Sea Ice Ridging Schemes

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Abstract

As sea ice is advected on the surface of the ocean, the ice concentration (0 ≤ A ≤ 1) and the mean ice thickness change in response to thermodynamic and mechanical forcing. In this paper the authors review the existing advection schemes and compare their properties in the absence of thermodynamic effects. In Hibler's classical scheme, the ice area fraction at a material particle changes due to the divergence of the large-scale horizontal velocity field, and a further constraint is applied in order to keep A ≤ 1. This scheme is used in almost all sea ice models, although Hibler and Shinohara have both since included a ridging sink term. In this paper the authors show that the Hibler advection scheme is a special case of Gray and Morland's ridging model and compare the ridging schemes of Hibler and Shinohara with the simple scheme of Gray and Morland. It is demonstrated that the Hibler scheme still allows ice concentrations to exceed unity in maintained convergence and that both Hibler and Shinohara schemes admit the possibility of negative ice concentrations during maintained shearing. A general framework is formulated for the functional form of the ridging sink term that guarantees 0 ≤ A ≤ 1. Finally, some elementary analytic solutions are derived, which imply that, if ridging is independent of shear effects, quantities are conserved along particle paths.

1. Introduction

Extreme atmospheric forcing enables sea water to freeze at the surface of the earth's polar oceans to form a thin horizontal covering of saline ice called sea ice. This layer is not continuous but is broken up into discrete coherent plates (fleets) separated by regions of open water (leads). On large enough scales the ice concentration A can be defined as the area fraction of ice floes per unit area (of pack), and the area fraction of open water equals 1 − A. It follows that at A = 0 only open water is present and there is a continuous distribution of ice concentrations up to A = 1, which represents a complete ice coverage. In response to the wind and ocean currents sea ice drifts with speeds of up to 4 km per day in the Arctic and up to 20 km per day in the Antarctic (Fischer 1995). The leads open and close due to the differential motion of the ice floes, and in response the ice concentration changes. Nikiforov (1957) was the first to derive an ice concentration equation to describe how A changes due to the large-scale horizontal velocity field $v$.

Nikiforov's model, however, neglects ridging, which is an important physical process. During a maintained converging flow adjacent floes interact and exert contact forces on one another. These stresses are relieved by fracture, crushing, and vertical displacement to form a ridge. During maintained convergence Nikiforov's (1957) equation predicts ice concentrations greater than unity! This clearly violates the definition that the ice concentration must lie in the range 0 ≤ A ≤ 1. Initially this problem was dealt with by coupling the velocity field in such a way as to bring the ice to rest as A approached unity. Often these were ad hoc schemes that did not conserve linear momentum (Nikiforov et al. 1967; Parkinson and Washington 1979). This approach prevented any redistribution of the ice mass by ridging and the thickness of the ice floes could only change by thermodynamic growth.

The AIDEX Project (Thorndike and Maykut, 1973; Thorndike et al. 1975) developed a complex theory for the evolution of a thickness distribution function that included a sink term (Rothrock 1975) to parameterize the effects of ridging. The simplified model that grew out of this work (Hibler 1979), however, used the same ice concentration equation as Nikiforov (1957) without any additional sink terms. The effect of ridging was parameterized by applying the constraint A ≤ 1, which reset the ice concentration to unity if A > 1 was predicted. Since the overall volume was kept constant, this
implied an increase in the mean thickness of the ice floes and as such represented a very simple ridging model. This scheme is used in almost all sea ice models, although Hibler (1984) and Shinohara (1990) have both since included a ridging sink term in their ice concentration equations.

Recently, Gray and Morland (1994) treated the ice pack as a structured mixture of ice floes and open water, and rationally reduced the three-dimensional mass, momentum, and energy balances to a depth integrated two-dimensional form. The analysis identified an extra contribution to the classical ice concentration equation (Nikiforov 1957) from the vertical velocity field in a ridging zone. This provided a link between the physical mechanism of redistribution by vertical displacement and the ridging sink terms in the ad hoc schemes of earlier models. Gray and Morland (1994) proposed a simple ridging model with a dependence on the rate of convergence and the ice concentration \( A \), which guarantees that the concentration stays in the range \( 0 \leq A \leq 1 \) consistent with its definition. This model can also be derived without the use of mixture theory (Schulkes 1995) by postulating a source term, but the link with the physical mechanism of vertical displacement is lost.

In this paper we show that the Hibler (1979) advection scheme is a special case of the Gray and Morland (1994) ridging model and compare the ridging schemes of Hibler (1984) and Shinohara (1990) with the simple scheme of Gray and Morland (1994). Finally, some elementary analytic solutions are derived, which imply that, if ridging is independent of the shear effects, there are quantities conserved along particle paths.

2. Classical formulation

In addition to the ice concentration the mean ice thickness is another important field that is advected with the ice and can be defined in two ways. The first is defined as the mean ice thickness per unit floe area \( h \), and the second is the mean ice thickness per unit area (of pack) \( h' \). These are related by a linear area fraction scaling

\[
h' = Ah. \tag{2.1}\]

The mean ice thickness per unit area \( h' \) is a measure of the total ice volume, whereas \( h \) is the mean ice thickness of the actual floes. A theory to describe the evolution of the thickness distribution of the ice pack was one of the major achievements of the AIDJEX Project (Thordike and Maykut 1973; Thordike et al. 1975). A simplification to the case when there is only one (mean) thickness level (Hibler 1979) led to the classical ice concentration (Nikiforov 1957) and ice thickness equations:

\[
\frac{DA}{Dt} + A\eta = S_A + \text{diffusion}, \quad A \leq 1 \quad \tag{2.2}\]

\[
\frac{Dh'}{Dt} + h'\eta = S_h + \text{diffusion}, \quad \tag{2.3}\]

where the horizontal material time derivative \( D/Dt \) and the divergence of the horizontal velocity field \( \eta \) are defined as

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + v_\alpha \frac{\partial}{\partial x_\alpha}, \quad \eta = \frac{\partial v_\alpha}{\partial x_\alpha}, \quad \tag{2.4}\]

respectively. The subscript \( \alpha (= 1, 2) \) is used to denote the horizontal in-plane coordinates \( x_1, x_2 \) only. Summation over 1 and 2 is assumed for repeated suffixes. The ice concentration following a material particle, therefore, changes due to the horizontal divergence of the floes on the surface of the ocean and by freezing of the open water, which is accounted for by the thermodynamic source term \( S_A \). The ice volume \( h' \) satisfies an analogous equation with a thermodynamic source term \( S_h \) that accounts for both the increase in volume through surface and basal accumulation on existing ice and through new ice forming in the leads. The diffusion terms were artificially introduced for numerical stability in model integrations with the viscous-plastic rheology (Hibler 1979) and do not represent any physical processes. They are therefore neglected in the rest of this paper.

When Eq. (2.2) predicts \( A > 1 \), the concentration is simply reset back to unity and the computation is continued. The constraint that \( A \leq 1 \) appears at first sight to be overly simplified. However, it does work because the ice volume is unchanged by the resetting process, so a lower ice concentration implies that the ice thickness \( h \) is larger by (2.1). The disadvantages are that the solution is dependent on the numerical time step, and that the effects of ridging are not fed into the momentum balance until \( A = 1 \).

3. A simple ridging model

A rigorous theory for the process of ridging was derived by Gray and Morland (1994) by treating sea ice as a horizontally structured three-dimensional mixture of coherent ice, ridging ice, and open water. By integrating the mass balance equations through the layer thickness the effect of the vertical velocity field, which is the physical mechanism by which mass is redistributed, was formally included in the two-dimensional ice concentration equation. The ice thickness equation was derived by differencing the layer surface and basal kinematic conditions, which also included the effects of vertical velocity. Their ice concentration and ice thickness equations are

\[
\frac{DA}{Dt} + A\eta[1 - \alpha H(-\eta)] = \frac{k}{\rho}, \quad \tag{3.1}\]

\[
\frac{Dh}{Dt} + h\eta\alpha H(-\eta) = q' - b', \quad \tag{3.2}\]
where $k$ is the mass transfer to ice from the water in the leads per unit volume per unit time, the constant $\rho$ is the ice density, and $q', b'$ are the volume fluxes of surface accumulation and basal melt per unit ice floe area. The Heaviside function $H$ is zero in divergence, $\eta \gg 0$, and unity during convergence, $\eta < 0$, and is used to model the irreversibility of the ridging process. That is deformation and redistribution of the ice thickness only occurs during convergence. The model function $\alpha$ defines the ratio of the vertical flux of ice redistributed by the ridging process to the horizontal flux of ice implied by the large-scale horizontal velocity field. The simplest ridging model assumes that $\alpha$ is a function only of the ice concentration, $\alpha = \alpha(A)$. Furthermore, if $\alpha$ satisfies the following properties

$$\alpha(0) = 0 \quad \alpha'(A) \geq 0 \quad \alpha \rightarrow 1 \text{ as } A \rightarrow 1,$$  \hspace{1cm} (3.3)

then in maintained convergence the requirement that the ice concentration tends to unity is satisfied and arises naturally from the balance laws (Gray and Morland 1994) without any further constraints. Note that (3.1) and (3.2) can be combined to yield an equation for the ice volume $h' = Ah$

$$\frac{Dh'}{Dt} + h' \eta = A(q' - b') + \frac{h'k}{A'},$$  \hspace{1cm} (3.4)

which is the same as (2.3) when the thermodynamic source terms have the interpretations $S''_h = k/\rho$ and $S''_b = A(q' - b') + h'k/\rho(A')$. In divergence ($\eta > 0$) and in convergence ($\eta < 0$) when $\alpha = 0$, the continuity equations (3.1) and (3.2) reduce to

$$\frac{DA}{Dt} + A \eta = \frac{k}{\rho},$$  \hspace{1cm} (3.5)

$$\frac{Dh}{Dt} = q' - b'.$$  \hspace{1cm} (3.6)

The ice concentration equation (3.5) is, therefore, identical to the classical formulation (2.2) without the constraint $A \leq 1$. The ice continuity equation (3.5) and ice volume equation (3.4) of Gray and Morland's (1994) theory are the same as the classical theory in divergence and in convergence when $\alpha = 0$. During convergence when $\alpha \neq 0$, however, Gray and Morland's theory introduces a vertical velocity field that transports the failed material to form ridges, which is not present in the classical theory.

Gray and Morland (1994) proposed two simple functional forms $\alpha = \alpha(A)$ that satisfy the requirements (3.3):

1) $\alpha(A) = A^m, \quad 0 \leq A \leq 1, \quad m > 0$ \hspace{1cm} (3.7)

2) $\alpha(A) = \begin{cases} 0, & 0 \leq A \leq A_f \\ \frac{A - A_f}{1 - A_f}, & 0 < A_f < A < 1, \end{cases}$ \hspace{1cm} (3.8)

and which provide the flexibility to adjust the ridging effects as a function of $A$ in at least a qualitatively plausible manner. The constant $A_f$ is the threshold concentration below which no ridging takes place. It is also instructive to consider the ridging function

$$\alpha(A) = \begin{cases} 0, & 0 \leq A < 1 \\ 1, & A = 1. \end{cases}$$  \hspace{1cm} (3.9)

In divergence and during convergence for ice concentrations $0 \leq A < 1$, this functional form for $\alpha$ yields the classical continuity equations (e.g., as used by Hibler 1979), without the constraint $A \leq 1$ as in (3.5) and (3.6). When $A$ reaches unity in convergence, the continuity equations reduce to

$$\frac{DA}{Dt} = \frac{k}{\rho},$$  \hspace{1cm} (3.10)

$$\frac{Dh}{Dt} + h' \eta = q' - b'.$$  \hspace{1cm} (3.11)

Note that since $h' = h$ when $A = 1$, these equations are precisely what was intended by the constraint $A \leq 1$ in the classical equations (2.2) and (2.3). That is, there is no further increase in the ice concentration due to convergence, so $A$ remains at unity (provided that there is no contribution from the thermodynamic source terms) and the ice thickness $h$ grows to accommodate the horizontal flux of material. It follows that the classical equations (2.2) and (2.3) are a special case of Gray and Morland's (1994) ridging theory, which is delivered by the third ridging function (3.9).

4. Riddging models with shear effects

Hibler (1984) and Shinhara (1990) have included ridging effects in their ice concentration equations. The treatments explicitly assume the existence of a sink term due to ridging in the ice concentration equation, but the source of this contribution from vertical velocity jumps is not identified. Both authors use an energy consistency argument (Rothrock 1975) with an assumed elliptic yield curve to derive the appropriate form for this source term. Shinhara (1990) also considers the cases of pure divergence and pure convergence to imply a further constraint on the ridging sink term. This further step has the very practical significance of preventing ice concentrations greater than unity. However, as we shall demonstrate, the scheme of Hibler (1984) still allows ice concentrations to exceed unity in maintained convergence and both schemes admit the possibility of negative ice concentrations during maintained shearing.

To discuss these models, we shall consider modified ice concentration and thickness equations of the form

$$\frac{DA}{Dt} + A \eta = \psi,$$  \hspace{1cm} (4.1)

$$\frac{Dh}{Dt} + \frac{h' \psi}{A} = 0.$$  \hspace{1cm} (4.2)
where $\psi$ is the ridging parameterization and all thermal effects have been neglected. In this discussion we shall adopt the final form of the Hibler (1984) theory as stated by Flato and Hibler (1991), which differs only in respect to the strength parameterization. The ridging functions of Hibler (1984) and Shinohara (1990) are

$$
\psi_H = C(A) \frac{1}{2} \left\{ |D_{11} + D_{22}| - [1.25(D_{11}^2
+ D_{22}) + D_{12}^2 + 1.5D_{11}D_{22}]^{1/2} \right\} \quad (4.2)
$$

$$
\psi_S = C(A) \frac{1}{2} \text{div}(v)
\times \left[ 1 - \left\{ (1 + e^{-2}) + (1 - e^{-2}) \sin 2\theta \right\}^{1/2} \cos \theta + \sin \theta \right], \quad (4.3)
$$

respectively, where the suffix $H$ and $S$ has been used to differentiate between the two cases. The ridging parameterization $\psi_S$ and $\psi_H$ are both expressed as functions of the ice concentration $A$ and the horizontal strain-rate tensor $D$. The functional dependence on the ice concentration is given by the function $C(A)$, which is defined as $C(A) = \exp[-C^*(1 - A)]$ in both cases, where $C^* = 20$. The functions of the strain-rate tensor are more difficult to compare as Hibler (1984) uses components in a Cartesian coordinate system, while Shinohara (1990) uses a mixture of vectorial notation and a polar representation in principal strain-rate space using the variable $\theta$. This is defined in (4.5) after a full explanation of the various representations. Note that Shinohara’s (1990) mixed notation could easily lead to a very bad numerical implementation, as the div and term appears (at first sight) to be uncoupled from the $\theta$ dependence. The term in square brackets has a singularity at $\theta = 3\pi/4$ and $7\pi/4$, which would imply unbounded ridging if one was not aware of the fact that this state corresponds to pure shear when div $v = 0$. The product gives a finite contribution, as we show in the coordinate independent form (4.12) and principal polar representation (4.16).

Several basic definitions are required before proceeding. In Cartesian coordinates the horizontal strain-rate tensor $D$ has components $D_{\alpha\beta} = \frac{1}{2}(\partial v_\alpha/\partial x_\beta + \partial v_\beta/\partial x_\alpha)$ where $\alpha, \beta = 1, 2$. In principal axes a rotated coordinate system is constructed in which the off-diagonal elements, $D_{12}$ and $D_{21}$, are identically zero. This leaves two components, $D_1$ and $D_2$, on the leading diagonal that are related to the components in Cartesian coordinates by

$$
D_1, D_2 = \frac{1}{2} (D_{11} + D_{22})
\pm \frac{1}{2} [(D_{11} - D_{22})^2 + 4D_{12}^2]^{1/2}. \quad (4.4)
$$

This has the advantage that an arbitrary strain-rate state can be represented by two components in principal axes, although the rotation necessary to diagonalize the strain-rate tensor is both a function of space and time. The state of strain rate is often represented in principle axes by the vector $(D_1, D_2)$. A parametric representation can also be used by defining rectangular polar coordinates $(\alpha, \theta)$, where

$$
D = (D_1^2 + D_2^2)^{1/2}, \quad \tan \theta = D_2/D_1, \quad (4.5)
$$

so that $D_1 = D \cos \theta$ and $D_2 = D \sin \theta$. It is precisely this representation that Shinohara (1990) used in defining his ridging function (4.3). The two-dimensional horizontal strain-rate tensor $D$ has two invariants, which are independent of the coordinate system, and these are defined as

$$
\eta = \text{tr}D, \quad \gamma^2 = \frac{1}{2} \text{tr}D^2, \quad (4.6)
$$

where $D = D - \frac{2}{3}I$ is the horizontal deviatoric strain-rate and $I$ is the two-dimensional unit tensor. The component representations of these invariants in the three different coordinate systems defined above are summarized in Table 1.

We introduce a special function $\Delta$, of the strain-rate invariants $\eta$ and $\gamma^2$, which arises as a proportionality factor in the normal flow rule for the elliptical yield curve (Hibler 1979) with the viscous-plastic rheology. This is defined as

$$
\Delta = (\eta^2 + \gamma^2)^{1/2}, \quad (4.7)
$$

where $\epsilon$ is the eccentricity of the ellipse. By substituting for the invariants from Table 1, we can express $\Delta$ in terms of the components in Cartesian, principal, and principal polar axes

$$
\Delta = \left\{ (D_{11} + D_{22})^2 + e^{-2}[(D_{11} - D_{22})^2 + 4D_{12}^2] \right\}^{1/2} \quad (4.8)
$$

$$
\Delta = \left\{ (D_1^2 + D_2^2) + e^{-2}(D_1 - D_2)^2 \right\}^{1/2} \quad (4.9)
$$

$$
\Delta = D \left\{ (1 + e^{-2}) + (1 - e^{-2}) \sin 2\theta \right\}^{1/2}, \quad (4.10)
$$

respectively. The function $\Delta$ is precisely the function that is used by both Hibler (1984) (for the case $\epsilon = 2$) and Shinohara (1990) in their definitions of $\psi$ in (4.2) and (4.3). It is therefore possible to express their ridging functions $\psi_H$ and $\psi_S$ entirely in terms of the invariants in a coordinate independent form.

<table>
<thead>
<tr>
<th>Coordinates</th>
<th>$\eta$</th>
<th>$4\gamma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian</td>
<td>$D_{11} + D_{22}$</td>
<td>$(D_{11} - D_{22})^2 + 4D_{12}^2$</td>
</tr>
<tr>
<td>Principal axes</td>
<td>$D_1 + D_2$</td>
<td>$(D_1 - D_2)^2$</td>
</tr>
<tr>
<td>Principal polar</td>
<td>$D(\cos \theta + \sin \theta)$</td>
<td>$D^2(\cos \theta - \sin \theta)^2$</td>
</tr>
</tbody>
</table>
\[ \psi_H = C(A) \left( \frac{1}{2} \left| \eta \right| - \Delta \right) \quad (4.11) \]
\[ \psi_S = C(A) \left( \frac{1}{2} \eta - \Delta \right) \quad (4.12) \]
\[ \psi_G = C(A) \eta H(-\eta), \quad (4.13) \]

where, for comparison, \( \psi_G \) is the ridging function required to recover Gray and Morland's (1994) ice concentration equation (3.1), provided we define \( C(A) = \alpha_0(A) \) in this case. If we assume that \( C(A) \) is the same function common to all three cases, we note the interesting property that
\[ \psi_H + \psi_G = \psi_S. \quad (4.14) \]

The Hibler (1984) ridging function \( \psi_H \) parameterizes the ridging due only to shear for the elliptic yield curve, while \( \psi_G \) expresses the ridging due to compression. Shinohara's (1990) ridging model combines these two effects. In divergence \( \psi_G = 0 \) and \( \psi_H = \psi_S \). It also follows from the definition of \( \Delta \), in (4.7), that \( \Delta \gg \left| \eta \right| \gg 0 \) and hence that \( \psi_H, \psi_S \ll 0 \) for all possible deformations. Equally the Heaviside function ensures that \( \psi_G \ll 0 \). A negative ridging function implies that the \( \psi \) term in (4.1) will tend to reduce the ice concentration as expected.

The properties of the various schemes are best presented by plotting them as a function of \( \theta \) in principal polar coordinates. This illustrates the magnitude of the ridging term in all possible states of the applied strain rate. The appropriate functional forms are obtained by substituting (4.10) and \( \eta \) (from Table 1) into (4.11) - (4.13) to give
\[ \psi_H = DC(A) \left( \frac{1}{2} \left| \cos \theta + \sin \theta \right| \right. \]
\[ \left. - \left[ (1 + e^{-2}) + (1 - e^{-2}) \sin 2 \theta \right]^{1/2} \right\}, \quad (4.15) \]
\[ \psi_S = DC(A) \left( \frac{1}{2} \left( \cos \theta + \sin \theta \right) \right. \]
\[ \left. - \left[ (1 + e^{-2}) + (1 - e^{-2}) \sin 2 \theta \right]^{1/2} \right\}, \quad (4.16) \]
\[ \psi_G = DC(A) (\cos \theta + \sin \theta) \]
\[ \times H(- (\cos \theta + \sin \theta)). \quad (4.17) \]

Figure 1 plots \( \psi/(DC) \) as a function of \( \theta \) for each of the three cases above. In the range \( -\pi/4 \leq \theta \leq 3\pi/4 \) the motion is divergent, while for \( 3\pi/4 \leq \theta \leq 7\pi/4 \) the motion is convergent. Pure divergence occurs when \( \theta = \pi/4 \), at the point marked \( D \) in Fig. 1, and pure convergence occurs when \( \theta = 5\pi/4 \) at the point marked \( C \). Pure shear occurs at the points marked \( S \) when \( \theta = 3\pi/4, 7\pi/4 \). With the assumption that ridging is isotropic each of the ridging functions should have reflective symmetry in the lines \( \theta = \pi/4 + n\pi/2 \), where \( n \) is integer.

The Hibler (1984) ridging function \( \psi_H \) is symmetric about the lines \( \theta = \pi/4 + n\pi/2 \), which indicates that it does not distinguish between the cases of convergence and divergence. The maximum amount of ridging occurs in the case of pure shear when \( \psi_H = -DC(A)/(\sqrt{2}) \). While in pure divergence and in pure convergence, \( \psi_H = 0 \) and there is no ridging. This implies that during maintained pure convergence, the ridging sink term is switched off in the ice concentration equation (4.1). Gray and Morland (1994) have shown that this situation will almost always lead to \( A \) greater than unity, violating the definition of the ice concentration. It should be noted that the Hibler (1984) ridging parameterization was never intended to replace the constraint \( A \leq 1 \) in the classical formulation, it was simply intended as a mechanism to produce more open water during shear. A number of authors (Lemke et al. 1990; Owens and Lemke 1990; Harder and Lemke 1994) use this ridging scheme in their models. However, although it is not explicitly mentioned, the constraint \( A > 1 \) is still applied (M. Harder 1995, personal communication).

The ridging function of Shinohara (1990) is identical to that of Hibler (1984) in divergence. In particular, in pure shear the divergence \( \eta = 0 \) and both \( \psi_S, \psi_H = -C(A)D^*, \) where \( D^* = D/(\sqrt{2}) > 0 \), so that the ice concentration equation (4.1) reduces to
\[ \frac{DA}{Dt} = -D^* \exp[-C^*(1 - A)]. \quad (4.18) \]
This has the solution
\[ A = 1 - \frac{1}{C^*} \ln \{ C^*D^*t + \exp[C^*(1 - A_0)] \}, \] (4.19)
which is strictly monotone decreasing \((DA/DT < 0)\)
and reaches zero at finite time
\[ t_{\text{zero}} = \frac{1}{D^*C^*} \{ \exp C^* - \exp[C^*(1 - A_0)] \}, \] (4.20)
before becoming negative. This special case demonstrates that both the Hibler (1984) and Shinohara (1990) schemes can produce negative ice concentrations if pure shear is maintained long enough, again violating the definition of the ice concentration. The amount of ridging implied by \(C(A)\) is so small at low concentrations, however, that the time \(t_{\text{zero}}\) (after which the ice concentration becomes negative) is so large that it is unlikely ever to be attained in sea ice simulations. While this example does not raise any serious computational issues, it is of mathematical interest to circumvent these problems and produce a ridging scheme that includes the effects of shear and is consistent with the definition \(0 \leq A \leq 1\). The ridging models of Hibler (1984) and Shinohara (1990) allow negative concentrations because ridging still occurs at the finite, albeit small, amount \(\exp(-C^*)\) when \(A\) reaches zero. A simple modification of \(C(A)\) so that \(C(0) = 0\) shuts off the ridging and prevents negative concentrations.

The theory presented by Gray and Morland (1994) was explicitly constructed to ensure that the mechanical redistribution due to ridging was always sufficient to keep \(A\) in the range \(0 \leq A \leq 1\). The curve \(\psi_g\) in Fig. 1 therefore represents the minimum amount of ridging that is required to prevent \(A > 1\), and additional shear-related effects must therefore increase the total amount of ridging. If a ridging function \(\psi > \psi_g\) (for the same state of strain rate and for the same function \(C(A)\), then a maintained deformation rate will eventually result in \(A\) greater than unity. It follows that the Hibler (1984) formulation \(\psi_H\) fails for all convergent states for which \(3|\eta| > \Delta\). This is seen in Fig. 1. For convergent strain-rate states close to pure shear, the ridging due to shear is sufficient to prevent \(A > 1\), but for higher convergence rates the \(\psi_H\) curve crosses the \(\psi_g\) curve and \(A > 1\) results. The curve of Shinohara (1990) on the other hand satisfies the constraints \(\psi_5 \leq \psi_g\) for all \(\theta\) and will therefore always prevent \(A > 1\).

It is physically reasonable that in pure divergence there should be no ridging and that in pure convergence the ridging is just enough to prevent further increase in \(A\) when \(A = 1\). It is therefore no coincidence that both Shinohara’s (1990) and Gray and Morland’s (1994) functions are equal at these two points. Thus, in maintained convergence \(A \to 1\), \(C(A) \to 1\) and \(\psi_5, \psi_g \to \eta\) (from above) to cancel out the \(A\eta\) term in (4.1). It is also sensible to assume that the shear effects can never produce ridging larger than in pure compression.

Gray and Killworth (1995) have shown that the elliptic yield curve leads to a linear instability in uniaxial divergence, which is both unphysical and implies that the problem is ill posed. Yield curves that remain in the third quadrant do not possess this deficiency. If in the future other yield curves are used, we envisage that a new function \(\Delta\) of the invariants will be formulated. In light of this, it is of interest to define the restrictions on an isotropic ridging function of the form
\[ \psi = C(A)\chi(\eta, \gamma^2), \] (4.21)
which will keep the concentration in the range \(0 \leq A \leq 1\). The function \(\chi\) need only be defined on the region \(\pi/4 \leq \theta \leq 5\pi/4\) as the reflective symmetry implied by isotropy automatically defines the function in all other ranges. We therefore require \(\chi\) to be a decreasing function in the range \(\pi/4 \leq \theta \leq 5\pi/4\) with the properties that
\[ \chi(\pi/4) = 0, \quad \chi(5\pi/4) = \eta, \]
\[ \chi(5\pi/4) \leq \chi \leq \chi_0 = \eta H(-\eta). \] (4.22)
In addition, the function \(C(A)\) should be defined as
\[ C(A) = A\alpha(A), \] (4.23)
where the function \(\alpha\) must satisfy the original conditions placed upon it in by Gray and Morland (1994) in (3.3). Two possible functional forms for \(\alpha\) are given in (3.7) and (3.8).

5. Conserved quantities

We now consider some elementary solutions of the ice concentration and thickness equations (4.1), in the absence of any thermodynamic source terms. When there is no ridging \(\psi = 0\), the thickness equation reduces to
\[ \frac{Dh}{Dt} = 0, \quad \Rightarrow \quad h = \text{const}, \] (5.1)
following a material particle. That is, the ice thickness \(h\) is conserved along a particle path. If at \(t = t_0\) the ice thickness \(h\) at a given material particle with label \(P\) is \(h_0\), then at all subsequent times the thickness \(h = h_0\) for that same labeled particle \(P\). The ice thickness is said to be conserved in the absence of ridging. This implies that the thickness is conserved in the model of Shinohara (1990) only when the ridging function \(\psi_5\) is zero in pure divergence, while in the model of Hibler (1984) it is also conserved during pure convergence! In the scheme of Hibler (1979) the thickness is conserved in both divergence and convergence until \(A = 1\) while in contrast (Gray and Morland 1994) it is conserved in divergence, and in convergence when \(\alpha = 0\).

In general, there are no such conserved quantities when ridging takes place except in the special case when the ridging is proportional to the divergence \(\eta\).
That is, there are no conserved quantities in the models of Hibler (1984) and Shinohara (1990) because of the additional shear effects, but there are conserved quantities in the model of Gray and Morland (1994). Thus, in convergence and in the absence of thermodynamic effects we may eliminate $\eta$ in (3.1) by subtracting a suitable multiple of (3.2) to obtain
\[
\frac{\alpha}{A(1 - \alpha)} \frac{DA}{Dt} = \frac{1}{h} \frac{Dh}{Dt}.
\] (5.2)

Now since $\alpha$ is a function of ice concentration $A$ and is independent of the remaining variables, it follows that
\[
\frac{D}{Dt} \left( \int \frac{\alpha}{A(1 - \alpha)} dA - \ln h \right) = 0,
\] (5.3)
which implies that
\[
\int \frac{\alpha}{A(1 - \alpha)} dA - \ln h = \text{const}
\] (5.4)

following a particle path. This is another conserved quantity, whose functional form depends on the nature of the ridging function $\alpha$. We have not been able to attribute a physical meaning to (5.4). It certainly does not express mass conservation (recall mass $\propto Ah$). For the two model functions (3.7) and (3.8) proposed by Gray and Morland (1994) the integral can be evaluated explicitly. However, the third function (3.9) has a singular integrand at $A = 1$, which prevents explicit integration. Note in this last case that (3.10) actually implies that, once $A$ reaches unity, the ice concentration is conserved following a material path providing there is no rate of mass supply to the ice from the lead water $k = 0$. Assuming that the integrals can be performed, then the constant in (5.4) can be evaluated from the initial conditions that at time $t = t_0$ the thickness $h$ at a labeled particle $P$ equals $h_0$ and the concentration $A = A_0$. Hence, provided convergence is maintained the ice thickness $h$ and ice concentration $A$ at some later time $t$ are related by Eq. (5.4). It follows that for the three ridging functions (3.7), (3.8), and (3.9) that the thickness is related to the concentration by

1) \[
\frac{h}{h_0} = \left( \frac{1 - A_0^2}{1 - A} \right)^{1/m}
\] (5.5)

2) \[
\frac{h}{h_0} = \left\{ \begin{array}{l}
1, \quad A < A_f \\
\left( \frac{1 - A_1}{1 - A} \right)^{1 - A_f} \left( \frac{A_1}{A} \right)^{A_f}, \quad A_f \leq A
\end{array} \right.
\] (5.6)

3) \[
\frac{h}{h_0} = \left\{ \begin{array}{l}
1, \quad A < 1 \\
\text{undefined}, \quad A = 1
\end{array} \right.
\] (5.7)

following a material particle $P$. The constant $A_1$ in (5.6) is defined in the following way, $A_1 = A_f$ if $A_0 < A_f$ and $A_1 = A_0$ if $A_0 \geq A_f$. Note, that in case 3 the ratio of the thickness to the initial thickness $h/h_0$ is undefined when $A = 1$. This is merely a statement that $h$ and $A$ are not related, as by (3.10) $A$ is conserved in the absence of thermodynamic effects. Similarly, in the absence of ridging there is again no explicit relation between $A$ and $h$, since $h$ is conserved. In the remaining cases there is an explicit relationship between the ice concentration and the floe thickness following a material particle.

The nondimensional thickness $h/h_0$ is plotted in Fig. 2 in each case as a function of $A$ for three different values of $A_0$. These lines represent the phase lines, along which a solution progresses in the direction of the arrows with increasing time. The phase lines in case 1 never cross or overlap and each approaches $A = 1$ asymptotically for large $h$. In contrast, the phase lines in case 3 are degenerate and the solution develops along the same phase line for all values of the initial concentration $A_0$. In the range $h/h_0 > 1$, the phase line in case 2 lies between the two extremes with degenerate phase lines for $A < A_f$ and well-defined phase lines for $A > A_f$, yielding a qualitatively similar behavior to case 1.
A < 1 there is no ridging and \( h = h_0 \). Subsequently at \( A = 1 \) the phase line progresses vertically, indicating that the thickness increase is uncoupled from the ice concentration. Case 2 lies somewhere between the two cases. For \( A \leq A_f \) the phase lines are degenerate so that all initial states in the range \( 0 \leq A_0 \leq A_f \) will evolve along the same phase line. For \( A \geq A_f \) the phase lines are well defined and the behavior again involves an asymptotic approach to \( A = 1 \), which is qualitatively similar to case 1.

The ice concentration equation of Gray and Morland (1994) can be integrated, for convergence, in all three cases from initial conditions \( A = A_0, h = h_0 \) at \( t = 0 \). We write a stretched time \( T \) by

\[
T = \int_0^t \eta(t') dt',
\]

so that (3.1) and (3.2) become

\[
\frac{DA}{DT} = A(1 - \alpha(A))
\]

\[
\frac{Dh}{DT} = \alpha(A) h,
\]

and one of cases 1 to 3 holds for the function \( \alpha(A) \). The solutions are

1) \[
A = \frac{A_0}{(1 - A_0^n)^{1/m}} = \frac{A_0}{(1 - A_0^n)^{1/m}} \exp(T)
\]

2) \[
A = \begin{cases}
A_0 \exp(T), & T < T_1 = \ln(A_f/A_0) \\
\frac{1}{1 + (A_f^{-1} - 1) \exp(- (T - T_1)/(1 - A_f))}, & T > T_1 
\end{cases}
\]

3) \[
A = \begin{cases}
A_0 \exp(T), & T < T_2 = \ln(1/A_0) \\
A = 1, & T > T_2 
\end{cases}
\]

and \( h \) is given by (5.5) – (5.7) as appropriate. When the ice concentration reaches unity in case 3 the evolution of \( h \) as a function of \( A \) is undefined in (5.7). In this case thickness is computed from (5.9) with \( \alpha(1) = 1 \), implying

\[
h = h_0 \exp T
\]

following a material path during maintained convergence. This is a relationship between the large-scale velocity field and the thickness rather than the ice concentration and thickness as in the remaining cases.

The \( A \) and \( h \) solutions are plotted as a function of time in Fig. 3 for the three cases of \( \alpha \). The temporal behavior of all three cases is qualitatively very similar and involves an asymptotic approach to \( A = 1 \) for long times. At this time, the behavior is so similar that it becomes difficult to differentiate between the models. For example, in case 1, for large \( T, A \approx 1 - \exp(-T)/A_0, h \approx h_0 A_0 \exp(T) \), which is independent of \( m \). This implies that choosing between ridging models using observations would be difficult. Put another way, there is a large error amplification caused by the behavior of the \( \alpha(A) \) functions near \( A = 1 \). Differentiating (5.4) w.r.t. \( A \) gives

\[
\frac{Dh}{DA} = \frac{h \alpha}{A(1 - \alpha)} \to \infty, \quad A \to 1.
\]

This high sensitivity may cause problems if the time step in numerical integration schemes is too long. The main difference between the cases is the time at which ridging is initiated. In case 1 ridging occurs continuously during convergence for all values of \( A \), while for case 2 ridging first begins when \( A = A_f \) at time \( T = \ln(A_f/A_0) \) (for initial \( A_0 < A_f \)). In the classical (Hibler 1979) ridging scheme there is no ridging until \( A = 1 \) at time \( T = \ln(1/A_0) \) after which lateral mass flux is immediately redistributed to increase the ice thickness. The jump in the ridging function (3.9) therefore causes a discontinuity in the derivatives of \( A \) and \( h \) at

![Fig. 3.](image-url) The qualitative temporal evolution of the concentration and thickness for each of the three ridging functions \( \alpha \) are plotted. The number on the line corresponds to the ridging functions 1–3 defined in Eqs. (3.7)–(3.9).
\[ T = \ln(1/A_0), \] which makes correct numerical implementation difficult.

It should be noted that even though the long time behavior of \( A \) and \( b \) is qualitatively similar for each of the ridging models, the highly sensitive feedback mechanism into the ice momentum balance (through the ice strength) means that the short-time flow regimes may be considerably altered. The ice strength is weakly coupled to the thickness but very strongly coupled to the concentration. In the Hibler (1979) scheme the concentration rapidly reaches unity and maximal strengths are quickly attained to prevent further convergence. In contrast, cases 1 and 2 have lower concentrations but higher thicknesses than case 3, which has the net effect of reducing the ice strength and increasing the mobility of the ice pack. This does not present a great problem but does mean that it may be necessary to increase the maximum ice strength, \( P^* \), [defined in the model of Hibler (1979)] in cases 1 and 2, so that similar ice strengths to case 3 are attained at lower concentrations.

6. Discussion

The ridging mechanism plays an important role in the evolution of the ice concentration and thickness, and ridging terms should be included to model these effects. It is as yet unclear what precise form this ridging parameterization should take, as it is very difficult to use observations to determine its properties. We have taken the view that the scheme should at the very least be consistent with the definition that the concentration lies in the range \( 0 \leq A \leq 1 \). A set of constraints that guarantees this, for a general ridging function dependent on two strain-rate invariants, has been determined in (4.21)–(4.23). Convergence effects are clearly of great importance, but it remains to be shown that a large amount of ridging takes place due to shear. If shear effects prove not to be of great importance, then certain quantities are conserved along particle paths during convergence. This fact may be of use in determining more precise properties of \( \psi \).

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