)(\bar{x}_1\bar{x}_2) = -(x_1\bar{x}_2 + 2\bar{x}_1x_2 + \bar{x}_1x_2)$ , so  $i_U^*(x\bar{y} + \bar{x}y)$  equals  $-2(x_1x_2 + \bar{x}_1x_2 + x_1\bar{x}_2 + \bar{x}_1\bar{x}_2)$ . Secondly,  $i_U^*(y\bar{y})$  equals  $x_1x_2\bar{x}_1\bar{x}_2$  which is equal to  $x_1x_2 + \bar{x}_1x_2 + x_1\bar{x}_2 + \bar{x}_1\bar{x}_2$ .  $\square$

**Lemma 7.1.12.** *The following equation holds in  $K^0(BU(2))$ .*

$$\bar{x}y - x\bar{y} = -2(y - \bar{y}).$$

*Proof.* Observe that  $i_U^*(\bar{x}y + 2y)$  equals

$$\begin{aligned} (\bar{x}_1 + \bar{x}_2)x_1x_2 + 2x_1x_2 &= -x_1x_2 - \bar{x}_1x_2 - x_1x_2 - x_1\bar{x}_2 + 2x_1x_2 \\ &= -(\bar{x}_1x_2 + x_1\bar{x}_2). \end{aligned}$$

Since conjugation commutes with  $i_U^*$ , it is the identity on  $\bar{x}y + 2y$ , and we conclude that  $\bar{x}y + 2y$  and  $x\bar{y} + 2\bar{y}$  are equal.  $\square$

**Lemma 7.1.13.** *The following equation holds in  $K^0(BU(2))$ .*

$$y^2 - \bar{y}^2 = (y - \bar{y} + x - \bar{x})y\bar{y}.$$

*Proof.* As in the preceding lemmas, we study the image of both sides of the equation under  $i_U^*$ , and use the relation  $x_i\bar{x}_i = -x_i - \bar{x}_i$  to simplify the right hand side.  $\square$

We will find it useful to have explicit expressions for the first few terms of  $\bar{x}$  and  $\bar{y}$  (see Remark 7.1.4):

**Lemma 7.1.14.** *The following equations hold in  $K^0(BU(2))$ .*

$$\begin{aligned}\bar{x} &= -x + x^2 - 2y - x^3 + 3xy \quad (\text{modulo terms of degree } 8), \\ \bar{y} &= y - xy - x^2y - y^2 \quad (\text{modulo terms of degree } 10).\end{aligned}$$

*Proof.*

$$\begin{aligned}i_U^*(\bar{x}) &= \overline{x_1 + x_2} \\ &= \sum_{i=1}^{\infty} (-1)^i (x_1^i + x_2^i) \\ &= -(x_1 + x_2) + (x_1 + x_2)^2 - 2x_1x_2 - (x_1 + x_2)^3 + 3(x_1 + x_2)x_1x_2 + \dots \\ &= i_U^*(-x + x^2 - 2y - x^3 + 3xy + \dots),\end{aligned}$$

and

$$\begin{aligned}i_U^*(\bar{y}) &= \overline{x_1x_2} \\ &= \sum_{r=2}^{\infty} (-1)^r \sum_{i+j=r} (x_1^i x_2^j) \\ &= x_1x_2 - (x_1 + x_2)x_1x_2 + (x_1 + x_2)^2x_1x_2 - (x_1 + x_2)^2 + \dots \\ &= i_U^*(y - xy - x^2y - y^2 + \dots).\end{aligned}$$

The result follows from injectivity of  $i_U^*$ . □

## 7.2 Real $K$ -theory

We define elements which generate the  $KO_*$ -algebra  $KO^*(BU(2)_+)$ . As in our calculations of  $KO^*(BU(1)_+)$  in Chapter 3, the images of the generators of the complex  $K$ -theory algebra under realification play an important role. In addition, we make use of the Pontryagin classes  $\rho_i \in KO^{4i}(BSp(n))$ .

### 7.2.1 Pontryagin Classes

The following construction of Pontryagin classes is analogous to that of Chern classes in Chapter 2.

Let  $\xi_n$  denote the canonical quaternionic  $n$ -plane bundle over the classifying space  $BSp(n)$  for every  $n \geq 1$ . For the canonical quaternionic line bundle  $\xi_1$  over  $\mathbb{H}P^\infty$  we

will just write  $\xi$ . For any quaternion oriented ring spectrum  $D$ , the orientation class  $v^D$  lies in  $D^4(\mathbb{H}P^\infty)$ , and determines an isomorphism

$$D^*(\mathbb{H}P_+^\infty) \cong D_*[[v^D]] \quad (7.5)$$

of  $D_*$ -algebras. The orientation leads to the construction of canonical Pontryagin classes  $\rho_i^D \in D^{4i}(BSp(n))$  for  $1 \leq i \leq n$ , and  $D^*(BSp(n)_+)$  is isomorphic to the formal power series algebra

$$D_*[[\rho_1^D, \dots, \rho_n^D]] \quad (7.6)$$

as  $D_*$ -algebras. For  $n = 1$ , the first Pontryagin class  $\rho_1^D(\eta)$  coincides with  $v^D$ , and (7.6) reduces to (7.5). References for this material are [30, Chapter 16] and [15].

For  $D = KO$ , we abbreviate  $\rho_i^{KO}$  to  $\rho_i$ .

**Example 7.2.1.** There is an isomorphism of  $KO_*$ -algebras [30, 16.34]

$$KO^*(BSp(n)_+) \cong KO_*[[\rho_1, \dots, \rho_n]].$$

We consider the image of the Pontryagin classes  $\rho_i \in KO^{4i}(BSp(n))$  under the homomorphism induced by the inclusion  $q_n: BU(n) \rightarrow BSp(n)$ .

**Definition 7.2.2.** Define  $\varrho := \beta q_2^*(\rho_2) \in KO^0(BU(2))$ .

As usual, we find it convenient to work with the image of  $\varrho$  in complex  $K$ -theory.

**Proposition 7.2.3.** *The image of  $\varrho$  under complexification is  $c(\varrho) = y\bar{y} \in K^0(BU(2))$ .*

*Proof.* Consider the following commutative diagram:

$$\begin{array}{ccc} KO^*(BSp(2)) & \xrightarrow{q_2^*} & KO^*(BU(2)) \\ \downarrow i_{Sp}^* & & \downarrow c \\ KO^*(\mathbb{H}P^\infty \times \mathbb{H}P^\infty) & & K^*(BU(2)) \\ \downarrow c & & \downarrow i_U^* \\ K^*(\mathbb{H}P^\infty \times \mathbb{H}P^\infty) & \xrightarrow{(q_1 \times q_1)^*} & K^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \end{array} \quad (7.7)$$

where  $c$  is complexification and  $i_G$  is the inclusion  $BG(1) \times BG(1) \rightarrow BG(2)$ . Since all of the homomorphisms are natural, the diagram is commutative.

Firstly, we observe that  $i_{S^p}^*(\rho_2)$  and  $\rho_1 \otimes \rho_1$  are equal. This is a standard argument, similar to that in the proof of Proposition 7.1.7.

Adapting [15, (9.1)], we firstly find that  $K^*(\mathbb{H}P_+^\infty)$  and  $K_*[[\xi_{\mathbb{C}} - 2\mathbb{C}]]$  are isomorphic, where  $\xi_{\mathbb{C}}$  is the complex 2-plane bundle obtained by neglecting the quaternionic structure of the canonical line bundle  $\xi$  over  $\mathbb{H}P^\infty$ . Secondly, the complexification of  $\rho_1$  is equal to  $z^{-2}(\xi_{\mathbb{C}} - 2\mathbb{C})$ .

Since  $c_1^H(q_1^*(\xi)) = 0$  and  $c_2^H(q_1^*(\xi)) = -(c_1^H)^2$  [29], the Chern character shows that  $\xi - 2\mathbb{C}$  and  $\eta \oplus \bar{\eta} - 2\mathbb{C}$  represent the same class in  $K^0(\mathbb{C}P^\infty)$ , namely  $x + \bar{x}$ .

Putting all this together, we obtain the following:

$$\begin{aligned} (q_1 \times q_1)^* \circ c \circ i_{S^p}^*(\rho_2) &= z^{-4}(x_1 + \bar{x}_1)(x_2 + \bar{x}_2) \\ &= z^{-4}x_1\bar{x}_1x_2\bar{x}_2 \\ &= i_U^*(z^{-4}y\bar{y}). \end{aligned}$$

Since  $i_U^*$  is injective, we refer to the diagram and observe that the complexification of  $q_2^*(\rho_2)$  is equal to  $z^{-4}y\bar{y}$ , proving the result.  $\square$

We define the following elements which, along with  $\varrho$ , generate the  $KO_*$ -algebra  $KO^*(BU(2))$ .

**Definition 7.2.4.** Define  $u_i, \tilde{u}_i \in KO^{-2i}(BU(2))$  by  $u_i := r(z^i x)$  and  $\tilde{u}_i := r(z^i y)$ .

We will use the same notation when discussing the restrictions of  $u_i, \tilde{u}_i$  and  $\varrho$  to  $KO^*(G_2(\mathbb{C}^n))$ . It should be clear from the context which is meant.

## 7.2.2 Basis Elements

We restate Hoggar's results concerning the graded abelian group  $KO^*(G_2(\mathbb{C}^n)_+)$ , then find a basis for  $KO^i(G_2(\mathbb{C}^{2n})_+)$  in terms of the elements  $u_i, \tilde{u}_i$  and  $\varrho$ .

The following result is due to Hoggar [23]:

**Theorem 7.2.5.** *The abelian groups  $KO^i(G_2(\mathbb{C}^n)_+)$  are as follows:*

$i$	$KO^i(G_2(\mathbb{C}^{2n})_+)$	$KO^i(G_2(\mathbb{C}^{2n+1})_+)$
0	$\mathbb{Z}^{n^2}$	$\mathbb{Z}^{n^2+n}$
-1	$\mathbb{Z}_2^n$	$\mathbb{Z}_2^n$
-2	$\mathbb{Z}^{n^2-n} \oplus \mathbb{Z}_2^n$	$\mathbb{Z}^{n^2} \oplus \mathbb{Z}_2^n$
-3	0	0
-4	$\mathbb{Z}^{n^2}$	$\mathbb{Z}^{n^2+n}$
-5	0	0
-6	$\mathbb{Z}^{n^2-n}$	$\mathbb{Z}^{n^2}$
-7	0	0

**Lemma 7.2.6.** *A basis of the abelian group  $KO^0(G_2(\mathbb{C}^{2n})_+)$  is given by the elements  $\{(u_0 + \tilde{u}_0)^j \varrho^k \mid 0 \leq j + k \leq n - 1\}$  and  $\{(u_0 + \tilde{u}_0)^j \varrho^k \tilde{u}_0 \mid 0 \leq j + k \leq n - 2\}$ .*

*Proof.* The proposed basis consists of  $n^2$  elements. From Lemma 7.1.14, we deduce that

$$c(u_0 + \tilde{u}_0) = x^2 - x^3 + 2xy, \quad c(\tilde{u}_0) = 2y - xy, \quad c(\varrho) = y^2 - xy^2$$

modulo terms of degrees 8, 8 and 12 respectively. Then

$$c((u_0 + \tilde{u}_0)^j \varrho^k) = x^{2j} y^{2k} - (j + k)x^{2j+1} y^{2k} + 2jx^{2j-1} y^{2k+1}$$

and

$$c((u_0 + \tilde{u}_0)^j \varrho^k \tilde{u}_0) = 2x^{2j} y^{2k+1} - (2j + 2k + 1)x^{2j+1} y^{2k+1} + 4jx^{2j-1} y^{2k+2}$$

modulo terms of degrees  $4j + 8k + 4$  and  $4j + 8k + 8$  respectively. If  $j + k \leq n - 1$ , then  $x^{2j} y^{2k} \in K^0(G_2(\mathbb{C}^{2n})_+)$  is a basis element, and if  $j + k \leq n - 2$ , then both  $x^{2j} y^{2k+1}$  and  $x^{2j+1} y^{2k+1}$  are basis elements of  $K^0(G_2(\mathbb{C}^{2n})_+)$  by Lemma 7.1.5. Then, since complexification is injective, the  $n^2$  elements satisfy the conditions of Lemma 4.2.3, and are indeed a basis.  $\square$

**Corollary 7.2.7.** *The elements  $\{\varrho^k \mid 1 \leq k \leq n - 1\} \in KO^0(G_2(\mathbb{C}^{2n}))$  are not in the image of the realification homomorphism.*

*Proof.* We examine a section of Bott's sequence for  $X = G_2(\mathbb{C}^{2n})$ :

$$K^0(G_2(\mathbb{C}^{2n})) \xrightarrow{r} KO^0(G_2(\mathbb{C}^{2n})) \xrightarrow{e} KO^{-1}(G_2(\mathbb{C}^{2n})) \rightarrow 0.$$

Recall that the image of  $r$  is isomorphic to the kernel of  $e$ . Since  $\text{Im } e \cong \mathbb{Z}_2^{n-1}$  is non-trivial, the generators of  $n-1$  copies of  $\mathbb{Z}$  in  $KO^0(G_2(\mathbb{C}^{2n}))$  are not in the image of  $r$ . The corollary follows by elimination. By definition,  $u_0$  and  $\tilde{u}_0$  are in the image of realification, and since complexification  $c: KO^0(G_2(\mathbb{C}^{2n})) \rightarrow K^0(G_2(\mathbb{C}^{2n}))$  is injective (2.1), it follows from Proposition 2.0.2 that any product  $u_0^i \tilde{u}_0^j$  is in the image of  $r$ . Also by Proposition 2.0.2,  $2\rho r(a) = r(y\bar{y})r(a) = 2r(ay\bar{y})$ . Since  $KO^0(G_2(\mathbb{C}^{2n}))$  is free of 2-torsion, we may deduce that  $\rho r(a)$  is in the image of  $r$  for any  $a \in K^0(G_2(\mathbb{C}^{2n}))$ . It follows that  $\rho^i(u_0 + \tilde{u}_0)^j \tilde{u}_0^k$  is in the image of  $r$  if  $j+k$  is non-zero. This leaves the  $n-1$  basis elements  $\rho, \dots, \rho^{n-1}$ .  $\square$

**Lemma 7.2.8.** *In  $KO^0(G_2(\mathbb{C}^{2n}))$ , any expression of the form  $(u_0 + \tilde{u}_0)^s u_0^t$  may be rewritten as a linear combination of elements  $u_0^j \rho^k \tilde{u}_0^l$  such that  $j+k+l \leq s+t$  and  $l \leq 1$ .*

*Proof.* When  $n \geq 3$  this relies on repeated use of the relation  $\tilde{u}_0^2 = \rho(u_0 + \tilde{u}_0 + 2)$ .

We prove this by complexifying the left hand side to get

$$\begin{aligned} (y + \bar{y})^2 &= y\bar{y}(x + \bar{x} + y + \bar{y} + 2) && \text{(by Lemma 7.1.9)} \\ &= c(\rho)c(u_0 + \tilde{u}_0 + 2). \end{aligned}$$

Note that we carry out the computation in  $K^0(BU(2))$  and pull back to  $K^0(G_2(\mathbb{C}^{2n}))$  along the homomorphism induced by inclusion. When  $n = 2$  the relation restricts to  $\tilde{u}^2 = 2\rho$ , which is sufficient to prove the lemma in this case. We do not need to consider the trivial case where  $n = 1$  and  $G_2(\mathbb{C}^{2n})$  is just a point.  $\square$

**Theorem 7.2.9.** *Bases for the abelian group  $KO^i(G_2(\mathbb{C}^{2n})_+)$  are as follows.*

$i$	<i>Basis elements</i>
0	$\{u_0^j \varrho^k \mid 0 \leq j+k \leq n-1\}$ and $\{u_0^j \varrho^k \tilde{u}_0 \mid 0 \leq j+k \leq n-2\}$ .
-1	$\{e \varrho^k \mid 0 \leq k \leq n-1\}$ .
-2	$\{u_1 u_0^j \varrho^k \mid 0 \leq j+k \leq n-2\}$ , $\{u_0^j \varrho^k \tilde{u}_1 \mid 0 \leq j+k \leq n-2\}$ and $\{e^2 \varrho^k \mid 0 \leq k \leq n-1\}$ .
-3	0
-4	$\{u_2 u_0^j \varrho^k \mid 0 \leq j+k \leq n-2\}$ , $\{u_0^j \varrho^k \tilde{u}_2 \mid 0 \leq j+k \leq n-2\}$ and $\{\alpha \varrho^k \mid 0 \leq k \leq n-1\}$ .
-5	0
-6	$\{u_3 u_0^j \varrho^k \mid 0 \leq j+k \leq n-2\}$ and $\{u_0^j \varrho^k \tilde{u}_3 \mid 0 \leq j+k \leq n-2\}$ .
-7	0

*Proof.* The bases for  $KO^{-1}(G_2(\mathbb{C}^{2n})_+)$  and  $\mathbb{Z}_2^n \subseteq KO^{-2}(G_2(\mathbb{C}^{2n})_+)$  follow from Corollary 7.2.7.

We found a basis for  $KO^0(G_2(\mathbb{C}^{2n})_+)$  in Lemma 7.2.6. We observe that the list of elements stated above are a spanning set by using Lemma 7.2.8, so we just check that we have the correct number of elements to deduce that they are indeed a basis.

For the remaining groups, we show that the summand  $\mathbb{Z}^{n^2-n}$  of  $KO^{-2}(G_2(\mathbb{C}^{2n}))$  has a basis consisting of the union of the sets  $\{u_1(u_0 + \tilde{u}_0)^j \varrho^k \mid 0 \leq j+k \leq n-2\}$  and  $\{(u_0 + \tilde{u}_0)^j \varrho^k \tilde{u}_1 \mid 0 \leq j+k \leq n-2\}$ , the union of sets  $\{\alpha \varrho^k \mid 0 \leq k \leq n-1\}$ ,  $\{u_2(u_0 + \tilde{u}_0)^j \varrho^k \mid 0 \leq j+k \leq n-2\}$  and  $\{(u_0 + \tilde{u}_0)^j \varrho^k \tilde{u}_2 \mid 0 \leq j+k \leq n-2\}$  is a basis for  $KO^{-4}(G_2(\mathbb{C}^{2n})_+)$ , and that the sets  $\{u_3(u_0 + \tilde{u}_0)^j \varrho^k \mid 0 \leq j+k \leq n-2\}$  and  $\{(u_0 + \tilde{u}_0)^j \varrho^k \tilde{u}_3 \mid 0 \leq j+k \leq n-2\}$  are a basis for  $KO^{-6}(G_2(\mathbb{C}^{2n})_+)$ . The result for  $KO^{-4}(G_2(\mathbb{C}^{2n})_+)$  then follows using Lemma 7.2.8 as above. For the remaining two cases, we additionally require the relation  $\tilde{u}_{2i+1} \tilde{u}_0 = (\tilde{u}_{2i+1} + u_{2i+1}) \varrho$ , proved by complexification and reference to Lemma 7.1.13.

For  $KO^{-4}(G_2(\mathbb{C}^{2n})_+)$  the argument is almost identical to that of Lemma 7.2.6 (after multiplying the complexification of any element by  $z^{-2}$ ). The only difference is that whereas  $c(\varrho)$  is not divisible by 2,  $c(\alpha \varrho)$  is. We just need to check that  $\alpha \varrho^k$  is not divisible by two.



Suppose 2 divides  $\alpha \varrho^k$ . Then  $c(\frac{1}{2}\alpha\varrho) = \frac{1}{2}c(\alpha)c(\varrho) = z^2y\bar{y}$ . Since realification  $r: K^{-4}(G_2(\mathbb{C}^{2n})_+) \rightarrow KO^{-4}(G_2(\mathbb{C}^{2n})_+)$  is surjective, there exists some  $b \in K^{-4}(G_2(\mathbb{C}^{2n})_+)$  such that  $cr(b) = z^2y\bar{y}$ . But then  $cr(z^{-2}b) = z^{-2}(b + \bar{b}) = y\bar{y} = c(\varrho)$ . This implies that  $\varrho$  and  $r(z^{-2}b)$  are equal, contradicting Corollary 7.2.7.

We use similar arguments for the remaining bases, for  $KO^{-6}(G_2(\mathbb{C}^{2n})_+)$  and  $\mathbb{Z}^{n^2-n} \subseteq KO^{-2}(G_2(\mathbb{C}^{2n})_+)$ . By restricting to this summand of  $KO^{-2}(G_2(\mathbb{C}^{2n})_+)$  we ensure that  $c$  is injective. Lemma 7.1.14 implies that

$$c(u_{2i+1}) = z^{2i+1}(2x - x^2 + 2y) \quad \text{and} \quad c(\tilde{u}_{2i+1}) = z^{2i+1}(xy - x^2y + y^2)$$

modulo terms of degrees 6 and 10 respectively. Then

$$c((u_0 + \tilde{u}_0)^j \varrho^k \tilde{u}_{2i+1}) = z^{2i+1} x^{2j+1} y^{2k+1}$$

and

$$c((u_0 + \tilde{u}_0)^j \varrho^k u_{2i+1}) = z^{2i+1} (2x^{2j+1}y^{2k} - (2j + 2k + 1)x^{2j+2}y^{2k} + (4j + 2)x^{2j}y^{2k+1})$$

modulo terms of degrees  $4j + 8k + 4$  and  $4j + 8k + 8$  respectively. Since our proposed basis has  $j + k \leq n - 2$ , Lemma 7.1.5 implies that  $x^{2j+1}y^{2k}$ ,  $x^{2j+1}y^{2k+1}$  and  $x^{2j+2}y^{2k}$  are basis elements in  $K^0(G_2(\mathbb{C}^{2n})_+)$ . Again, injectivity of  $c$  implies that the  $n^2 - n$  elements satisfy the conditions of Lemma 4.2.3, and are a basis.  $\square$

### 7.2.3 Multiplicative Relations

We describe the interaction of  $u_i$  and  $\tilde{u}_i \in KO^{-2i}(G_2(\mathbb{C}^{2n}))$  with the coefficient ring  $KO_*$ , then the products of these elements. This furnishes us with a partial description of  $KO^*(G_2(\mathbb{C}^{2n})_+)$  as a  $KO_*$ -algebra.

Complexification is not a monomorphism on  $KO^{-i}(G_2(\mathbb{C}^{2n})_+)$  when  $i \equiv 1, 2 \pmod{8}$ , as it annihilates the elements  $e^i \varrho^k$  ( $0 \leq k \leq n - 1$ ), which generate the summand  $\mathbb{Z}_2^n \subset KO^{-i}(G_2(\mathbb{C}^{2n})_+)$  (2.1). Consequently, several of our relations are proved modulo these elements. Let  $(e) \subset KO^*(G_2(\mathbb{C}^{2n})_+)$  be the ideal generated by  $e$ . Since both  $u_i$  and  $\tilde{u}_i$  are in the image of realification, both  $eu_i$  and  $e\tilde{u}_i$  equal zero, and  $(e) \in KO^*(G_2(\mathbb{C}^{2n})_+)$  consists of the elements  $0 \in KO^0(G_2(\mathbb{C}^{2n})_+)$ ,  $\beta^j e \varrho^k \in$

$KO^{-1-8j}(G_2(\mathbb{C}^{2n})_+)$ , and  $\beta^j e^2 \varrho^k \in KO^{-2-8j}(G_2(\mathbb{C}^{2n})_+)$  for  $0 \leq k \leq n-1$ . By stating a result modulo  $(e)$  we mean that the result holds in the quotient  $KO^*(G_2(\mathbb{C}^{2n})_+)/ (e)$ .

The following sequences will be useful throughout the rest of the chapter.

**Proposition 7.2.10.** *Let  $\eta_2$  be the canonical 2-plane bundle over  $G_2(\mathbb{C}^{2n-1})$ . Then there exist split short exact sequences of  $KO_*$ -modules (mod  $(e)$ ).*

$$0 \rightarrow KO^*(T\eta_2) \rightarrow KO^*(G_2(\mathbb{C}^{2n})) \rightarrow KO^*(\mathbb{C}P^{2n-2}) \rightarrow 0$$

and

$$0 \rightarrow KO^*(MU(2)) \rightarrow KO^*(BU(2)) \rightarrow KO^*(\mathbb{C}P^\infty) \rightarrow 0$$

*Proof.* We examine the  $KO$ -theory long exact sequences obtained from Corollary 7.0.3 and Lemma 7.0.5. The arguments involved in deriving the short exact sequences are similar, so we only discuss the latter in detail.

Since  $KO^{2i+1}(\mathbb{C}P^\infty)$  is zero for all  $i \in \mathbb{Z}$ , we need to show that the homomorphism  $\delta: KO^{2i}(\mathbb{C}P^\infty) \rightarrow KO^{2i+1}(BU(2))$  is the zero homomorphism to obtain the required short exact sequence. Let  $\delta'$  be the corresponding homomorphism in the  $K$ -theory long exact sequence, and consider the commutative diagram

$$\begin{array}{ccc} KO^{2i}(\mathbb{C}P^\infty) & \xrightarrow{\delta} & KO^{2i+1}(MU(2)) \\ \uparrow r & & \uparrow r \\ K^*(\mathbb{C}P^\infty) & \xrightarrow{\delta'} & K^{2i+1}(MU(2)) \end{array}$$

Since  $K^{2i+1}(MU(2))$  is zero and  $r: K^*(\mathbb{C}P^\infty) \rightarrow KO^{2i}(\mathbb{C}P^\infty)$  is surjective, commutativity implies that  $\delta$  must be the zero homomorphism. The sequence splits because  $KO^*(\mathbb{C}P^\infty)$  is torsion free, and since we are working modulo  $(e)$ , we complexify to see that the splitting respects the action of  $\alpha$  and  $\beta \in K_*$ .  $\square$

**Proposition 7.2.11.** *The following relations hold in  $KO^*(G_2(\mathbb{C}^{2n})_+)$ .*

$$eu_i = e\tilde{u}_i = 0, \quad \alpha u_i = 2u_{i+2} \pmod{(e)}, \quad \alpha \tilde{u}_i = 2\tilde{u}_{i+2} \pmod{(e)},$$

$$\beta u_i = u_{i+4}, \quad \beta \tilde{u}_i = \tilde{u}_{i+4}.$$

*Proof.* Both  $eu_i$  and  $e\tilde{u}_i$  are zero by (2.1). The remaining relations are proved using complexification.  $\square$

**Proposition 7.2.12.** *The following relations hold in  $KO^{-2(i+j)}(G_2(\mathbb{C}^{2n}))/\langle e \rangle$ .*

$$u_i u_j = u_{i+2} u_{j-2}, \quad u_i \tilde{u}_j = u_{i+2} \tilde{u}_{j-2}, \quad \tilde{u}_i \tilde{u}_j = \tilde{u}_{i+2} \tilde{u}_{j-2}.$$

*Proof.* The relations are proved using complexification. □

We now consider the image of  $r: K^*(G_2(\mathbb{C}^{2n})) \rightarrow KO^*(G_2(\mathbb{C}^{2n}))$ . The following two lemmas allow us to present an arbitrary element  $r(z^i x^j y^k) \in KO^{-2i}(G_2(\mathbb{C}^{2n}))$  in terms of our basis.

**Lemma 7.2.13.** *The following relations hold in  $KO^{-4i}(G_2(\mathbb{C}^{2n}))$ .*

1.  $r(z^{2i} x^j y^k) = u_0 r(z^{2i} x^{j-1} y^k) + (u_0 + \tilde{u}_0 - \varrho) r(z^{2i} x^{j-2} y^k), \quad \text{if } j \geq 2,$
2.  $r(z^{2i} x^j y^k) = \tilde{u}_0 r(z^{2i} x^j y^{k-1}) - \varrho r(z^{2i} x^j y^{k-2}), \quad \text{if } k \geq 2,$
3.  $r(z^{2i} x y) = \begin{cases} u_{2i} \tilde{u}_0 + 2\beta^{\frac{i}{2}} \varrho, & i \equiv 0 \pmod{2}, \\ u_{2i} \tilde{u}_0 + \alpha \beta^{\frac{i-1}{2}} \varrho, & i \equiv 1 \pmod{2}. \end{cases}$
4.  $r(z^{2i} x^2) = \begin{cases} u_{2i} u_0 + 2u_{2i} + 2\tilde{u}_{2i} - 2\beta^{\frac{i}{2}} \varrho, & i \equiv 0 \pmod{2}, \\ u_{2i} u_0 + 2u_{2i} + 2\tilde{u}_{2i} - \alpha \beta^{\frac{i-1}{2}} \varrho, & i \equiv 1 \pmod{2}. \end{cases}$
5.  $r(z^{2i} y^2) = \begin{cases} \tilde{u}_{2i} \tilde{u}_0 - 2\beta^{\frac{i}{2}} \varrho, & i \equiv 0 \pmod{2}, \\ \tilde{u}_{2i} \tilde{u}_0 - \alpha \beta^{\frac{i-1}{2}} \varrho, & i \equiv 1 \pmod{2}. \end{cases}$

*Proof.* We complexify the left hand sides:

1. 
$$\begin{aligned} z^{2i}(x^j y^k + \overline{x^j y^k}) &= z^{2i} \{ (x + \bar{x})(x^{j-1} y^k + \overline{x^{j-1} y^k}) - x\bar{x}(x^{j-2} y^k + \overline{x^{j-2} y^k}) \} \\ &= (x + \bar{x}) z^{2i} (x^{j-1} y^k + \overline{x^{j-1} y^k}) \\ &\quad + (x + \bar{x} + y + \bar{y} - y\bar{y}) z^{2i} (x^{j-2} y^k + \overline{x^{j-2} y^k}) \\ &\hspace{15em} \text{(using Lemma 7.1.10)} \\ &= c(u_0 r(z^{2i} x^{j-1} y^k) + (u_0 + \tilde{u}_0 - \varrho) r(z^{2i} x^{j-2} y^k)). \end{aligned}$$
2. 
$$\begin{aligned} z^{2i}(x^j y^k + \overline{x^j y^k}) &= (y + \bar{y}) z^{2i} (x^j y^{k-1} + \overline{x^j y^{k-1}}) - y\bar{y} z^{2i} (x^j y^{k-2} + \overline{x^j y^{k-2}}) \\ &= c(\tilde{u}_0 r(z^{2i} x^j y^{k-1}) - \varrho r(z^{2i} x^j y^{k-2})). \end{aligned}$$

$$\begin{aligned}
3. \quad z^{2i}(xy + \overline{xy}) &= z^{2i}\{(x + \overline{x})(y + \overline{y}) - (x\overline{y} + \overline{x}y)\} \\
&= z^{2i}(x + \overline{x})(y + \overline{y}) + 2z^{2i}y\overline{y} \quad (\text{using Lemma 7.1.11}) \\
&= \begin{cases} c(u_{2i}\tilde{u}_0 + 2y^{\frac{i}{2}}\varrho), & i \equiv 0 \pmod{2}, \\ c(u_{2i}\tilde{u}_0 + xy^{\frac{i-1}{2}}\varrho), & i \equiv 1 \pmod{2}. \end{cases}
\end{aligned}$$

Complexification  $c: KO^{-4i}(G_2(\mathbb{C}^{2n})) \rightarrow K^{-4i}(G_2(\mathbb{C}^{2n}))$  is injective for any  $i \in \mathbb{Z}$  (2.1), so relations 1-3 hold. Parts 4. and 5. follow from 1. and 2. respectively, using the relation

$$r(z^{2i}) = \begin{cases} 2\beta^{\frac{i}{2}}, & i \equiv 0 \pmod{2}, \\ \alpha\beta^{\frac{i-1}{2}}, & i \equiv 1 \pmod{2} \end{cases}$$

from Lemma 2.0.3. □

**Lemma 7.2.14.** *The following relations hold in  $KO^{-4i-2}(G_2(\mathbb{C}^{2n}))/\langle e \rangle$  when  $i$  is even, and in  $KO^{-4i-2}(G_2(\mathbb{C}^{2n}))$  when  $i$  is odd.*

1.  $r(z^{2i+1}x^jy^k) = u_0r(z^{2i+1}x^{j-1}y^k) + (u_0 + \tilde{u}_0 - \varrho)r(z^{2i+1}x^{j-2}y^k)$ , if  $j \geq 2$ ,
2.  $r(z^{2i+1}x^jy^k) = \tilde{u}_0r(z^{2i+1}x^jy^{k-1}) - \varrho r(z^{2i+1}x^jy^{k-2})$ , if  $k \geq 2$ ,
3.  $r(z^{2i+1}xy) = u_{2i+1}\tilde{u}_0 - 2\tilde{u}_{2i+1}$   
 $= \tilde{u}_{2i+1}u_0 + 2\tilde{u}_{2i+1}$ ,
4.  $r(z^{2i+1}x^2) = u_{2i+1}u_0$ ,
5.  $r(z^{2i+1}y^2) = \tilde{u}_{2i+1}\tilde{u}_0$ .

*Proof.* We complexify the left hand sides:

1.  $z^{2i+1}(x^jy^k - \overline{x^jy^k}) = (x + \overline{x})z^{2i+1}(x^{j-1}y^k - \overline{x^{j-1}y^k})$   
 $- x\overline{x}z^{2i+1}(x^{j-2}y^k - \overline{x^{j-2}y^k})$   
 $= (x + \overline{x})z^{2i+1}(x^{j-1}y^k - \overline{x^{j-1}y^k})$   
 $+ (x + \overline{x} + y + \overline{y} - y\overline{y})z^{2i+1}(x^{j-2}y^k - \overline{x^{j-2}y^k})$   
(using Lemma 7.1.10)  
 $= c(u_0r(z^{2i+1}x^{j-1}y^k) + (u_0 + \tilde{u}_0 - \varrho)r(z^{2i+1}x^{j-2}y^k)).$
2.  $z^{2i+1}(x^jy^k - \overline{x^jy^k}) = (y + \overline{y})z^{2i+1}(x^jy^{k-1} - \overline{x^jy^{k-1}})$   
 $- y\overline{y}z^{2i+1}(x^jy^{k-2} - \overline{x^jy^{k-2}})$   
 $= c(\tilde{u}_0r(z^{2i+1}x^jy^{k-1}) - \varrho r(z^{2i+1}x^jy^{k-2})).$

$$\begin{aligned}
3. \quad z^{2i+1}(xy - \bar{x}\bar{y}) &= z^{2i+1}(x - \bar{x})(y + \bar{y}) + z^{2i+1}(\bar{x}y - x\bar{y}) \\
&= z^{2i+1}(x - \bar{x})(y + \bar{y}) - 2z^{2i+1}(y - \bar{y}) \quad (\text{using Lemma 7.1.12}) \\
&= c(u_{2i+1}\tilde{u}_0 - 2\tilde{u}_{2i+1}),
\end{aligned}$$

or

$$\begin{aligned}
z^{2i+1}(xy - \bar{x}\bar{y}) &= (x + \bar{x})z^{2i+1}(y - \bar{y}) - z^{2i+1}(\bar{x}y - x\bar{y}) \\
&= (x + \bar{x})z^{2i+1}(y - \bar{y}) + 2z^{2i+1}(y - \bar{y}) \\
&= c(u_0\tilde{u}_{2i+1} + 2\tilde{u}_{2i+1}).
\end{aligned}$$

4. refer to Lemma 3.2.3 (restricting to the summand  $KO^*(\mathbb{C}P^{2n})$  using Proposition 7.2.10)

$$5. \quad z^{2i+1}(y^2 - \bar{y}^2) = (y + \bar{y})z^{2i+1}(y - \bar{y}) = c(\tilde{u}_0\tilde{u}_{2i+1}).$$

Complexification  $c: KO^{-4i-2}(G_2(\mathbb{C}^{2n})) \rightarrow K^{-4i-2}(G_2(\mathbb{C}^{2n}))$  is injective if and only if  $i \equiv 1 \pmod{2}$ . When  $i \equiv 0 \pmod{2}$ , complexification will kill any torsion.  $\square$

We move on to multiplicative relations between  $u_i$ ,  $\tilde{u}_i$  and  $\varrho$ :

**Proposition 7.2.15.** *The following relations hold in  $KO^*(G_2(\mathbb{C}^{2n}))/\langle e \rangle$ .*

1.  $u_{2i+1}u_{2j-1} = u_{2(i+j)}(u_0 + 4),$
2.  $u_{2i+1}\tilde{u}_{2j-1} = \begin{cases} u_{2(i+j)}\tilde{u}_0 + 4\beta^{\frac{i+j}{2}}\varrho, & i \equiv j \pmod{2}, \\ u_{2(i+j)}\tilde{u}_0 + 2\alpha\beta^{\frac{i+j-1}{2}}\varrho, & \text{else,} \end{cases}$
3.  $u_{2i+1}\tilde{u}_{2j} = u_0\tilde{u}_{2i+2j+1} + 4\tilde{u}_{2i+2j+1},$
4.  $\tilde{u}_{2i+1}\tilde{u}_{2j} = \varrho(u_{2(i+j)} + \tilde{u}_{2(i+j)}) + \begin{cases} 2\varrho\beta^{\frac{i+j}{2}}, & i \equiv j \pmod{2}, \\ \varrho\alpha\beta^{\frac{i+j-1}{2}}, & \text{otherwise,} \end{cases}$
5.  $\tilde{u}_{2i+1}\tilde{u}_j = \varrho(u_{2i+j+1} + \tilde{u}_{2i+j+1}).$

*Proof.* The relation  $r(z^{2i+1}a)r(z^{2j-1}b) = 2r(z^{2(i+j)}ab) - r(a)r(z^{2(i+j)}b)$  follows from the Proposition 2.0.2, and is used in the proofs of 2. and 5.

1. We restrict to the summand  $KO^*(\mathbb{C}P^{2n})$  (see Proposition 7.2.10) and refer to Proposition 3.2.7.

2. Referring to 2.0.2, it is clear that  $u_{2i+1}\tilde{u}_{2j-1}$  and  $2r(z^{2(i+j)}xy) - u_{2(i+j)}\tilde{u}_0$  are equal. We refer to Lemma 7.2.13 (3.) to obtain the required solution.

3. This follows from 3. in Lemma 7.2.14.

$$\begin{aligned}
 4. \quad z^{2i}(y + \bar{y})z^{2j}(y + \bar{y}) &= y\bar{y}z^{2(i+j)}(x + \bar{x} + y + \bar{y} + 2) \quad (\text{by Lemma 7.1.9}) \\
 &= \begin{cases} c(\varrho(u_{2(i+j)} + \tilde{u}_{2(i+j)} + 2\beta^{\frac{i+j}{2}})), & i \equiv j \pmod{2}, \\ c(\varrho(u_{2(i+j)} + \tilde{u}_{2(i+j)} + \alpha\beta^{\frac{i+j-1}{2}})), & \text{otherwise.} \end{cases}
 \end{aligned}$$

5. If  $j$  is odd, we have  $r(z^{2i+1}y)r(z^jy) = 2r(z^{2i+j+1}y^2) - \tilde{u}_{2i+j+1}\tilde{u}_0$ . The result follows by applying the formula for  $r(z^{2i}y^2)$  from Lemma 7.2.13 and the formula for  $\tilde{u}_{2i}\tilde{u}_0$  proved above.

If  $j$  is even, complexification yields  $z^{2i+j+1}(y^2 - \bar{y}^2)$ . Using Lemma 7.1.13, this is equal to  $c((u_{2i+j+1} + \tilde{u}_{2i+j+1})\varrho)$  as required.

We have proved 1., 2. and 4. since complexification is injective. Both 3. and 5. are true up to 2-torsion.  $\square$

## 7.2.4 $KO_*$ -algebras

It is clear from Proposition 7.2.9 that  $KO^*(G_2(\mathbb{C}^{2n}))$  is generated as a polynomial algebra over  $KO_*$  by the elements  $u_i, \tilde{u}_i$  ( $i \in \mathbb{Z}$ ) and  $\varrho$ . The relations of Propositions 7.2.11, 7.2.12 and 7.2.15 allow us to rewrite any polynomial  $P \in KO_*[u_i, \tilde{u}_i, \varrho]$  in terms of the basis for  $KO^*(G_2(\mathbb{C}^{2n})_+)$  given in Theorem 7.2.9 for sufficiently large  $n$ . However to identify  $KO^*(G_2(\mathbb{C}^{2n})_+)$  as a quotient algebra of  $KO_*[u_i, \tilde{u}_i, \varrho]$ , we require further relations describing the elements  $u_i u_0^j \varrho^k$  and  $\tilde{u}_i u_0^j \varrho^k$  where  $j+k \geq n-1$ , and  $\varrho^k$  where  $k \geq n$  as a (possibly zero) linear combination of the basis elements of  $KO^i(G_2(\mathbb{C}^{2n})_+)$  and  $KO^0(G_2(\mathbb{C}^{2n})_+)$  respectively. It is important to note that each such relation depends on  $n$ , since for sufficiently large  $n$ , each of  $u_i u_0^j \varrho^k$ ,  $\tilde{u}_i u_0^j \varrho^k$  and  $\varrho^k$  will be a basis element in the graded group  $KO^*(G_2(\mathbb{C}^{2n}))$ .

Describing  $BU(2)$  as the union of the sequence of subcomplexes

$$G_2(\mathbb{C}^n) \subset G_2(\mathbb{C}^{n+1}) \subset G_2(\mathbb{C}^{n+2}) \dots$$

we apply Theorem 2.0.17 to describe  $KO^i(BU(2))$ .

**Theorem 7.2.16.** *The  $KO_*$ -algebra  $KO^*(BU(2)_+)$  is generated by the elements  $u_i, \tilde{u}_i$  ( $i \in \mathbb{Z}$ ) and  $\varrho$ , and  $KO^i(BU(2)_+)$  is isomorphic to the following direct sums of power series rings and modules.*

$i$	$KO^i(BU(2)_+)$
0	$\mathbb{Z}[[u_0, \varrho]] \oplus \tilde{u}_0 \mathbb{Z}[[u_0, \varrho]].$
-1	$e \mathbb{Z}[[\varrho]].$
-2	$u_1 \mathbb{Z}[[u_0, \varrho]] \oplus \tilde{u}_1 \mathbb{Z}[[u_0, \varrho]] \oplus e^2 \mathbb{Z}[[\varrho]].$
-3	0
-4	$u_2 \mathbb{Z}[[u_0, \varrho]] \oplus \tilde{u}_2 \mathbb{Z}[[u_0, \varrho]] \oplus \alpha \mathbb{Z}[[\varrho]].$
-5	0
-6	$u_3 \mathbb{Z}[[u_0, \varrho]] \oplus \tilde{u}_3 \mathbb{Z}[[u_0, \varrho]].$
-7	0

The action of  $KO_*$  on  $KO^*(BU(2))/(e)$  is given by the relations

$$eu_i = e\tilde{u}_i = 0, \quad \alpha u_i = 2u_{i+2}, \quad \alpha \tilde{u}_i = 2\tilde{u}_{i+2}, \quad \beta u_i = u_{i+4} \quad \text{and} \quad \beta \tilde{u}_i = \tilde{u}_{i+4}$$

for all  $i \in \mathbb{Z}$ .

The algebra relations (modulo  $(e)$ ) are given by

$$\begin{aligned} u_i u_j &= u_{i+2} u_{j-2}, & u_i \tilde{u}_j &= u_{i+2} \tilde{u}_{j-2}, & \tilde{u}_i \tilde{u}_j &= \tilde{u}_{i+2} \tilde{u}_{j-2}, \\ u_{2i+1} u_{2j-1} &= u_{2(i+j)}(u_0 + 4), & u_1 \tilde{u}_{4j-1} &= u_0 \tilde{u}_{4i} - 4\beta^i, & u_1 \tilde{u}_{4j+1} &= u_0 \tilde{u}_{4i+2} - 2\alpha\beta^i, \\ u_{2i+1} \tilde{u}_{2j} &= u_0 \tilde{u}_{2i+2j+1} + 4\tilde{u}_{2i+2j+1}, & \tilde{u}_{2i+1} \tilde{u}_j &= \varrho(u_{2i+j+1} + \tilde{u}_{2i+j+1}), \\ \tilde{u}_0 \tilde{u}_{4i} &= \varrho(u_{4i} + \tilde{u}_{4i} + 2\beta^i) \quad \text{and} \quad \tilde{u}_0 \tilde{u}_{4i+2} &= \varrho(u_{4i+2} + \tilde{u}_{4i+2} + \alpha\beta^i) \end{aligned}$$

for all  $i, j \in \mathbb{Z}$ .

*Proof.* Since the inclusion of  $G_2(\mathbb{C}^{2n})$  in  $G_2(\mathbb{C}^{2n+2})$  induces a surjection in real  $K$ -theory for all  $n$ , Theorem 2.0.17 implies that  $KO^i(BU(2)) \cong \varprojlim KO^i(G_2(\mathbb{C}^{2n}))$  is isomorphic to  $\prod \mathbb{Z}$  when  $i \equiv 0, -4, -6 \pmod{8}$ , to  $\prod \mathbb{Z}_2$  when  $i \equiv -1 \pmod{8}$ , to  $\prod \mathbb{Z} \oplus \prod \mathbb{Z}_2$  when  $i \equiv -2 \pmod{8}$  and is isomorphic to the trivial group 0 when  $i \equiv -3, -5, -7 \pmod{8}$ . We write the elements of  $KO^{-2i}(BU(2)_+)$  in the form  $\sum_{m,n=0}^{\infty} a_{m,n} u_i u_0^m \varrho^n + \sum_{m,n=0}^{\infty} a_{m,n} \tilde{u}_i u_0^m \varrho^n + KO_{\frac{i}{2}} \sum_{n=0}^{\infty} c_n \varrho^n$ , and the elements of  $KO^{-8i-1}(BU(2)_+)$  in the form  $e\beta^i \sum_{n=0}^{\infty} c_n \varrho^n$ , where  $a_{m,n}, b_{m,n}$  and  $c_n$  are integers.

We equip  $KO^*(BU(2)_+)$  with multiplication (2.12), and it follows that each  $KO^i(BU(2))$  is isomorphic to the direct sum of rings,  $\mathbb{Z}[[u_0, \varrho]]$ -modules and  $\mathbb{Z}[[\varrho]]$ -modules stated.

Clearly  $KO^*(BU(2)_+)$  is generated over  $KO_*$  as an algebra by the elements  $u_i, \tilde{u}_i$  ( $i \in \mathbb{Z}$ ) and  $\varrho$ . We retain multiplicative relations from  $KO^*(G_2(\mathbb{C}^{2n})_+)$ . Recall that, by the definition of inverse limits, an element in  $KO^i(BU(2))$  equals the zero element if and only if it is annihilated by restriction to  $KO^i(G_2(\mathbb{C}^{2n}))$  for every  $n \geq 0$ . Thus we retain only the relations which do not depend on  $n$ , i.e. those proved in Propositions 7.2.11, 7.2.12 and 7.2.15.  $\square$

## 7.2.5 Characteristic Classes

We extend our definitions from Section 3.2.6. We call  $\varrho, \tilde{u}_i \in KO^{-2i}(BU(2))$  the  *$i$ th universal second order Fujii classes*, and denote the pullbacks along the map classifying an arbitrary 2-plane bundle  $\theta: X \rightarrow BU(2)$  as  $\varrho(\theta)$  and  $\tilde{u}_i(\theta) \in KO^{-2i}(X)$ , for any integer  $i$ . We call  $\tilde{u}_i(\theta)$  the  *$i$ th second order Fujii class* of  $\theta$ . The Fujii classes  $u_i(\theta), \tilde{u}_i(\theta), \varrho(\theta)$  of  $\theta$  generate a  $KO_*$ -subalgebra of  $KO^*(X)$ , which we call the *Fujii subalgebra* of  $\theta$ , written as  $FS(\theta)$ . Due to the periodicity relation,  $FS(\theta)$  is unaffected when we restrict to  $0 \leq i \leq 3$ . Again, we measure  $FS(\theta)$  by defining the *Fujii quotient* as the quotient of  $KO_*$ -modules

$$FQ(\theta) = KO^*(X)/FS(\theta).$$

**Example 7.2.17.** It is immediate that  $FS(\eta_2)$ , the Fujii subalgebra of  $\eta_2$ , coincides with  $KO^*(BU(2))$ . It follows that  $FQ(\eta_2)$  is isomorphic to the trivial group.

**Example 7.2.18.** Let  $\eta_2$  denote the canonical 2-plane bundle over  $G_2(\mathbb{C}^{2n})$ . Using the basis for  $KO^*(G_2(\mathbb{C}^{2n})_+)$  given in Theorem 7.2.9, we see that the Fujii subalgebra of  $\eta_2$  coincides with  $KO^*(G_2(\mathbb{C}^{2n}))$  and that  $FQ(\eta_2)$  is isomorphic to the trivial group.

## 7.2.6 Thom Algebra

For the remainder of the chapter, we work modulo  $(e)$ .



Consider the second split exact sequence of Proposition 7.2.10. The zero section  $\iota: BU(2) \rightarrow MU(2)$  induces the monomorphism  $\iota^*: KO^*(MU(2)) \rightarrow KO^*(BU(2))$  with cokernel  $KO^*(\mathbb{C}P^\infty)$ . Then the Euler ideal is the subalgebra of  $KO^*(BU(2))$  generated by the elements  $\tilde{u}_i$  ( $i \in \mathbb{Z}$ ) and  $\varrho$ . Each element  $u$  in the subalgebra corresponds to a unique element  $(\iota^*)^{-1}(u) = \langle u \rangle \in KO^*(MU(2))$ , and we have the relations

$$v \cdot \langle u \rangle = \langle vu \rangle \quad \text{and} \quad \langle uu' \rangle = \langle u \rangle \langle u' \rangle \quad (7.8)$$

for every Eulerian  $u, u' \in KO^*(BU(2))$  and every  $v \in KO^*(BU(2)_+)$ .

We read off the structure of the Euler ideal directly from Theorem 7.2.16, and lift the results back to  $KO^*(MU(2))$ . Using the notation of (2.10), we have the following two results.

**Theorem 7.2.19.** *The Thom module  $KO^*(MU(2))$  over  $KO^*(BU(2)_+)$  is generated by the elements  $\langle \tilde{u}_0 \rangle, \langle \tilde{u}_1 \rangle, \langle \tilde{u}_2 \rangle, \langle \tilde{u}_3 \rangle, \langle \varrho \rangle$  with relations*

1.  $e \cdot \langle \tilde{u}_0 \rangle = e \cdot \langle \tilde{u}_1 \rangle = e \cdot \langle \tilde{u}_2 \rangle = e \cdot \langle \tilde{u}_3 \rangle = 0,$
2.  $\alpha \cdot \langle \tilde{u}_0 \rangle = 2\langle \tilde{u}_2 \rangle, \quad \alpha \cdot \langle \tilde{u}_1 \rangle = 2\langle \tilde{u}_3 \rangle, \quad \alpha \cdot \langle \tilde{u}_2 \rangle = 2\beta \cdot \langle \tilde{u}_0 \rangle, \quad \alpha \cdot \langle \tilde{u}_3 \rangle = 2\beta \cdot \langle \tilde{u}_1 \rangle,$
3.  $u_j \cdot \langle \tilde{u}_0 \rangle = u_{j-2} \cdot \langle \tilde{u}_2 \rangle, \quad u_j \cdot \langle \tilde{u}_1 \rangle = u_{j-2} \cdot \langle \tilde{u}_3 \rangle,$   
 $\tilde{u}_j \cdot \langle \tilde{u}_0 \rangle = \tilde{u}_{j-2} \cdot \langle \tilde{u}_2 \rangle, \quad \tilde{u}_j \cdot \langle \tilde{u}_1 \rangle = \tilde{u}_{j-2} \cdot \langle \tilde{u}_3 \rangle,$
4.  $u_{4j-1} \cdot \langle \tilde{u}_1 \rangle = u_{4j} \cdot \langle \tilde{u}_0 \rangle + 4\beta^j \cdot \langle \varrho \rangle,$   
 $u_{4j+1} \cdot \langle \tilde{u}_1 \rangle = u_{4j} \cdot \langle \tilde{u}_2 \rangle + 2\alpha\beta^j \cdot \langle \varrho \rangle,$   
 $u_{4j+1} \cdot \langle \tilde{u}_0 \rangle = (u_{4j} + 2\alpha\beta^j) \cdot \langle \tilde{u}_1 \rangle,$   
 $u_{4j+3} \cdot \langle \tilde{u}_0 \rangle = (u_{4j} + 4\beta^j) \cdot \langle \tilde{u}_3 \rangle,$
5.  $\tilde{u}_j \cdot \langle \tilde{u}_1 \rangle = u_{j+1} \cdot \langle \varrho \rangle + \varrho \cdot \langle \tilde{u}_{j+1} \rangle,$   
 $\tilde{u}_{2j+1} \cdot \langle \tilde{u}_0 \rangle = u_{2j+1} \cdot \langle \varrho \rangle + \varrho \cdot \langle \tilde{u}_{2j+1} \rangle,$   
 $\tilde{u}_{4j} \cdot \langle \tilde{u}_0 \rangle = (u_{4j} + 2\beta^j) \cdot \langle \varrho \rangle + \beta^j \varrho \cdot \langle \tilde{u}_0 \rangle,$   
 $\tilde{u}_{4j+2} \cdot \langle \tilde{u}_0 \rangle = (u_{4j+2} + \alpha\beta^j) \cdot \langle \varrho \rangle + \beta^j \varrho \cdot \langle \tilde{u}_2 \rangle.$

The Thom algebra is generated by the same elements, with the additional relations

$$\langle \tilde{u}_i \rangle \langle \tilde{u}_j \rangle = \tilde{u}_i \cdot \langle \tilde{u}_j \rangle, \text{ and}$$

$$\langle \varrho \rangle \langle \tilde{u}_j \rangle = \varrho \cdot \langle \tilde{u}_j \rangle$$

for all integers  $0 \leq i, j \leq 3$ .

Alternative choices of generators are of course possible, since the periodicity relation  $\beta^k \cdot \langle \tilde{u}_i \rangle = \langle \tilde{u}_{i+4k} \rangle$  holds for all integers  $i$  and  $k$ .

**Theorem 7.2.20.**  *$KO^i(MU(2))$  is isomorphic to the following direct sums of power series rings and modules.*

$i$	$KO^i(MU(2))$
0	$\mathbb{Z}[[u_0, \varrho]] \cdot \langle \varrho \rangle \oplus \mathbb{Z}[[u_0, \varrho]] \cdot \langle \tilde{u}_0 \rangle.$
-1	$e\mathbb{Z}[[\varrho]] \cdot \langle \varrho \rangle.$
-2	$u_1\mathbb{Z}[[u_0, \varrho]] \cdot \langle \varrho \rangle \oplus \mathbb{Z}[[u_0, \varrho]] \cdot \langle \tilde{u}_1 \rangle \oplus e^2\mathbb{Z}[[\varrho]] \cdot \langle \varrho \rangle.$
-3	0
-4	$u_2\mathbb{Z}[[u_0, \varrho]] \cdot \langle \varrho \rangle \oplus \mathbb{Z}[[u_0, \varrho]] \cdot \langle \tilde{u}_2 \rangle \oplus \alpha\mathbb{Z}[[\varrho]] \cdot \langle \varrho \rangle.$
-5	0
-6	$u_3\mathbb{Z}[[u_0, \varrho]] \cdot \langle \varrho \rangle \oplus \mathbb{Z}[[u_0, \varrho]] \cdot \langle \tilde{u}_3 \rangle.$
-7	0

### 7.2.7 Comments on Torsion

It should be noted that the existence of 2-torsion, precisely, the existence of summands isomorphic to  $\mathbb{Z}_2$  in  $KO^*(BU(2))$  and  $KO^*(G_2(\mathbb{C}^n))$  is due to the fact that the element  $\varrho \in KO^0(BU(2))$  and its powers are free over the coefficient ring  $KO_*$ . By contrast,  $KO^*(BU(1))$  is free from torsion, as is  $KO^*(\mathbb{C}P^{2n})$ . Although  $KO^*(\mathbb{C}P^{2n+1})$  contains torsion in the summand  $KO^*(S^{4n+2}) \subseteq KO^*(\mathbb{C}P^{2n+1})$  (Proposition 3.2.1), this is associated with the top cell of  $\mathbb{C}P^{2n+1}$ , and does not survive in an inverse limit argument.

We defined  $\varrho \in KO^0(BU(2))$  in terms of the second Pontryagin class  $\rho_2 \in KO^8(BSp(2))$ . The Pontryagin classes  $\rho_i \in KO^{4i}(BSp(n))$  are free over the coefficient ring  $KO_*$ . The image of  $\rho_2$  in  $KO^*(BU(2))$  is non-zero, and retains the property of having no relations over the coefficients. In contrast,  $q_2^*(\rho_1) = u_{-2} \in KO^4(BU(2))$ .

Since  $u_{-2}$  is in the image of realification, its image under multiplication by  $e$  is the zero element.

We recall Section 7.2.1, and generalise our definition of  $\varrho$ . Recall the inclusion  $BU(n) \xrightarrow{q_n} BSp(n)$ .

**Definition 7.2.21.** We define the elements  $\varrho_i \in KO^0(BU(n))$  as follows:

$$\varrho_i := \beta^i q_n^*(\rho_{2i})$$

where  $\rho_{2i} \in KO^{8i}(BSp(n))$  is the  $2i$ th Pontryagin class.

Note that the restriction of  $\varrho_1 \in KO^0(BU(n))$  to  $KO^*(BU(2))$  coincides with  $\varrho \in KO^0(BU(2))$ .

**Conjecture 7.2.22.** *The subring*

$$T_n \cong \mathbb{Z}[\![\varrho_1, \dots, \varrho_{\lfloor \frac{n}{2} \rfloor}]\!] \subset KO^0(BU(n)_+)$$

*is free over the coefficient ring  $KO_*$ .*

In particular,  $e^i a$  ( $i \in \{0, 1, 2\}$ ) is non-zero for any element  $a \in T_n$ . The author believes that all torsion in  $KO^*(BU(n))$  is described in this way.

This appears to be consistent with Kono and Hara's table [26, Main Theorem] describing the abelian groups  $KO^i(G_m(\mathbb{C}^{m+n}))$ . When  $m$  is even,  $KO^i(G_m(\mathbb{C}^{m+n}))$  contains an increasing amount of torsion as  $n$  increases when  $i = -1, -2$ . When  $m$  is odd, there is torsion dimensions other than  $-1$  and  $-2$ , occurring if and only if  $n$  is odd. This behaviour seems to suggest that the torsion is associated with the top cell, as in  $KO^*(\mathbb{C}P^{2n+1}) \cong KO^*(G_1(\mathbb{C}^{(2n+1)+1}))$ .

# Chapter 8

## Products of Complex Projective Spaces

We examine the product of two complex projective spaces, computing the  $KO_*$ -algebras  $KO^*((\mathbb{C}P^m \times \mathbb{C}P^n)_+)$ . Our calculations highlight a difficulty of real  $K$ -theory computations: the lack of a Künneth Theorem [4, Introduction]. The  $KO_*$ -algebras  $KO^*((\mathbb{C}P^m \times \mathbb{C}P^n)_+)$  and  $KO^*(\mathbb{C}P_+^m) \otimes_{KO_*} KO^*(\mathbb{C}P_+^n)$  are not isomorphic.

After a review of the complex  $K$ -theory, we compute  $KO^*(\mathbb{C}P^m \times \mathbb{C}P^n)$  as a graded abelian group for all  $m, n \in \mathbb{Z}^{\geq 0}$ , before finding bases for the groups  $KO^i(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$ . Further calculations allow us to express  $KO^*((\mathbb{C}P^m \times \mathbb{C}P^n)_+)$  as a  $KO_*$ -algebra for all  $m, n \in \mathbb{Z}^{\geq 0}$ . Using inverse limits we deduce the  $KO_*$ -algebra  $KO^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$ .

The above results have been used by Dubajic [16] to calculate the  $KO$ -theory of Milnor hypersurfaces, of which  $\mathbb{C}P^m \times \mathbb{C}P^n$  is a trivial example. We believe these  $KO_*$ -algebras are not in the literature, although the calculations of  $KO^*((\mathbb{C}P^m \times \mathbb{C}P^n)_+)$  as a  $KO_*$ -algebra (for  $m, n \in \mathbb{Z}^{\geq 0}$ ) have recently been made independently in [32].

The Thom complex of  $\eta \times \eta$  over  $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$  has the homotopy type of the smash product  $\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty$  [30], and the zero section  $\iota: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow T\eta \times \eta$  induces a monomorphism. We use  $\iota^*$  to interpret the preceding results in light of the action (2.4), giving generators and relations for the Thom algebra of  $\eta \times \eta$  over

$\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ . Using the pullbacks of the generators to the bundles  $\eta(m) \times \eta(n)$  over  $\mathbb{C}P^m \rightarrow \mathbb{C}P^n$ , we describe the Thom algebra of  $\eta(m) \times \eta(n)$  for all positive integers  $m$  and  $n$ .

## 8.1 Complex $K$ -theory

We begin by reviewing the complex  $K$ -theory algebras before examining the real  $K$ -theory in detail.

Recalling (7.3) and (7.4), we have the isomorphisms of  $K_*$ -algebras

$$K^*((\mathbb{C}P^m \times \mathbb{C}P^n)_+) \cong K_*[x_1, x_2]/(x_1^{m+1}, x_2^{n+1})$$

and

$$K^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+) \cong K_*[[x_1, x_2]].$$

where  $x_1, x_2 \in K^0(\mathbb{C}P^m \times \mathbb{C}P^n)$  are defined by  $x_1 := x \otimes 1$  and  $x_2 := 1 \otimes x$ .

Describing  $\eta \times \eta$  as the direct sum  $\pi_1(\eta) \oplus \pi_2(\eta)$  [20] (where  $\pi_1$  and  $\pi_2$  are the projections of  $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$  onto the first and second factor respectively), we see that  $c_2(\eta \times \eta)$  equals  $z^{-2}x_1x_2$ . There are no multiplicative relations in  $K^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ , so the elements  $\kappa_2(\eta \times \eta)$  and  $\overline{c_2(\eta \times \eta)}/c_2(\eta \times \eta)$  of  $K^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  are equal. Using (2.9), it follows that  $\kappa_2(\eta \times \eta)$  is equal to  $(1 + \bar{x}_1)(1 + \bar{x}_2)$ . Referring to Corollary 2.0.11, we describe the Thom algebras.

Firstly  $K^*(T\eta \times \eta)$  is a free  $K^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$ -module on one generator  $t$ . The Thom algebra is generated by  $t$ , with the single relation

$$t^2 = x_1x_2 \cdot t. \tag{8.1}$$

Conjugation interacts with the Thom class via the relation

$$\bar{t} = (1 + \bar{x}_1)(1 + \bar{x}_2) \cdot t. \tag{8.2}$$

Restricting to the complex vector bundle  $\eta(m) \times \eta(n)$  over  $\mathbb{C}P^m \times \mathbb{C}P^n$ , we observe that  $K^*(T\eta(m) \times \eta(n))$  is a free  $K^*((\mathbb{C}P^m \times \mathbb{C}P^n)_+)$ -module on a single generator  $t \in K^0(T\eta(m) \times \eta(n))$ . The Thom algebra of  $\eta(m) \times \eta(n)$  is generated by  $t$ , and  $t$  satisfies the relations (8.1) and (8.2).

## 8.2 Real $K$ -theory

### 8.2.1 Group Structure

We first calculate  $KO^*(\mathbb{C}P^m \times \mathbb{C}P^n)$  as a graded abelian group. Recall that for any non-negative integer  $n$ , we have the cofibre sequence

$$\mathbb{C}P^{n-1} \xrightarrow{j_{n-1}} \mathbb{C}P^n \xrightarrow{p} S^{2n} \quad (8.3)$$

where  $j_{n-1}$  is the inclusion of the subspace  $\mathbb{C}P^{n-1}$ , and  $p$  is projection onto the top cell.

**Lemma 8.2.1.** *As graded abelian groups,  $KO^*(\mathbb{C}P^{2m} \wedge \mathbb{C}P^n)$  and  $\mathbb{Z}(mn)$  are isomorphic.*

*Proof.* We proceed by induction on  $n$ . The case  $n = 0$  is Proposition 3.2.1. Suppose the proposition holds for  $n - 1$ . We smash (8.3) with  $\mathbb{C}P^{2m}$ , and obtain the cofibre sequence

$$\mathbb{C}P^{2m} \wedge \mathbb{C}P^{n-1} \xrightarrow{1 \wedge j_{n-1}} \mathbb{C}P^{2m} \wedge \mathbb{C}P^n \xrightarrow{1 \wedge p} \mathbb{C}P^{2m} \wedge S^{2n}.$$

We apply the functor  $KO^*(-)$  and examine the resulting  $KO$ -theory long exact sequence of abelian groups.

For any  $i$ , both  $KO^{2i-1}(\mathbb{C}P^{2m} \wedge \mathbb{C}P^{n-1})$  and  $KO^{2i+1}(\mathbb{C}P^{2m} \wedge S^{2n})$  are zero (by the induction hypothesis and Proposition 3.2.1 respectively), so for any  $i \in \mathbb{Z}$ , the long exact sequence breaks up into sequences

$$0 \rightarrow KO^{2i}(\mathbb{C}P^{2m} \wedge S^{2n}) \rightarrow KO^{2i}(\mathbb{C}P^{2m} \wedge \mathbb{C}P^n) \rightarrow KO^{2i}(\mathbb{C}P^{2m} \wedge \mathbb{C}P^{n-1}) \rightarrow 0$$

and

$$0 \rightarrow KO^{2i+1}(\mathbb{C}P^{2m} \wedge \mathbb{C}P^n) \rightarrow 0.$$

The first sequence splits, as  $KO^{2i}(\mathbb{C}P^{2m} \wedge \mathbb{C}P^{n-1})$  is torsion free, and so  $KO^{2i}(\mathbb{C}P^{2m} \wedge \mathbb{C}P^n)$  and  $KO^{2i}(\mathbb{C}P^{2m} \wedge \mathbb{C}P^{n-1}) \oplus KO^{2i}(\mathbb{C}P^{2m} \wedge S^{2n+2})$  are isomorphic. Then we have isomorphisms  $KO^{2i}(\mathbb{C}P^{2m} \wedge \mathbb{C}P^n) \cong \mathbb{Z}^{m(n-1)} \oplus \mathbb{Z}^m$  and  $KO^{2i+1}(\mathbb{C}P^{2m} \wedge \mathbb{C}P^n) = 0$ , completing the inductive step.  $\square$

**Lemma 8.2.2.** *There is an isomorphism of graded abelian groups*

$$KO^*(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1}) \cong KO^*(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n}) \oplus KO^*(\mathbb{C}P^{2m+1} \wedge S^{4n+2}).$$

*Proof.* We examine the  $KO$ -theory long exact sequence induced by the cofibre sequence

$$\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n} \xrightarrow{1 \wedge j_{2n}} \mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1} \xrightarrow{1 \wedge p} \mathbb{C}P^{2m+1} \wedge S^{4n+2}.$$

Lemma 8.2.1 states that  $KO^{2i+1}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n})$  is zero for any integer  $i$ , and from Proposition 3.2.1,  $KO^{2i+1}(\mathbb{C}P^{2m+1} \wedge S^{4n+2}) \cong KO^{2i-4n-1}(\mathbb{C}P^{2m+1})$  is isomorphic to  $\mathbb{Z}_2$  if  $i \equiv 2n + 2m + 1 \pmod{4}$ , but zero otherwise. We examine these two cases in turn.

If  $KO^{2i+1}(\mathbb{C}P^{2m+1} \wedge S^{4n+2})$  is zero, the long exact sequence breaks up into sequences

$$0 \rightarrow KO^{2i-4n-2}(\mathbb{C}P^{2m+1}) \rightarrow KO^{2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1}) \rightarrow KO^{2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n}) \rightarrow 0$$

and

$$0 \rightarrow KO^{2i+1}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1}) \rightarrow 0,$$

the first of which splits, as  $KO^{2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n})$  is free abelian.

Let  $i \equiv 2n + 2m + 1 \pmod{4}$ . Then  $KO^{2i+1}(\mathbb{C}P^{2m+1} \wedge S^{4n+2})$  is isomorphic to  $\mathbb{Z}_2$ , and we have the exact sequence

$$0 \rightarrow KO^{2i-4n-2}(\mathbb{C}P^{2m+1}) \xrightarrow{(1 \wedge p)^*} KO^{2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1}) \rightarrow KO^{2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n}) \rightarrow KO^{2i-4n-1}(\mathbb{C}P^{2m+1}) \xrightarrow{(1 \wedge p)^*} KO^{2i+1}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1}) \rightarrow 0.$$

From Proposition 3.2.1, we have group isomorphisms  $KO^{2i-4n-2}(\mathbb{C}P^{2m+1}) \cong \mathbb{Z}^m \oplus \mathbb{Z}_2$  and  $KO^{2i-4n-1}(\mathbb{C}P^{2m+1}) \cong \mathbb{Z}_2$ . Using the first of these, the sequence shows that  $KO^{2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1})$  has exactly one  $\mathbb{Z}_2$ -summand. It follows from Proposition 2.0.1 that  $KO^{2i+1}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1})$  is isomorphic to  $\mathbb{Z}_2$ , and the epimorphism  $(1 \wedge p)^*: KO^{2i-4n-1}(\mathbb{C}P^{2m+1}) \rightarrow KO^{2i+1}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1})$  is an isomorphism  $\mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ . We are left with the short exact sequence

$$0 \rightarrow KO^{2i-4n-2}(\mathbb{C}P^{2m+1}) \rightarrow KO^{2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1}) \rightarrow KO^{2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n}) \rightarrow 0$$

which splits as before.  $\square$

Using the two lemmas above and Proposition 3.2.1, the isomorphism

$$KO^*(\mathbb{C}P^m \times \mathbb{C}P^n) \cong KO^*(\mathbb{C}P^m \wedge \mathbb{C}P^n) \oplus KO^*(\mathbb{C}P^m) \oplus KO^*(\mathbb{C}P^n) \quad (8.4)$$

(the weak homotopy equivalence  $\Sigma(X \times Y) \simeq \Sigma(X \wedge Y) \vee \Sigma X \vee \Sigma Y$  [31] and periodicity imply (8.4)) gives the following result.

**Proposition 8.2.3.** *As a graded abelian group,  $KO^*(\mathbb{C}P^m \times \mathbb{C}P^n)$  is isomorphic to*

$\mathbb{Z}(\lfloor \frac{mn}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor) \oplus S$ , where

$$S \cong \begin{cases} 0 & \text{both } m \text{ and } n \text{ even,} \\ KO^*(S^{2m}) & m \text{ odd, } n \text{ even,} \\ KO^*(S^{2n}) & m \text{ even, } n \text{ odd,} \\ KO^*(S^{2m} \vee S^{2n} \vee S^{2(m+n)}) & \text{both } m \text{ and } n \text{ odd.} \end{cases}$$

## 8.2.2 Basis Elements

We generalise Definition 3.2.2 and define the elements  $u_i, v_i, w_i$  which generate the  $KO_*$ -algebras  $KO^*((\mathbb{C}P^m \times \mathbb{C}P^n)_+)$ . We then prove a series of lemmas which are a step towards giving bases for  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  in terms of these elements.

**Definition 8.2.4.** Define

$$u_i = r(z^i x_1), \quad v_i = r(z^i x_2), \quad w_i = r(z^i x_1 x_2).$$

These elements of  $KO^{-2i}(\mathbb{C}P^m \times \mathbb{C}P^n)$  are defined for any integer  $i$ . For brevity, we will use  $\lambda_i \in KO^{-2i}(\mathbb{C}P^m \times \mathbb{C}P^n)$  to denote any of  $u_i, v_i$  or  $w_i$ .

We refer to (3.5) to describe the complexifications  $c(u_i)$  and  $c(v_i)$ , and use (2.8) to describe  $c(w_i)$ .

$$z^{-i}c(u_i) = x_1 + (-1)^i \bar{x}_1 = \begin{cases} x_1^2 - x_1^3 + x_1^4 - \cdots + (-1)^m x_1^m & i \text{ even,} \\ 2x_1 - x_1^2 + x_1^3 - \cdots + (-1)^{m+1} x_1^m & i \text{ odd.} \end{cases} \quad (8.5)$$

$$z^{-i}c(v_i) = x_2 + (-1)^i \bar{x}_2 = \begin{cases} x_2^2 - x_2^3 + x_2^4 - \cdots + (-1)^n x_2^n & i \text{ even,} \\ 2x_2 - x_2^2 + x_2^3 - \cdots + (-1)^{n+1} x_2^n & i \text{ odd.} \end{cases} \quad (8.6)$$



$$z^{-i}c(w_i) = x_1x_2 + (-1)^i\bar{x}_1\bar{x}_2 = x_1x_2 + \sum_{\substack{j+k \neq 2 \\ 1 \leq j \leq 2m, 1 \leq k \leq 2n}} (-1)^{i+j+k}x_1^jx_2^k \quad (8.7)$$

The elements  $r(z^i x_1^j x_2^k)$ ,  $1 \leq j \leq 2m$ ,  $1 \leq k \leq 2n$  span  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$ .

The following lemmas are a step towards reducing this spanning set to a basis:

**Lemma 8.2.5.** *The following equations hold in  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for any integer  $i$ .*

$$\begin{aligned} r(z^i x_1^j) &= u_0(r(z^i x_1^{j-1}) + r(z^i x_1^{j-2})), \quad j \geq 3, \\ r(z^i x_1^2) &= \begin{cases} u_i(u_0 + 2) & i \text{ even,} \\ u_i u_0 & i \text{ odd,} \end{cases} \\ r(z^i x_2^k) &= v_0(r(z^i x_2^{k-1}) + r(z^i x_2^{k-2})), \quad k \geq 3, \\ r(z^i x_2^2) &= \begin{cases} v_i(v_0 + 2) & i \text{ even,} \\ v_i v_0 & i \text{ odd.} \end{cases} \end{aligned}$$

*Proof.* The proof is similar to that of Lemma 3.2.3. □

**Lemma 8.2.6.** *If  $j, k \geq 2$  and  $j + k \geq 5$  then the following relation holds in  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for any integer  $i$ .*

$$r(z^i x_1^j x_2^k) = w_0 r(z^i x_1^{j-1} x_2^{k-1}) - u_0 v_0 r(z^i x_1^{j-2} x_2^{k-2}).$$

*Proof.* Complexifying  $r(z^i x_1^j x_2^k)$ , we obtain:

$$\begin{aligned} z^i x_1^j x_2^k + \overline{z^i x_1^j x_2^k} &= \left(x_1 x_2 + \overline{x_1 x_2}\right) \left(z^i x_1^{j-1} x_2^{k-1} + \overline{z^i x_1^{j-1} x_2^{k-1}}\right) \\ &\quad - x_1 \overline{x_1} x_2 \overline{x_2} \left(z^i x_1^{j-2} x_2^{k-2} + \overline{z^i x_1^{j-2} x_2^{k-2}}\right) \\ &= c \left(r(x_1 x_2) r(z^i x_1^{j-1} x_2^{k-1}) - r(x_1) r(x_2) r(z^i x_1^{j-2} x_2^{k-2})\right). \end{aligned}$$

Recall that complexification is monomorphic. □

**Lemma 8.2.7.** *Assuming  $k \geq 3$  and  $j \geq 3$  respectively, we have the following relations in  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for any integer  $i$ .*

$$\begin{aligned} r(z^i x_1 x_2^k) &= w_0 r(z^i x_2^{k-1}) + v_0 (u_0 r(z^i x_2^{k-2}) - r(z^i x_1 x_2^{k-2})), \text{ and} \\ r(z^i x_1^j x_2) &= w_0 r(z^i x_1^{j-1}) + u_0 (v_0 r(z^i x_1^{j-2}) - r(z^i x_1^{j-2} x_2)). \end{aligned}$$

*Proof.* We prove the first relation only, as the proof for the second relation is similar.

Complexifying  $r(z^i x_1 x_2^k)$ , we obtain

$$\begin{aligned}
z^i x_1 x_2^k + \overline{z^i x_1 x_2^k} &= (x_1 x_2 + \overline{x_1 x_2}) (z^i x_2^{k-1} + \overline{z^i x_2^{k-1}}) - x_2 \overline{x_2} (z^i \overline{x_1} x_2^{k-2} + \overline{z^i x_1} \overline{x_2}^{k-2}) \\
&= c \left( r(x_1 x_2) r(z^i x_2^{k-1}) \right) \\
&\quad + c r(x_2) \left( (x_1 + \overline{x_1}) (z^i x_2^{k-2} + \overline{z^i x_2^{k-2}}) - (z^i x_1 x_2^{k-2} + \overline{z^i x_1} \overline{x_2}^{k-2}) \right) \\
&= c \left( w_0 r(z^i x_2^{k-1}) + v_0 \left( r(x_1) r(z^i x_2^{k-2}) - r(z^i x_1 x_2^{k-2}) \right) \right).
\end{aligned}$$

Again, we note that complexification is monomorphic.  $\square$

**Lemma 8.2.8.** *The following relation holds in  $KO^{-4(i+j)}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for all integers  $i$  and  $j$ .*

$$w_{2i} w_{2j} = u_{2(i+j)} v_0 (u_0 + v_0 + w_0 + 4).$$

*Proof.* We require the following equations:

- $r(z^{2i} x_1 \overline{x_1} x_2 \overline{x_2}) = 2u_{2i} v_0,$
- $r(z^{2i} x_1^2) r(x_2^2) = r(z^{2i} x_1) (r(x_1) + 2) r(x_2) (r(x_2) + 2)$   
 $= u_{2i} v_0 (u_0 + 2) (v_0 + 2),$
- $r(z^{2i} x_1^2 \overline{x_2}^2) = r(z^{2i} x_1 \overline{x_2}) r(x_1 \overline{x_2}) - r(z^{2i} x_1 \overline{x_1} x_2 \overline{x_2})$   
 $= (r(z^{2i} x_1) r(\overline{x_2}) - r(z^{2i} x_1 x_2)) (r(x_1) r(\overline{x_2}) - r(x_1 x_2)) - 2u_{2i} v_0$   
 $= (u_{2i} v_0 - w_{2i}) (u_0 v_0 - w_0) - 2u_{2i} v_0.$

These are proved using complexification and the equation (2.9), Lemma 8.2.5, and Proposition 2.0.2 respectively. Then we combine

$$\begin{aligned}
r(z^{2(i+j)} x_1^2 x_2^2) &= r(z^{2(i+j)} x_1 x_2) r(x_1 x_2) - r(z^{2(i+j)} x_1 \overline{x_1} x_2 \overline{x_2}) \quad (\text{Proposition 2.0.2}) \\
&= w_{2(i+j)} w_0 - 2u_{2(i+j)} v_0
\end{aligned}$$

and

$$\begin{aligned}
r(z^{2(i+j)} x_1^2 x_2^2) &= r(z^{2(i+j)} x_1^2) r(x_2^2) - r(z^{2(i+j)} x_1^2 \overline{x_2}^2) \quad (\text{Proposition 2.0.2}) \\
&= u_{2(i+j)} v_0 (u_0 + 2) (v_0 + 2) \\
&\quad - (u_{2(i+j)} v_0 - w_{2(i+j)}) (u_0 v_0 - w_0) + 2u_{2(i+j)} v_0 \\
&= u_{2(i+j)} v_0 (2u_0 + 2v_0 + 6 + w_0) + w_{2(i+j)} u_0 v_0 - w_{2(i+j)} w_0
\end{aligned}$$

to get  $2w_{2(i+j)} w_0 = u_{2(i+j)} v_0 (2u_0 + 2v_0 + 2w_0 + 8)$ . As  $KO^{-4i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  is free of 2-torsion, we may divide by two.  $\square$

**Lemma 8.2.9.** *The following relations hold in  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for any integer  $i$ .*

$$r(z^i x_1^2 x_2) = u_0 w_i + u_0 v_i$$

and

$$r(z^i x_1 x_2^2) = v_0 w_i + u_i v_0.$$

*Proof.* We prove the second relation only. The proof for the first relation is similar.

$$\begin{aligned} cr(z^i x_1 x_2^2) &= (z^i x_1 x_2^2 + \overline{z^i x_1 x_2^2}) \\ &= (x_2 + \overline{x_2})(z^i x_1 x_2 + \overline{z^i x_1 x_2}) - (z^i x_1 + \overline{z^i x_1}) x_2 \overline{x_2} \\ &= c(r(x_2) r(z^i x_1 x_2) + r(z^i x_1) r(x_2)). \end{aligned}$$

and complexification is monomorphic.  $\square$

**Corollary 8.2.10.** *The following relation holds in  $KO^{-4i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for any integer  $i$ .*

$$r(z^{2i} x_1^2 x_2^2) = u_{2i} v_0 (u_0 + v_0 + w_0 + 2)$$

*Proof.* We show that  $r(z^{2i} x_1^2 x_2^2) = w_{2i} w_0 - 2u_{2i} v_0$ . The result then follows from Lemma 8.2.8.

$$\begin{aligned} cr(z^{2i} x_1^2 x_2^2) &= z^{2i} (x_1^2 x_2^2 + \overline{x_1^2 x_2^2}) \\ &= (x_1 x_2 + \overline{x_1 x_2}) z^{2i} (x_1 x_2 + \overline{x_1 x_2}) - 2z^{2i} x_1 \overline{x_1} x_2 \overline{x_2} \\ &= c(r(x_1 x_2) r(z^{2i} x_1 x_2) - 2r(z^{2i} x_1) r(x_2)), \end{aligned}$$

and complexification is monomorphic.  $\square$

**Lemma 8.2.11.** *The following relation holds in  $KO^{-4i-2}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for any integer  $i$ .*

$$r(z^{2i+1} x_1^2 x_2^2) = u_0 (v_{2i+1} w_0 + v_{2i+1} v_0 - u_{2i+1} v_0)$$

*Proof.* Firstly, we observe that

$$r(z^{2i+1} x_1^2 x_2^2) = r(x_1^2 x_2) r(z^{2i+1} x_2) - r(\overline{z^{2i+1} x_1^2 x_2} \overline{x_2})$$

by Proposition 2.0.2. But since

$$r(x_1^2 x_2) r(z^{2i+1} x_2) = (u_0 w_0 + u_0 v_0) v_{2i+1} \quad (\text{Lemma 8.2.9})$$

and

$$\begin{aligned} r(\overline{z^{2i+1} x_1^2 x_2} \overline{x_2}) &= r(z^{2i+1} x_1^2 (x_2 + \overline{x_2})) \\ &= r(z^{2i+1} x_1^2) r(x_2) \quad (\text{Proposition 2.0.2}) \\ &= u_{2i+1} u_0 v_0 \quad (\text{Proposition 8.2.5}), \end{aligned}$$

this simplifies to the required equation.  $\square$

We would now like to prove an equivalent result to that of Lemma 3.2.4, and give a basis for  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$ . Since realification is an epimorphism on  $K^*(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  (2.1), the elements  $\{r(z^i x_1^j x_2^k) \mid i \in \mathbb{Z}, 0 \leq j \leq 2m, 0 \leq k \leq 2n, j+k \neq 0\}$  span the abelian group  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$ . For any integers  $i, j$  such that  $i+j \geq 1$ , the above results (Lemmas and Corollary 8.2.5 to 8.2.11) allow us to rewrite  $r(z^i x_1^j x_2^k)$  as a polynomial in  $\mathbb{Z}[\lambda_l \mid l \in \mathbb{Z}]$ . The monomials obtained by this process are not, in general, linearly independent and we require further calculations before stating a basis.

### 8.2.3 Multiplicative Relations

We now describe the interaction of  $\lambda_i \in KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  with the coefficient ring for each  $\lambda$ , then the products  $\lambda_i \lambda_j \in KO^{-2(i+j)}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for all  $\lambda$  and integers  $i, j$ . This furnishes us firstly with a basis for the abelian group  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  in terms of the elements  $\lambda_j \in KO^{-2j}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$ , and secondly with a description of  $KO^*((\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})_+)$  as a  $KO_*$ -algebra.

**Proposition 8.2.12.** *For any integers  $i, j$  and any  $\lambda$ , the following relations hold in  $KO^*((\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})_+)$ .*

$$e\lambda_i = 0, \quad \alpha\lambda_i = 2\lambda_{i+2}, \quad \beta\lambda_i = \lambda_{i+4}, \quad \lambda_i \lambda'_j = \lambda_{i+2} \lambda'_{j-2}.$$

*Proof.* As  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  is torsion free for all  $i \in \mathbb{Z}$ , complexification is injective, and the relations are proved as in Proposition 3.2.5.  $\square$

**Lemma 8.2.13.** *The following relations hold in  $KO^{-4(k+l)}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for any integers  $k, l$ .*

$$u_{2k+1} u_{2l-1} = u_{2(k+l)}(u_0 + 4), \quad v_{2k+1} v_{2l-1} = v_{2(k+l)}(v_0 + 4).$$

*Proof.* These two equations follow from Proposition 3.2.10 by pulling back along the inclusions  $\mathbb{C}P^{2m} \times \{0\} \hookrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$  and  $\{0\} \times \mathbb{C}P^{2n} \hookrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ .  $\square$

**Lemma 8.2.14.** *The following relations hold in  $KO^*(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for any integers  $i, j$ .*

1.  $v_{2i+1}u_j = 2w_{2i+j+1} - u_{2i+j+1}v_0$ ,
2.  $u_{2i+1}w_{2j-1} = u_0w_{2(i+j)} + 2u_{2(i+j)}v_0$ ,
3.  $u_{2i+1}w_{2j} = (u_0 + 4)w_{2i+2j+1} - 2u_{2i+2j+1}v_0$ ,
4.  $v_{2i+1}w_j = v_0w_{2i+j+1} + 2u_{2i+j+1}v_0$ ,
5.  $w_{2i}w_{2j} = u_{2(i+j)}v_0(u_0 + v_0) + u_0v_0w_{2(i+j)}$ ,
6.  $w_{2i+1}w_{2j-1} = u_{2(i+j)}v_0(u_0 + v_0) + u_0v_0w_{2(i+j)}$ ,
7.  $w_{2i+1}w_{2j} = u_{2i+2j+1}v_0(u_0 - v_0) + (u_0 + 2)v_0w_{2i+2j+1}$ .

*Proof.* Complexification is injective, and we use Proposition 2.0.2 to deduce relations  $r(a^{2i+1})r(z^jb) = r(z^{2i+j+1}ab) - r(z^{2i+j+1}\bar{a}b)$  and  $r(a)r(z^{2i+j+1}b) = r(z^{2i+j+1}ab) + r(z^{2i+j+1}\bar{a}b)$  for any  $a, b \in K^*(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$ . Summing these, we obtain the equation  $r(z^{2i+1}a)r(z^jb) = 2r(z^{2i+j+1}ab) - r(a)r(z^{2i+j+1}b)$ . This proves 1. immediately. The remainder are proved using same relation and consulting previous results. For 2. we need Lemmas 8.2.9 and 8.2.12. For 3. we need Lemma 8.2.9 and 1. For 4. we need Lemma 8.2.9. Relation 5. was proved in Lemma 8.2.8. We have rewritten it using Proposition 8.2.12. For 6. we need Lemmas 8.2.8 and 8.2.10. For 7. we need Lemma 8.2.11, and relations 1. and 4.  $\square$

## 8.2.4 Products of Finite Projective Spaces

**Lemma 8.2.15.** *The following relations hold in  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  for all  $i \in \mathbb{Z}$ .*

$$u_i u_0^m = u_0^m w_i = v_i v_0^n = v_0^n w_i = 0.$$

*Proof.* The relations are clear from (8.5), (8.6) and (8.7) if we complexify, since the complexification of each of  $u_i u_0^m$ ,  $u_0^m w_i$ ,  $v_i v_0^n$  and  $v_0^n w_i$  has either  $x_1^{2m+1}$  or  $x_2^{2n+1}$  as a factor for any  $i$ , and  $c$  is injective.  $\square$

We are now in a position to state a basis for  $KO^*(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$ , described as a graded abelian group in Proposition 8.2.3.

**Lemma 8.2.16.** *As graded abelian groups,  $KO^*(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  and  $\mathbb{Z}(2mn + m + n)$  are isomorphic. A basis for  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$  is the union of the sets  $\{u_0^j v_0^k w_i \mid 0 \leq j \leq m-1, 0 \leq k \leq n-1\}$ ,  $\{u_i u_0^j v_0^k \mid 0 \leq j \leq m-1, 0 \leq k \leq n\}$  and  $\{v_i v_0^k \mid 0 \leq k \leq n-1\}$ .*

*Proof.* Recall (see comment after Lemma 8.2.11) that the elements  $r(z^i x_1^j x_2^k)$  (where  $i \in \mathbb{Z}$ ,  $1 \leq j \leq 2m$ ,  $1 \leq k \leq 2n$ , and  $j+k \neq 0$ ) span the abelian group  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$ , and that we may rewrite each  $r(z^i x_1^j x_2^k)$  as a polynomial in  $\mathbb{Z}[\lambda_l \mid l \in \mathbb{Z}]$ . Given an arbitrary product of  $\lambda_l$ s in  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$ , Lemmas 8.2.8, 8.2.13, 8.2.14 and the relation  $\lambda_i \lambda'_j = \lambda_{i+2} \lambda'_{j-2}$  of Proposition 8.2.12 are sufficient to rewrite the product as a linear combination of elements  $w_i u_0^j v_0^k$  ( $j \geq 0$ ,  $k \geq 0$ ),  $u_i u_0^j v_0^k$  ( $j \geq 0$ ,  $k \geq 0$ ) and  $v_i v_0^k$  ( $k \geq 0$ ). The relations of Lemma 8.2.15 imply that the  $2mn + m + n$  elements in the proposed basis span  $KO^{-2i}(\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})$ .  $\square$

**Proposition 8.2.17.** *There is an isomorphism of  $KO_*$ -algebras*

$$KO^*((\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})_+) \cong KO_*[u_i, v_i, w_i \mid i \in \mathbb{Z}] / K(2m, 2n)$$

where  $K(2m, 2n)$  is the ideal

$$\begin{aligned} & (e\lambda_i, \quad \alpha\lambda_i - 2\lambda_{i+2}, \quad \beta\lambda_i - \lambda_{i+4}, \quad \lambda_i \lambda'_j - \lambda_{i+2} \lambda'_{j-2}, \quad u_{2i+1} u_{2j-1} - u_{2(i+j)}(u_0 + 4), \\ & \quad v_{2i+1} u_j - 2w_{2i+j+1} + u_{2i+j+1} v_0, \quad u_{2i+1} w_{2j-1} - u_0 w_{2(i+j)} - 2u_{2(i+j)} v_0, \\ & \quad u_{2i+1} w_{2j} - (u_0 + 4)w_{2i+2j+1} + 2u_{2i+2j+1} v_0, \quad v_{2i+1} v_{2j-1} - v_{2(i+j)}(v_0 + 4), \\ & \quad v_{2i+1} w_j - v_0 w_{2i+j+1} - 2u_{2i+j+1} v_0, \quad w_{2i} w_{2j} - u_{2(i+j)} v_0 (u_0 + v_0) - u_0 v_0 w_{2(i+j)}, \\ & \quad \quad \quad w_{2i+1} w_{2j-1} - u_{2(i+j)} v_0 (u_0 + v_0) - u_0 v_0 w_{2(i+j)}, \\ & \quad \quad \quad w_{2i+1} w_{2j} - u_{2i+2j+1} v_0 (u_0 - v_0) - (u_0 + 2)v_0 w_{2i+2j+1} \\ & \quad \quad \quad u_i u_0^m, u_0^m w_i, v_i v_0^n, v_0^n w_i). \end{aligned}$$

*Proof.* The relations (Proposition 8.2.12, Lemmas 8.2.8, 8.2.13, 8.2.14 and 8.2.15) are sufficient to rewrite a polynomial  $P \in KO_*[u_i, v_i, w_i \mid i \in \mathbb{Z}]$  in terms of the basis given in Lemma 8.2.16.  $\square$

Due to the periodicity relation  $\beta\lambda_i = \lambda_{i+4}$ , the  $KO_*$ -algebra requires only twelve

generators:  $u_0, u_1, u_2, u_3, v_0, v_1, v_2, v_3, w_0, w_1, w_2, u_3$ , for example. We retain the other  $\lambda_i$  for notational convenience.

We now examine  $KO^*(\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n})$  and  $KO^*(\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n+1})$  using the homomorphisms induced by the inclusions of  $\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n}$  and  $\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n+1}$  in  $\mathbb{C}P^{2m+2} \times \mathbb{C}P^{2n}$  and  $\mathbb{C}P^{2m+2} \times \mathbb{C}P^{2n+2}$  respectively. Since the inclusions induce homomorphisms of  $KO_*$ -algebras, the relations of Proposition 8.2.17 hold in the images.

Recall Definition 3.2.8. We define  $\sigma_{-2m-1} \in KO^{4m+2}(\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n})$  by pulling  $\sigma_{-2m-1} \in KO^{4m+2}(\mathbb{C}P^{2m+1})$  back using the projection  $\pi_1: \mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n} \rightarrow \mathbb{C}P^{2m+1}$  onto the first factor.

**Proposition 8.2.18.** *There is an isomorphism of  $KO_*$ -algebras*

$$KO^*((\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n})_+) \cong KO^*((\mathbb{C}P^{2m+2} \times \mathbb{C}P^{2n})_+)[\sigma_{-2m-1}]/K(2m+1, 2n)$$

where  $K(2m+1, 2n)$  is the ideal

$$\begin{aligned} &(\sigma_{-2m-1}^2, \quad u_{-2m-1}u_0^m - 2\sigma_{-2m-1}, \quad u_{-2m}u_0^m - e^2\sigma_{-2m-1}, \quad u_{-2m+1}u_0^m - \alpha\sigma_{-2m-1}, \\ &u_{-2m+2}u_0^m, \quad \sigma_{-2m-1}u_i, \quad \sigma_{-2m-1}v_i - w_{i-2m-1}u_0^m, \quad \sigma_{-2m-1}w_i, \quad v_0^n w_i). \end{aligned}$$

*Proof.* We recall the splitting (8.4) and refer to Propositions 3.2.9 and 3.2.7 for the structure of the summands  $KO^*(\mathbb{C}P^{2m+1})$  and  $KO^*(\mathbb{C}P^{2n})$ . We obtain the relations  $2\sigma_{-2m-1} = u_{-2m-1}u_0^m$ ,  $e^2\sigma_{-2m-1} = u_{-2m}u_0^m$ ,  $\alpha\sigma_{-2m-1} = u_{-2m+1}u_0^m$ ,  $u_{-2m+2}u_0^m = \sigma_{-2m-1}u_i = 0$  and  $v_i v_0^n = 0$ .

Complexification is a monomorphism on  $KO^*(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n})$  and since each of  $\sigma_{-2m-1}v_i$ ,  $u_0^m v_0^k w_{2i+1}$  and  $\sigma_{-2m-1}w_i$  lies in the summand  $KO^*(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n})$ , we deduce the remaining relations by complexifying.

We examine the remaining summand:  $KO^*(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n}) \cong \mathbb{Z}((2m+1)n)$ . Fix  $i$ . As in Lemma 8.2.16 we rewrite  $r(z^i x_1^j x_2^k) \in KO^{-2i}(\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n})$  as a polynomial in the  $\lambda_i$ s such that the subscripts of each monomial sum to  $i$ . Our relations are sufficient to rewrite such a polynomial as a linear combination of elements  $u_0^j v_0^k w_i$  ( $0 \leq j \leq m$ ,  $0 \leq k \leq n-1$ ),  $u_i u_0^j v_0^k$  ( $0 \leq j \leq m-1$ ,  $0 \leq k \leq n$ ), and  $v_i v_0^k$  ( $0 \leq k \leq n-1$ ). But  $u_i u_0^j$  ( $i \in \mathbb{Z}$ ,  $0 \leq j \leq m-1$ ) and  $\sigma_{-2m-1} KO_*$  are a

basis for  $KO^*(\mathbb{C}P^{2m+1})$  (Proposition 3.2.9), and  $v_i v_0^k$  ( $i \in \mathbb{Z}$ ,  $0 \leq k \leq n-1$ ) are a basis for  $KO^*(\mathbb{C}P^{2n})$  (Proposition 3.2.7). So the remaining  $(2m+1)n$  elements  $u_0^j v_0^k w_i$  ( $0 \leq j \leq m$ ,  $0 \leq k \leq n-1$ ) and  $u_i u_0^j v_0^k$  ( $0 \leq j \leq m-1$ ,  $1 \leq k \leq n$ ) span  $KO^{-2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n}) \cong \mathbb{Z}^{(2m+1)n}$ , and hence are an additive basis for the summand.  $\square$

Similarly to above, we define the elements  $\sigma_{-2m-1} \in KO^{4m+2}(\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n+1})$  and  $\sigma_{-2n-1} \in KO^{4n+2}(\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n+1})$  by pulling  $\sigma_{-2m-1} \in KO^{4m+2}(\mathbb{C}P^{2m+1})$  and  $\sigma_{-2n-1} \in KO^{4n+2}(\mathbb{C}P^{2n+1})$  back using the projections of  $\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n+1}$  onto its first and second factors respectively.

**Proposition 8.2.19.** *There is an isomorphism of  $KO_*$ -algebras*

$$KO^*((\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2n+1})_+) \cong KO^*(\mathbb{C}P^{2m+2} \times \mathbb{C}P^{2n+2})[\sigma_{-2m-1}, \sigma_{-2n-1}] / K(2m+1, 2n+1)$$

where  $K(2m+1, 2n+1)$  is the ideal

$$\begin{aligned} & (\sigma_{-2m-1}^2, \sigma_{-2n-1}^2, u_{-2m-1} u_0^m - 2\sigma_{-2m-1}, u_{-2m} u_0^m - e^2 \sigma_{-2m-1}, \\ & u_{-2m+1} u_0^m - \alpha \sigma_{-2m-1}, u_{-2m+2} u_0^m, \sigma_{-2m-1} u_i, \sigma_{-2m-1} w_i, \sigma_{-2m-1} v_i - w_{i-2m-1} u_0^m, \\ & v_{-2n-1} v_0^n - 2\sigma_{-2n-1}, v_{-2n} v_0^n - e^2 \sigma_{-2n-1}, v_{-2n+1} v_0^n - \alpha \sigma_{-2n-1}, \\ & v_{-2n+2} v_0^n, \sigma_{-2n-1} v_i, \sigma_{-2n-1} w_i, \sigma_{-2n-1} u_i - w_{i-2m-1} v_0^n, \\ & w_{-2(m+n+1)} u_0^m v_0^n - 2\sigma_{-2m-1} \sigma_{-2n-1}, w_{-2m-2n-1} u_0^m v_0^n - e^2 \sigma_{-2m-1} \sigma_{-2n-1}, \\ & w_{-2(m+n)} u_0^m v_0^n - \alpha \sigma_{-2m-1} \sigma_{-2n-1}, w_{-2m-2n+1} u_0^m v_0^n). \end{aligned}$$

*Proof.* Recall the splitting (8.4) and refer to Proposition 3.2.9 for the structure of the summands  $KO^*(\mathbb{C}P^{2m+1})$  and  $KO^*(\mathbb{C}P^{2n+1})$ . From  $KO^*(\mathbb{C}P^{2m+1})$  we obtain the relations  $2\sigma_{-2m-1} u_{-2m-1} u_0^m$ ,  $e^2 \sigma_{-2m-1} = u_{-2m} u_0^m$ ,  $\alpha \sigma_{-2m-1} = u_{-2m+1} u_0^m$ , and  $u_{-2m+2} u_0^m = \sigma_{-2m-1} u_i = 0$ . We obtain similar relations from  $KO^*(\mathbb{C}P^{2n+1})$ .

We examine the remaining summand,  $KO^*(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1})$ , which is isomorphic to  $\mathbb{Z}(2mn + m + n) \oplus KO^*(S^{4(m+n+1)})$  as a graded group. Using the homomorphism induced in  $KO$ -theory by the inclusion of  $\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1}$  into  $\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+2}$ , we obtain the relations  $\sigma_{-2m-1} v_i = w_{i-2m-1} u_0^m$  and  $\sigma_{-2m-1} w_i = 0$ . Similarly we obtain  $\sigma_{-2n-1} w_i = 0$  and  $\sigma_{-2n-1} u_i = w_{i-2m-1} v_0^n$  from  $K^*(\mathbb{C}P^{2m+2} \times \mathbb{C}P^{2n+1})$ .



Copying the proof of Lemma 8.2.16, we establish that the image of the realification homomorphism in  $KO^*(\mathbb{C}P^{2m+1} \times \mathbb{C}P^{2m+1})$  is spanned by the union of the sets  $\{w_i u_0^j v_0^k \mid j \geq 0, k \geq 0\}$ ,  $\{u_i u_0^j v_0^k \mid j \geq 0, k \geq 0\}$  and  $\{v_i v_0^k \mid k \geq 0\}$ .

Since  $v_0^{n+1}$  and  $u_{2i} u_0^m$  each have either  $e^2$  or 0 as a factor, we have the equation  $w_i v_0^{n+1} = w_i u_0^{m+1} = u_i v_0^{n+1} = u_{2i} u_0^m v_0 = 0$ . From  $\sigma_{-2m-1} v_i = w_{i-2m-1} u_0^m$ , we deduce the equation  $u_{2i+1} u_0^m v_0 = 2w_{2i+1} u_0^m$  (multiply by two, pick an appropriate value of  $i$ , and use the relation  $\lambda_i \lambda_j = \lambda_{i+2} \lambda_{j-2}$ ). Using these relations, and discarding those elements which lie in the summands  $KO^*(\mathbb{C}P^{2m+1})$  and  $KO^*(\mathbb{C}P^{2n+1})$ , we see that the image of the realification homomorphism in  $KO^{-2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2m+1})$  is spanned by the union of the sets  $\{w_i u_0^j v_0^k \mid 0 \leq j \leq m, 0 \leq k \leq n\}$  and  $\{u_i u_0^j v_0^k \mid 0 \leq j \leq m-1, 1 \leq k \leq n\}$ .

Examining Bott's sequence (2.1), we establish that realification is surjective on  $K^{2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1})$  unless  $i \equiv 2(m+n+1) \pmod{4}$ , whereas complexification is injective on  $KO^{2i}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1})$  unless  $i \equiv 2(m+n)+1 \pmod{4}$ .

By complexification, we see that  $2\sigma_{-2m-1}\sigma_{-2n-1}$  and  $w_{-2(n+m+1)} u_0^m v_0^n$  are equal. We check by complexification that the elements making up our spanning set for the image of the  $r$  in  $KO^{4(n+m+1)}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1})$  are linearly independent, so  $\sigma_{-2m-1}\sigma_{-2n-1}$  is not in the image of  $r$ , and by Bott's sequence both  $e\sigma_{-2m-1}\sigma_{-2n-1}$  and  $e^2\sigma_{-2m-1}\sigma_{-2n-1}$  are non-zero. Complexification immediately gives us the relations  $\alpha\sigma_{-2m-1}\sigma_{-2n-1} = w_{-2(n+m)} u_0^m v_0^n$  and  $w_{-2n-2m+1} u_0^m v_0^n = 0$ . Finally, we note that since the image of  $r$  spans  $KO^{4n+4m+2}(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1})$ , which is isomorphic to  $\mathbb{Z}^{2mn+m+n} \oplus \mathbb{Z}_2$ , some element of the spanning set  $\{w_{-2n-2m-1} u_0^j v_0^k \mid 0 \leq j \leq m, 0 \leq k \leq n\}$  and  $\{u_{-2n-2m-1} u_0^j v_0^k \mid 0 \leq j \leq m-1, 1 \leq k \leq n\}$  must be equal to  $e^2\sigma_{-2m-1}\sigma_{-2n-1}$ . Since only  $w_{-2n-2m-1} u_0^m v_0^n$  is equal to zero when complexified, it must equal  $e^2\sigma_{-2m-1}\sigma_{-2n-1}$ .

In summary, for  $KO^*(\mathbb{C}P^{2m+1} \wedge \mathbb{C}P^{2n+1})$ , which is isomorphic to the direct sum  $\mathbb{Z}(2mn+m+n) \oplus KO^*(S^{4(m+n+1)})$ , we have found a basis consisting of the sets  $\{w_i u_0^j v_0^k \mid 0 \leq j \leq m, 0 \leq k \leq n, j+k < m+n\}$  and  $\{u_i u_0^j v_0^k \mid 0 \leq j \leq m-1, 1 \leq k \leq n\}$  for  $\mathbb{Z}(2mn+m+n)$ , and that  $KO^*(S^{4(m+n+1)})$  is isomorphic to  $KO_*[\sigma_{-2m-1}\sigma_{-2n-1}]/((\sigma_{-2m-1}\sigma_{-2n-1})^2)$ . Our relations are sufficient to rewrite an

arbitrary product of  $\lambda_i$ s in terms of this basis and of the bases for  $KO^*(\mathbb{C}P^{2m+1})$  and  $KO^*(\mathbb{C}P^{2n+1})$ .  $\square$

### 8.2.5 Product of Infinite Projective Spaces

We now deduce the  $KO_*$ -algebra  $KO^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$  from Proposition 8.2.17 using inverse limits. The homomorphism induced by the inclusion of  $\mathbb{C}P^{2n} \times \mathbb{C}P^{2n}$  in  $\mathbb{C}P^{2n+2} \times \mathbb{C}P^{2n+2}$  is surjective (Definition 8.2.4 and Proposition 8.2.17), so applying Theorem 2.0.17, we see that  $KO^{-2i}(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \varprojlim KO^{-2i}(\mathbb{C}P^{2n} \times \mathbb{C}P^{2n})$  is isomorphic to  $\prod \mathbb{Z}$  and we write the elements in the form  $\sum_{m,n=0}^\infty a_{mn} w_i u_0^m v_0^n + \sum_{m,n=0}^\infty b_{mn} u_i u_0^m v_0^n + \sum_{n=0}^\infty c_n v_i v_0^n$ , where  $a_{mn}$ ,  $b_{mn}$  and  $c_n$  are integers.

Equipping  $KO^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$  with multiplication (2.12), we obtain isomorphism of  $KO^{-2i}((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$  with the following direct sums of power series rings and modules:

$$KO^0((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+) \cong w_0 \mathbb{Z}[[u_0, v_0]] \oplus \mathbb{Z}[[u_0, v_0]]$$

and

$$KO^{-2i}(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong w_i \mathbb{Z}[[u_0, v_0]] \oplus u_i \mathbb{Z}[[u_0, v_0]] \oplus v_i \mathbb{Z}[[v_0]].$$

The above demonstrates that  $KO^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$  is generated over  $KO_*$  as an algebra by the elements  $\lambda_i$ ,  $i \in \mathbb{Z}$ . Certain multiplicative relations are carried over from  $KO^*((\mathbb{C}P^{2n} \times \mathbb{C}P^{2n})_+)$ . Recall that, by the definition of inverse limits, an element in  $KO^i((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$  equals the zero element if and only if it is annihilated by restriction to  $KO^i((\mathbb{C}P^{2n} \times \mathbb{C}P^{2n})_+)$  for every  $n \geq 0$ . Referring to Proposition 8.2.17, we retain those relations which depend on neither  $m$  nor  $n$ , i.e. those proved in Proposition 8.2.12 and Lemmas 8.2.8, 8.2.13 and 8.2.14.

Summarising the above, we have the following result.

**Proposition 8.2.20.** *The  $KO_*$ -algebra  $KO^{-2i}((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$  is generated by the elements  $u_i$ ,  $v_i$  and  $w_i$  ( $i \in \mathbb{Z}$ ), and  $KO^{-2i}(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  is isomorphic to the following direct sums of power series rings and modules.*

$$KO^0((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+) \cong w_0 \mathbb{Z}[[u_0, v_0]] \oplus \mathbb{Z}[[u_0, v_0]]$$

and

$$KO^{-2i}(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong w_i\mathbb{Z}[[u_0, v_0]] \oplus u_i\mathbb{Z}[[u_0, v_0]] \oplus v_i\mathbb{Z}[[v_0]].$$

For any  $i \in \mathbb{Z}$ ,  $KO^{-2i-1}(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  is isomorphic to the trivial group.

The action of  $KO_*$  on  $KO^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  is given by the relations

$$e\lambda_i = 0, \quad \alpha\lambda_i = 2u_{i+2} \quad \text{and} \quad \beta\lambda_i = u_{i+4},$$

for all  $i \in \mathbb{Z}$ , where  $\lambda_i$  represents any of  $u_i$ ,  $v_i$  or  $w_i$ .

The algebra relations are given by

$$\begin{aligned} \lambda_i\lambda'_j &= \lambda_{i+2}\lambda'_{i-2}, & u_{2i+1}u_{2j-1} &= u_{2(i+j)}(u_0 + 4), \\ v_{2i+1}u_j &= 2w_{2i+j+1} + u_{2i+j+1}v_0, & u_{2i+1}w_{2j-1} &= u_0w_{2(i+j)} - 2u_{2(i+j)}v_0, \\ u_{2i+1}w_{2j} &= (u_0 + 4)w_{2i+2j+1} + 2u_{2i+2j+1}v_0, & v_{2i+1}v_{2j-1} &= v_{2(i+j)}(v_0 + 4), \\ v_{2i+1}w_j &= v_0w_{2i+j+1} - 2u_{2i+j+1}v_0, & w_{2i}w_{2j} &= u_{2(i+j)}v_0(u_0 + v_0) - u_0v_0w_{2(i+j)}, \\ w_{2i+1}w_{2j-1} &= u_{2(i+j)}v_0(u_0 + v_0) - u_0v_0w_{2(i+j)}, \\ \text{and } w_{2i+1}w_{2j} &= u_{2i+2j+1}v_0(u_0 - v_0) - (u_0 + 2)v_0w_{2i+2j+1} \end{aligned}$$

for all  $i, j \in \mathbb{Z}$ .

*Remark 8.2.21.* The multiplication map

$$m: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$$

induces a cohomology homomorphism

$$m^*: D^*(\mathbb{C}P^\infty) \longrightarrow D^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

For a complex oriented cohomology theory, the image of the first Chern class under  $m^*$  is the formal group law of the cohomology theory. In particular, if  $D = K$ , we obtain the multiplicative formal group law. Applying realification, we see that for  $D = KO$ ,

$$m^*(u_i) = u_i + v_i + w_i.$$

### 8.3 Thom Algebra

We now discuss the Thom algebras of  $\eta \times \eta$  over  $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$  and of  $\eta(m) \times \eta(n)$  over  $\mathbb{C}P^m \times \mathbb{C}P^n$ . Since these bundles are not *Spin*, the Thom algebras have more than one generator and several relations.

The zero section  $\iota: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow T\eta \times \eta$  induces a monomorphism in complex  $K$ -theory. Since  $\iota^*(t) = x_1x_2$ , the module  $z^i\mathbb{Z}[[x_1, x_2]] \cdot t \cong K^{-2i}(T\eta \times \eta)$  has image  $z^ix_1x_2\mathbb{Z}[[x_1, x_2]] \cong K^{-2i}(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty)$  in  $K^{-2i}(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ . Complexification is a monomorphism on both  $KO^*(T\eta \times \eta)$  and  $KO^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ , and commutativity of  $c$  with  $\iota^*$  demonstrates that  $\iota^*: KO^{-2i}(T\eta \times \eta) \rightarrow KO^{-2i}(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$  is a monomorphism with image  $KO^{-2i}(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty)$ .

Referring to Propositions 8.2.20 and 3.2.10, (8.4) gives the following description of  $KO^{-2i}(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty)$ .

**Lemma 8.3.1.** *For each integer  $i$ , we have the isomorphism of direct sums of power series modules*

$$KO^{-2i}(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty) \cong w_i\mathbb{Z}[[u_0, v_0]] \oplus u_iv_0\mathbb{Z}[[u_0, v_0]].$$

We showed above that  $KO^*(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty) \subset KO^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$  is the Euler ideal, so every element  $u \in KO^*(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty)$  is Eulerian and corresponds to a unique element  $(\iota^*)^{-1}(u) = \langle u \rangle \in KO^*(T\eta \times \eta)$ . We have the relations

$$v \cdot \langle u \rangle = \langle vu \rangle \quad \text{and} \quad \langle uu' \rangle = \langle u \rangle \langle u' \rangle \tag{8.8}$$

for every  $u, u' \in KO^*(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty)$  and every  $v \in KO^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$ . Thus from Lemma 8.3.1 and (8.8) we obtain the following result.

**Corollary 8.3.2.** *For each integer  $i$ , we have the isomorphism of direct sums of power series modules*

$$KO^{-2i}(T\eta \times \eta) \cong \mathbb{Z}[[u_0, v_0]] \cdot \langle w_i \rangle \oplus \mathbb{Z}[[u_0, v_0]] \cdot \langle u_iv_0 \rangle.$$

We read off the structure of the Euler ideal directly from Proposition 8.2.20, and lift the results back to  $KO^*(T\eta \times \eta)$ . Using the notation of (2.10), we have the

following result. For brevity we will use  $\lambda'_i$  and  $\langle \lambda'_i \rangle$  to denote  $w_i$  or  $u_i v_0$  and  $\langle w_i \rangle$  or  $\langle u_i v_0 \rangle$  respectively. We still use  $\lambda_i$  to denote any of  $u_i$ ,  $v_i$  and  $w_i$  as before.

**Proposition 8.3.3.** *The Thom module  $KO^*(T(\eta \times \eta))$  is generated over  $KO^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$  by the elements  $\langle w_0 \rangle$ ,  $\langle w_1 \rangle$ ,  $\langle w_2 \rangle$ ,  $\langle w_3 \rangle$ ,  $\langle u_0 v_0 \rangle$ ,  $\langle u_1 v_0 \rangle$ ,  $\langle u_2 v_0 \rangle$  and  $\langle u_3 v_0 \rangle$ , with relations*

1.  $e \cdot \langle \lambda'_0 \rangle = e \cdot \langle \lambda'_1 \rangle = e \cdot \langle \lambda'_2 \rangle = e \cdot \langle \lambda'_3 \rangle = 0$ ,
2.  $\alpha \cdot \langle \lambda'_0 \rangle = 2\langle \lambda'_2 \rangle$ ,  $\alpha \cdot \langle \lambda'_1 \rangle = 2\langle \lambda'_3 \rangle$ ,  $\alpha \cdot \langle \lambda'_2 \rangle = 2\beta \cdot \langle \lambda'_0 \rangle$ ,  $\alpha \cdot \langle \lambda'_3 \rangle = 2\beta \cdot \langle \lambda'_1 \rangle$ ,
3.  $\lambda_j \cdot \langle \lambda'_0 \rangle = \lambda_{j-2} \cdot \langle \lambda'_2 \rangle$ ,  $\lambda_j \cdot \langle \lambda'_1 \rangle = \lambda_{j-2} \cdot \langle \lambda'_3 \rangle$ ,
4.  $u_{4j+k} \cdot \langle w_0 \rangle = (u_{4j} + 4\beta^j) \cdot \langle w_k \rangle - 2\beta^j \cdot \langle u_k v_0 \rangle$ , ( $k = 1$  or  $3$ ),  
 $u_{4j+k} \cdot \langle w_1 \rangle = u_{4j} \cdot \langle w_{k+1} \rangle - 2\beta^j \cdot \langle u_{k+1} v_0 \rangle$ , ( $k = \pm 1$ ),
5.  $v_{4j+k} \cdot \langle w_0 \rangle = v_{4j} \cdot \langle w_k \rangle + 2\beta^j \cdot \langle u_k v_0 \rangle$ , ( $k = 1$  or  $3$ ),  
 $v_{4j+k} \cdot \langle w_1 \rangle = v_{4j} \cdot \langle w_{k+1} \rangle + 2\beta^j \cdot \langle u_{k+1} v_0 \rangle$ , ( $k = \pm 1$ ),
6.  $w_{4j+k} \cdot \langle w_0 \rangle = u_{4j} v_0 \cdot \langle w_k \rangle + (u_{4j} + v_{4j}) \cdot \langle u_k v_0 \rangle$ , ( $k = 0$  or  $2$ ),  
 $w_{4j+k} \cdot \langle w_0 \rangle = (u_{4j} v_0 + 2v_{4j}) \cdot \langle w_k \rangle + (u_{4j} - v_{4j}) \cdot \langle u_k v_0 \rangle$ , ( $k = 1$  or  $3$ ),  
 $w_{4j+k} \cdot \langle w_1 \rangle = u_{4j+2} v_0 \cdot \langle w_{k-1} \rangle + (u_{4j+2} + v_{4j+2}) \cdot \langle u_{k-1} v_0 \rangle$ , ( $k = 1$  or  $3$ ),
7.  $u_{4j+k} \cdot \langle u_1 v_0 \rangle = (u_{4j} + 4\beta^j) \cdot \langle u_{k+1} v_0 \rangle$ , ( $k = \pm 1$ ),  
 $u_{4j+k} \cdot \langle u_0 v_0 \rangle = u_{4j} \cdot \langle u_k v_0 \rangle$ , ( $k = 1$  or  $3$ ),
8.  $v_{4j+k} \cdot \langle u_0 v_0 \rangle = 2v_{4j} \cdot \langle w_k \rangle - v_{4j} \cdot \langle u_k v_0 \rangle$ , ( $k = 1$  or  $3$ ),  
 $v_{4j+k} \cdot \langle u_1 v_0 \rangle = 2v_{4j} \cdot \langle w_{k+1} \rangle - v_{4j} \cdot \langle u_{k+1} v_0 \rangle$ , ( $k = \pm 1$ ),
9.  $w_{4j+k} \cdot \langle u_0 v_0 \rangle = u_{4j} v_0 \cdot \langle w_k \rangle$ , ( $k = 0, 1, 2$  or  $3$ ),  
 $w_{4j+k} \cdot \langle u_1 v_0 \rangle = u_{4j} v_0 \cdot \langle w_{k+1} \rangle + 2v_{4j} \cdot \langle u_{k+1} v_0 \rangle$ , ( $k = \pm 1$ ),  
 $w_{4j+k} \cdot \langle u_1 v_0 \rangle = (u_{4j} v_0 + 4v_{4j}) \cdot \langle w_{k+1} \rangle - 2v_{4j} \cdot \langle u_{k+1} v_0 \rangle$ , ( $k = 0$  or  $2$ ).

The Thom algebra is generated by the same elements, with the additional relation

$$\langle \lambda'_i \rangle \langle \lambda'_j \rangle = \lambda'_i \cdot \langle \lambda'_j \rangle$$

for all integers  $0 \leq i, j \leq 3$ .

Alternative choices of generators are of course possible, since the periodicity relation  $\beta^k \cdot \langle \lambda_i \rangle = \langle \lambda_{i+4k} \rangle$  holds for all integers  $i$  and  $k$ .

For the complex 2-plane bundle  $\eta(m) \times \eta(n)$  over  $\mathbb{C}P^m \times \mathbb{C}P^n$ , we pull the generators  $\langle u_i v_0 \rangle, \langle w_i \rangle \in KO^{-2i}(\mathbb{C}P^m \times \mathbb{C}P^n)$  back along the map of Thom complexes induced by the inclusion  $\mathbb{C}P^m \times \mathbb{C}P^n \rightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ . We denote the pullbacks as  $\langle u_i v_0 \rangle, \langle w_i \rangle \in KO^*(T(\eta(m) \times \eta(n)))$ .

**Proposition 8.3.4.** *The Thom module  $KO^*(T(\eta(2m-1) \times \eta(2n-1)))$  is generated over  $KO^*((\mathbb{C}P^{2m-1} \times \mathbb{C}P^{2n-1})_+)$  by the elements  $\langle w_0 \rangle, \langle w_1 \rangle, \langle w_2 \rangle, \langle w_3 \rangle, \langle u_0 v_0 \rangle, \langle u_1 v_0 \rangle, \langle u_2 v_0 \rangle$  and  $\langle u_3 v_0 \rangle$  with the relations listed in Proposition 8.3.3 plus the following additional relations.*

1.  $\sigma_{1-2m} \cdot \langle u_0 v_0 \rangle = \sigma_{1-2m} \cdot \langle u_2 v_0 \rangle = 0,$   

$$\sigma_{1-2m} \cdot \langle u_1 v_0 \rangle = \begin{cases} \beta^{-\frac{m}{2}} u_0^{m-1} \cdot \langle u_2 v_0 \rangle, & m \text{ even,} \\ \beta^{\frac{1-m}{2}} u_0^{m-1} \cdot \langle u_0 v_0 \rangle, & m \text{ odd,} \end{cases}$$
  

$$\sigma_{1-2m} \cdot \langle u_3 v_0 \rangle = \begin{cases} \beta^{\frac{2-m}{2}} u_0^{m-1} \cdot \langle u_0 v_0 \rangle, & m \text{ even,} \\ \beta^{\frac{1-m}{2}} u_0^{m-1} \cdot \langle u_2 v_0 \rangle, & m \text{ odd,} \end{cases}$$
2.  $\sigma_{1-2n} \cdot \langle u_0 v_0 \rangle = \sigma_{1-2n} \cdot \langle u_1 v_0 \rangle = \sigma_{1-2n} \cdot \langle u_2 v_0 \rangle = \sigma_{1-2n} \cdot \langle u_3 v_0 \rangle = 0,$
3. 
$$\sigma_{1-2m} \cdot \langle w_0 \rangle = \begin{cases} 2\beta^{-\frac{m}{2}} u_0^{m-1} \cdot \langle w_1 \rangle - \beta^{-\frac{m}{2}} u_0^{m-1} \cdot \langle u_1 v_0 \rangle, & m \text{ even,} \\ 2\beta^{-\frac{1-m}{2}} u_0^{m-1} \cdot \langle w_3 \rangle - \beta^{-\frac{1-m}{2}} u_0^{m-1} \cdot \langle u_3 v_0 \rangle, & m \text{ odd,} \end{cases}$$
  

$$\sigma_{1-2m} \cdot \langle w_1 \rangle = \begin{cases} \beta^{-\frac{m}{2}} u_0^{m-1} \cdot \langle u_2 v_0 \rangle, & m \text{ even,} \\ \beta^{\frac{1-m}{2}} u_0^{m-1} \cdot \langle u_0 v_0 \rangle, & m \text{ odd,} \end{cases}$$
  

$$\sigma_{1-2m} \cdot \langle w_2 \rangle = \begin{cases} 2\beta^{-\frac{m}{2}} u_0^{m-1} \cdot \langle w_3 \rangle - \beta^{-\frac{m}{2}} u_0^{m-1} \cdot \langle u_3 v_0 \rangle, & m \text{ even,} \\ 2\beta^{\frac{1-m}{2}} u_0^{m-1} \cdot \langle w_1 \rangle - \beta^{\frac{1-m}{2}} u_0^{m-1} \cdot \langle u_1 v_0 \rangle, & m \text{ odd,} \end{cases}$$
  

$$\sigma_{1-2m} \cdot \langle w_3 \rangle = \begin{cases} \beta^{\frac{2-m}{2}} u_0^{m-1} \cdot \langle u_0 v_0 \rangle, & m \text{ even,} \\ \beta^{\frac{1-m}{2}} u_0^{m-1} \cdot \langle u_2 v_0 \rangle, & m \text{ odd,} \end{cases}$$
4. 
$$\sigma_{1-2n} \cdot \langle w_0 \rangle = \begin{cases} \beta^{-\frac{n}{2}} v_0^{n-1} \cdot \langle u_1 v_0 \rangle, & n \text{ even,} \\ \beta^{-\frac{1-n}{2}} v_0^{n-1} \cdot \langle u_3 v_0 \rangle, & n \text{ odd,} \end{cases}$$
  

$$\sigma_{1-2n} \cdot \langle w_1 \rangle = \begin{cases} \beta^{-\frac{n}{2}} v_0^{n-1} \cdot \langle u_2 v_0 \rangle, & n \text{ even,} \\ \beta^{\frac{1-n}{2}} v_0^{n-1} \cdot \langle u_0 v_0 \rangle, & n \text{ odd,} \end{cases}$$

$$\begin{aligned}\sigma_{1-2n} \cdot \langle w_2 \rangle &= \begin{cases} \beta^{\frac{-n}{2}} v_0^{n-1} \cdot \langle u_3 v_0 \rangle, & n \text{ even,} \\ \beta^{\frac{1-n}{2}} v_0^{n-1} \cdot \langle u_1 v_0 \rangle, & n \text{ odd,} \end{cases} \\ \sigma_{1-2n} \cdot \langle w_3 \rangle &= \begin{cases} \beta^{\frac{2-n}{2}} v_0^{n-1} \cdot \langle u_0 v_0 \rangle, & n \text{ even,} \\ \beta^{\frac{1-n}{2}} v_0^{n-1} \cdot \langle u_2 v_0 \rangle, & n \text{ odd.} \end{cases}\end{aligned}$$

The Thom algebra is generated by the same elements, with the additional relation

$$\langle \lambda'_i \rangle \langle \lambda'_j \rangle = \lambda'_i \cdot \langle \lambda'_j \rangle$$

for all integers  $0 \leq i, j \leq 3$ .

Alternative choices of generators are of course possible, since the periodicity relation  $\beta^k \cdot \langle \lambda'_i \rangle = \langle \lambda'_{i+4k} \rangle$  holds for all integers  $i$  and  $k$ .

Since  $u_0^m$  and  $v_0^n$  in  $KO^0((\mathbb{C}P^{2m-1} \times \mathbb{C}P^{2n-1})_+)$  have either  $e^2$  or 0 as a factor, we have the relations  $u_0^m \cdot \langle w_i \rangle = v_0^n \cdot \langle w_i \rangle = u_0^m \cdot \langle u_i v_0 \rangle = v_0^n \cdot \langle u_i v_0 \rangle = 0$  for any integer  $i$  (since  $e \cdot \langle \lambda'_i \rangle = 0$ ).

*Proof.* Each additional relation may be proved by complexification, which is a monomorphism on  $KO^{-2i}(T(\eta(2m-1) \times \eta(2n-1)))$ . Alternatively, since  $2\sigma_{1-2m} = u_{1-2m} u_0^{m-1}$  and  $2\sigma_{1-2n} = v_{1-2n} v_0^{n-1}$ , we compute  $2\sigma_i \cdot \langle \lambda'_j \rangle$  using Lemma 8.2.14 and Proposition 8.2.17, then divide by 2. We use the fact that  $KO^*(T(\eta(2m-1) \times \eta(2n-1)))$  is free from 2-torsion.

Applying the additional relations  $u_0^m \cdot \langle w_i \rangle = v_0^n \cdot \langle w_i \rangle = u_0^m \cdot \langle u_i v_0 \rangle = v_0^n \cdot \langle u_i v_0 \rangle = 0$  to the description of  $KO^{-2i}(T\eta \times \eta)$  given in Corollary 8.3.2, it follows that the union of  $\{\beta^{\frac{i-i'}{4}} u_0^j v_0^k \cdot \langle w_{i'} \rangle \mid 0 \leq j \leq m-1, 0 \leq k \leq n-1\}$  and  $\{\beta^{\frac{i-i'}{4}} u_0^j v_0^k \cdot \langle u_{i'} v_0 \rangle \mid 0 \leq j \leq m-1, 0 \leq k \leq n-1\}$ , where  $i'$  is the mod 4 reduction of  $i$ , is a basis for  $KO^{-2i}(T(\eta(2m-1) \times \eta(2n-1)))$ .  $\square$

**Proposition 8.3.5.** *The Thom module  $KO^*(T(\eta(2m) \times \eta(2n-1)))$  is generated over  $KO^*((\mathbb{C}P^{2m} \times \mathbb{C}P^{2n-1})_+)$  by the elements  $\langle w_0 \rangle, \langle w_1 \rangle, \langle w_2 \rangle, \langle w_3 \rangle, \langle u_0 v_0 \rangle, \langle u_1 v_0 \rangle, \langle u_2 v_0 \rangle$  and  $\langle u_3 v_0 \rangle$ , with the relations listed in Proposition 8.3.3 plus the following additional relations.*

1.  $u_0^m \cdot \langle u_0 v_0 \rangle = u_0^m \cdot \langle u_1 v_0 \rangle = u_0^m \cdot \langle u_2 v_0 \rangle = u_0^m \cdot \langle u_3 v_0 \rangle = 0,$

$$2. \sigma_{1-2n} \cdot \langle u_0 v_0 \rangle = \sigma_{1-2n} \cdot \langle u_1 v_0 \rangle = \sigma_{1-2n} \cdot \langle u_2 v_0 \rangle = \sigma_{1-2n} \cdot \langle u_3 v_0 \rangle = 0$$

$$3. \sigma_{1-2n} \cdot \langle w_0 \rangle = \begin{cases} \beta^{-\frac{n}{2}} v_0^{n-1} \cdot \langle u_1 v_0 \rangle, & n \text{ even,} \\ \beta^{-\frac{1-n}{2}} v_0^{n-1} \cdot \langle u_3 v_0 \rangle, & n \text{ odd,} \end{cases}$$

$$\sigma_{1-2n} \cdot \langle w_1 \rangle = \begin{cases} \beta^{-\frac{n}{2}} v_0^{n-1} \cdot \langle u_2 v_0 \rangle, & n \text{ even,} \\ \beta^{\frac{1-n}{2}} v_0^{n-1} \cdot \langle u_0 v_0 \rangle, & n \text{ odd,} \end{cases}$$

$$\sigma_{1-2n} \cdot \langle w_2 \rangle = \begin{cases} \beta^{-\frac{n}{2}} v_0^{n-1} \cdot \langle u_3 v_0 \rangle, & n \text{ even,} \\ \beta^{\frac{1-n}{2}} v_0^{n-1} \cdot \langle u_1 v_0 \rangle, & n \text{ odd,} \end{cases}$$

$$\sigma_{1-2n} \cdot \langle w_3 \rangle = \begin{cases} \beta^{\frac{2-n}{2}} v_0^{n-1} \cdot \langle u_0 v_0 \rangle, & n \text{ even,} \\ \beta^{\frac{1-n}{2}} v_0^{n-1} \cdot \langle u_2 v_0 \rangle, & n \text{ odd.} \end{cases}$$

The Thom algebra is generated by the same elements, with the additional relation

$$\langle \lambda'_i \rangle \langle \lambda'_j \rangle = \lambda'_i \cdot \langle \lambda'_j \rangle$$

for all integers  $0 \leq i, j \leq 3$ .

Alternative choices of generators are of course possible, since the periodicity relation  $\beta^k \cdot \langle \lambda'_i \rangle = \langle \lambda'_{i+4k} \rangle$  holds for all integers  $i$  and  $k$ .

Since  $u_0^{m+1}$  and  $v_0^n$  in  $KO^0((\mathbb{C}P^{2m} \times \mathbb{C}P^{2m-1})_+)$  have either  $e^2$  or 0 as a factor, we have the additional relations  $u_0^{m+1} \cdot \langle w_i \rangle = v_0^n \cdot \langle w_i \rangle = v_0^n \cdot \langle u_i v_0 \rangle = 0$  (since  $e \cdot \langle \lambda'_i \rangle = 0$ ).

*Proof.* Firstly, we note that the relations listed in Proposition 8.3.3 are still applicable. The relation  $u_0^m \cdot \langle u_i v_0 \rangle = 0$  is immediate by complexification. The remaining relations, of the form  $\sigma_{1-2n} \cdot \langle \lambda'_i \rangle$ , may be proved either by complexification (note that complexification is injective), or by multiplying  $\sigma_{1-2n}$  by 2, using the relations of Proposition 8.3.3 and then dividing by 2 (note that  $KO^*(T(\eta(2m) \times \eta(2m-1)))$  is free of 2-torsion).

Applying the additional relations to the description of  $KO^{-2i}(T\eta \times \eta)$  given in Corollary 8.3.2, we deduce that the union of the sets  $\{\beta^{\frac{i-i'}{4}} u_0^j v_0^k \cdot \langle w_{i'} \rangle \mid 0 \leq j \leq m, 0 \leq k \leq n-1\}$  and  $\{\beta^{\frac{i-i'}{4}} u_0^j v_0^k \cdot \langle u_{i'} v_0 \rangle \mid 0 \leq j \leq m-1, 0 \leq k \leq n-1\}$ , where  $i'$  is the mod 4 reduction of  $i$ , is a basis for  $KO^{-2i}(T(\eta(2m) \times \eta(2m-1)))$ . Our relations are sufficient to write any product  $\lambda_j \cdot \langle \lambda'_i \rangle$  as a linear combination of basis elements.  $\square$



**Proposition 8.3.6.** *The Thom module  $KO^*(T(\eta(2m) \times \eta(2n)))$  is generated over  $KO^*((\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})_+)$  by the elements  $\langle w_0 \rangle, \langle w_1 \rangle, \langle w_2 \rangle, \langle w_3 \rangle, \langle u_0 v_0 \rangle, \langle u_1 v_0 \rangle, \langle u_2 v_0 \rangle, \langle u_3 v_0 \rangle$  and  $\sigma_{-2(m+n+1)}$ . Relations consist of those listed in Proposition 8.3.3 plus the following additional relations:*

1.  $u_0^m \cdot \langle u_0 v_0 \rangle = u_0^m \cdot \langle u_2 v_0 \rangle = 0,$   
 $u_0^m \cdot \langle u_1 v_0 \rangle = 2u_0^m \cdot \langle w_1 \rangle,$   
 $u_0^m \cdot \langle u_3 v_0 \rangle = 2u_0^m \cdot \langle w_3 \rangle,$
2.  $v_0^n \cdot \langle u_0 v_0 \rangle = v_0^n \cdot \langle u_1 v_0 \rangle = v_0^n \cdot \langle u_2 v_0 \rangle = v_0^n \cdot \langle u_3 v_0 \rangle = 0,$
3.  $u_0^m v_0^n \cdot \langle w_0 \rangle = \begin{cases} \alpha \beta^{\frac{m+n}{2}} \cdot \sigma_{-2(m+n+1)}, & m+n \text{ even,} \\ 2\beta^{\frac{m+n+1}{2}} \cdot \sigma_{-2(m+n+1)}, & m+n \text{ odd,} \end{cases}$   
 $u_0^m v_0^n \cdot \langle w_1 \rangle = \begin{cases} 0, & m+n \text{ even,} \\ e^2 \beta^{\frac{m+n+1}{2}} \cdot \sigma_{-2(m+n+1)}, & m+n \text{ odd,} \end{cases}$   
 $u_0^m v_0^n \cdot \langle w_2 \rangle = \begin{cases} 2\beta^{\frac{m+n+2}{2}} \cdot \sigma_{-2(m+n+1)}, & m+n \text{ even,} \\ \alpha \beta^{\frac{m+n+1}{2}} \cdot \sigma_{-2(m+n+1)}, & m+n \text{ odd,} \end{cases}$   
 $u_0^m v_0^n \cdot \langle w_3 \rangle = \begin{cases} e^2 \beta^{\frac{m+n+2}{2}} \cdot \sigma_{-2(m+n+1)}, & m+n \text{ even,} \\ 0, & m+n \text{ odd.} \end{cases}$

The Thom algebra is generated by the same elements, with the additional relations

$$\langle \lambda'_i \rangle \langle \lambda'_j \rangle = \lambda'_i \cdot \langle \lambda'_j \rangle$$

$$\langle \lambda'_i \rangle \sigma_{-2(m+n+1)} = 0 \pmod{e^2 \cdot \sigma_{-2(m+n+1)}}$$

and

$$(\sigma_{-2(m+n+1)})^2 = 0$$

for all integers  $0 \leq i, j \leq 3$ .

*Proof.* Similar to the proof of Proposition 8.3.4. We note that the relations listed in Proposition 8.3.3 are applicable.

We obtain the relation  $\sigma_{-2(m+n+1)} \langle \lambda'_i \rangle = 0$  by complexifying, which annihilates  $e^2 \cdot \sigma_{-2(m+n+1)} \in KO^{4m+4n+2}(T\eta(2n))$ , but is injective on the quotient  $KO^*(T\eta(2m) \times \eta(2n)) / (e^2 \cdot \sigma_{-2(m+n+1)})$ .

From previous results, we deduce the isomorphism  $KO^*(T(\eta(2m) \times \eta(2m))) \cong \mathbb{Z}(2(mn + m + n)) \oplus KO^*(S^{4(m+n+1)})$ . Applying the additional relations to Corollary 8.3.2, we deduce that the union of the sets  $\{\beta^{\frac{i-i'}{4}} u_0^j v_0^k \cdot \langle w_{i'} \rangle \mid 0 \leq j \leq m, 0 \leq k \leq n, j + k < m + n\}$  and  $\{\beta^{\frac{i-i'}{4}} u_0^j v_0^k \cdot \langle u_{i'} v \rangle_0 \mid 0 \leq j \leq m - 1, 0 \leq k \leq n - 1\}$  for all  $i \in \mathbb{Z}$  give us a basis for the summand  $\mathbb{Z}(2(mn + m + n))$ . For the remaining summand,  $KO^*(S^{4(m+n+1)})$ , we identify  $\sigma_{-2(m+n+1)} := r(z^{-2(m+n+1)} x_1^{2m} x_2^{2n} \cdot t)$  with  $\sigma_{-2m-1} \sigma_{-2n-1} = r(z^{-2(m+n+1)} x_1^{2m+1} x_2^{2n+1})$  and referring to Proposition 8.2.19, we have the isomorphism  $KO^*(S^{4(m+n+1)}) \cong KO_*[\sigma_{-2(m+n+1)}] / (\sigma_{-2(m+n+1)}^2)$ , and the relations  $2 \cdot \sigma_{-2(m+n+1)} = u_0^m v_0^n \cdot \langle w_{-2(m+n+1)} \rangle$ ,  $e^2 \cdot \sigma_{-2(m+n+1)} = u_0^m v_0^n \cdot \langle w_{-2m-2n-1} \rangle$ ,  $x \cdot \sigma_{-2(m+n+1)} = u_0^m v_0^n \cdot \langle w_{-2(m+n)} \rangle$  and  $u_0^m v_0^n \cdot \langle w_{-2m-2n+1} \rangle = 0$ . So we require the nine generators  $\langle \lambda'_0 \rangle, \langle \lambda'_1 \rangle, \langle \lambda'_2 \rangle, \langle \lambda'_3 \rangle$  and  $\sigma_{-2(m+n+1)}$ .

As before, we cannot express any  $\langle \lambda'_i \rangle$  as a product, and  $\sigma_{-2(m+n+1)}$  cannot be expressed as a product of  $\langle \lambda'_i \rangle$ s (Proposition 8.2.19). Thus our generators are minimal.

The relations we have so far allow us to write an arbitrary product  $\lambda_j \cdot \langle \lambda'_i \rangle$  as a linear combination of elements  $\beta^{\frac{i-i'}{4}} u_0^j v_0^k \cdot \langle \lambda'_{i'} \rangle$  where  $j, k \geq 0$  and  $i'$  is the mod 4 reduction of  $i$ . But since  $u_0^{m+1}$  and  $v_0^{n+1}$  are equal to the zero element in  $KO^0((\mathbb{C}P^{2m} \times \mathbb{C}P^{2n})_+)$ , we have the additional relations  $u_0^{m+1} \cdot \langle w_i \rangle = v_0^{n+1} \cdot \langle w_i \rangle = 0$ . We also have the relations  $u_0^m \cdot \langle u_{2i} v_0 \rangle = v_0^n \cdot \langle u_i v_0 \rangle = 0$  and  $u_0^m \cdot \langle u_{2i+1} v_0 \rangle = 2u_0^m \cdot \langle w_{2i+1} \rangle$ , which may be proved by complexification, since the products lie in the summand  $\mathbb{Z}(2(mn + m + n))$  which is torsion free. So our relations are sufficient, as we may write an arbitrary product of elements as a linear combination of basis elements.

□

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