The effect of change in thermal properties on the propagation of a periodic thermal wave: Application to a snow-buried rocky outcrop

J. M. N. T. Gray¹ and L. W. Morland
School of Mathematics, University of East Anglia, Norwich, England

S. C. Colbeck
Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire

Abstract.
The propagation of a periodic thermal wave into snow is significantly altered by the presence of a shallow rock interface because of the large difference in thermal properties of the two media. The temperature distribution is modeled using classical heat conduction equations subject to a periodic diurnal or seasonal surface heat flux condition, jump conditions at the interface, and insulating conditions in the far-field. The natural length scale (skin depth) over which order unity changes in temperature occur is proportional to the square root of the timescale of the surface temperature variations. If the interface lies close to or within the skin depth then large temperature gradients can be sustained in the snow before temperature oscillations are forced through to the underlying rock. These features are explained by an analytic one-dimensional periodic solution. A numerical algorithm is constructed to solve for the temperature around plane two-dimensional rock geometries. The results show that during a period of atmospheric cooling the presence of a buried rocky outcrop increases the snow temperature and temperature gradients simultaneously to produce very favorable conditions for crystal growth and avalanche formation.

Introduction

Field observations [Logan, 1992; Jamieson and Johnston, 1993] in subfreezing dry snowpacks suggest that the temperature distribution is significantly altered around buried rocky outcrops. In particular, enhanced temperature gradients develop in the snow immediately above the rock and, since crystal growth rates increase linearly with the temperature gradient [Colbeck, 1983] and nonlinearly with temperature [Lamb and Hobbs, 1971], increased crystal growth rates are also expected. Sufficiently large temperature gradients produce faceted depth hoar crystals with very little cohesion. The presence of a rocky outcrop may either initiate the formation of depth hoar in a zone where it would not otherwise form, or may enhance the rate of formation in areas where depth hoar is already being produced. In this event either a "weak spot" or a "super weak spot" in the snow is formed which can trigger a slab avalanche release [Armstrong and Williams, 1992]. Hørkeland et al. [1995] state that "the snow overlying rocks was found to have significantly weaker resistance than adjacent areas that were not over rocks". Nine Colorado avalanche accidents were directly attributed to such trigger zones by Logan [1992].

A quantitative understanding of the temperature distribution around snow-buried objects is required to assess the implications of this in predicting avalanche risk. In this paper we present some elementary solutions to the classical heat conduction equations [Carslaw and Jaeger, 1959] with associated energy jump and boundary conditions. This allows us to determine the relative importance of the geometrical effects, of snow depth, bump height, and bump length, as well as the physical properties, such as the snow density and the skin depth associated with the temperature wave propagation on various atmospheric forcing timescales.

Theory

As a first approximation it is reasonable to assume that heat transfer in dry subfreezing snow is dominated by conduction [Gray and Morland, 1994] which neglects external energy supply from solar radiation and latent

---

¹Now at Institut für Mechanik, Technische Hochschule Darmstadt, Darmstadt, Germany.
heat effects [Gray et al., 1995] as well as thermal convection of the pore gas [Bader et al., 1939; Sturm and Johnson, 1991]. The energy balance is therefore

\[ \rho C_p \frac{\partial T_s}{\partial t} - \nabla \cdot (K^e \nabla T_s) = 0, \]  

(1)

where \( T_s \) is the temperature in the snow layer, \( \rho \) is the snow density, \( C_p^I = 2.031 \times 10^3 \, \text{J} \, \text{kg}^{-1} \, \text{K}^{-1} \) is the specific heat of ice and \( K^e \) is the effective thermal conductivity of snow. The effective thermal conductivity is determined empirically and is dependent on the temperature, grain size, and snow density. Normally, the effects of grain size and temperature are neglected and an algebraic expression is used to parameterize the thermal conductivity as a function of snow density. These formulas are reviewed by Mellor [1977] and Hukusabo [1990]. The relation

\[ K^e = K_0 + K_1 (\rho)^2 \]  

(2)

suggested by Anderson [1976] is used here, where the constants \( K_0 = 2 \times 10^{-2} \, \text{W} \, \text{m}^{-1} \, \text{K}^{-1} \) and \( K_1 = 2.5 \times 10^{-6} \, \text{W} \, \text{m}^{-1} \, \text{K}^{-1} \) for a seasonal snow cover. Thus for low-density snow (\( \rho = 400 \, \text{kg} \, \text{m}^{-3} \)) the thermal conductivity is \( K^e \approx 0.42 \, \text{W} \, \text{m}^{-1} \, \text{K}^{-1} \), and for fairly high-density snow (\( \rho = 100 \, \text{kg} \, \text{m}^{-3} \)) the thermal conductivity rises to \( K^e \approx 0.2 \, \text{W} \, \text{m}^{-1} \, \text{K}^{-1} \), an enhancement of nearly 10 times.

The rock is assumed to have uniform density and homogeneous thermal properties. External energy supply due to solar radiation and the associated albedo effects caused by shallow dark colored objects are neglected. The energy balance for the rock with temperature \( T_r \) is therefore

\[ \rho R C_p \frac{\partial T_r}{\partial t} - \nabla \cdot (K^R \nabla T_r) = 0, \]  

(3)

where the rock density \( \rho R = 2703 \, \text{kg} \, \text{m}^{-3} \), the specific heat \( C_p^R = 0.796 \times 10^3 \, \text{J} \, \text{kg}^{-1} \, \text{K}^{-1} \), and the thermal conductivity \( K^R = 2.768 \, \text{W} \, \text{m}^{-1} \, \text{K}^{-1} \) are the values for granite [Dziak, 1980].

Let \( Oxyz \) be a rectangular Cartesian coordinate system such that the plane snow surface is defined by \( z = d^* \) and the plane \( z = 0 \) coincides with the rock surface at its greatest distance from the snow surface; \( d^* \) is therefore the maximum snow layer thickness. Here only plane configurations are considered in which there is no variation of geometry or surface conditions with the coordinate \( y \). In avalanche applications the surface \( z = d^* \) may be steeply inclined. The rock surface is defined by \( z = g(x) \), and the snow domain \( R^s \) and the rock domain \( R^r \) are defined by \( z > g(x) \) and \( z < g(x) \), respectively. Furthermore, the snow cover is continuous, so \( g(x) < d^* \). Figure 1 illustrates the three different rock surface profiles considered. These consist of a single step, an isolated bump, and repeated bumps, which are represented by the functional forms

\[ g(x) = \frac{h^*}{2} \left\{ 1 + \tanh(\pi x/l^*) \right\}, \]  

(4)

Case 1

\[ g(x) = \frac{h^*}{1 + (\pi x/2l^*)^2}, \]  

Case 2

Case 3

\[ g(x) = h^* \left\{ 1 - \cos(\pi x/l^*) \right\}, \]

in the range \( -\infty < x < \infty \), respectively. Each interface has an associated maximum height \( h^* \) above the \( z = 0 \) plane and a characteristic width \( l^* \) over which order unity changes in the geometry take place. The geometry therefore introduces three length scales, \( d^*, h^*, \) and \( l^* \) which all play an important role in the nature of the solution. Note that in order to draw comparisons between the three cases in (4) the slope of case 1 is chosen to be the same as that of case 3 at \( z = h/2 \), and the curvature at the apex of the single bump (case 2) is the same as that in case 3. That is, the profiles have been scaled so that they represent similar geometries for the same \( h^* \) and \( l^* \).

The snow/rock interface \( z = g(x) \) is a singular surface at which the energy balance reduces to the jump condition

\[ z = g(x) : \quad K^s \frac{\partial T_s}{\partial n} = K^R \frac{\partial T_r}{\partial n}. \]  

(5)

The unit normal \( n \) to the interface is given by \( \Delta n = (-\partial g/\partial x, 0, 1) \), where the normalization factor \( \Delta = \{1 + (\partial g/\partial x)^2\}^{1/2} \), and the normal temperature derivative is given by \( \partial T/\partial n = n \cdot \nabla T \). A good thermal contact between the two media is assumed so the tem-

![Figure 1](image-url) Figure 1. Cross sections of each of the three bump profiles. (1) a step, (2) an isolated bump, and (3) repeated bumps. The functional forms have been scaled so that they all have similar lengths \( l^* \) and heights \( h^* \). The rock domain \( R^r \) lies below the interface and extends vertically downward to infinity. The snow domain \( R^s \) lies above the interface and has a flat surface which lies a distance \( d^* \) above the lowest point. The dot-dash line at \( x = 0 \) in case 2 is the line about which the geometry is symmetric. The repeated bumps are symmetric about all the dot-dash lines.
perature is continuous across the interface [Carslaw and Jaeger, 1959]

\[ z = g(x) : \quad T_s = T_r. \]  \tag{6}

This may not be a good assumption if void space develops at the snow/rock interface, which occurs during surface cooling when there is a diffusive vapor flux into the pack at an impermeable lower boundary [Gray et al., 1995].

In the horizontal far-field the interface in cases 1 and 2 is flat, and in the absence of any density or surface boundary varieties there is no lateral diffusion of heat. The temperature is therefore one-dimensional in the \( x \) coordinate, and the far-field boundary condition is

\[ \frac{\partial T}{\partial x} \to 0 \quad \text{as} \quad x \to +\infty, \]  \tag{7}

in both the snow and rock domains. The single bump (case 2) has a symmetry line (the dot-dash line in Figure 1) at \( x = 0 \). If the snow density and the initial and boundary conditions are also symmetric about \( x = 0 \), then the heat flux across the symmetry line is identically zero, and the boundary condition (7) can be applied at \( x = 0 \). This reduces the problem from an infinite to a semi infinite horizontal domain. Similarly, in case 3 the geometry is symmetric about the lines \( x = nL^*, \) for integer \( n \). If the density, initial conditions, and boundary conditions are also symmetric about these lines, the problem reduces from an infinite horizontal domain to a single finite range, \( 0 < x < l^* \), with zero heat flux conditions (7) at either end.

During a period of atmospheric cooling snow surfaces temperatures were observed to be warmer over a buried rock than elsewhere in the snow [Logan, 1992]. Qualitatively similar results are obtained in this paper by assuming a laterally uniform temperature gradient surface boundary condition varying sinusoidally in time

\[ z = d^* : \quad \frac{\partial T_s}{\partial z} = \frac{T^*}{z_s^*} \cos \left( \frac{\pi t}{2T^*} \right), \]  \tag{8}

which approximates the heating and cooling phases on a diurnal or seasonal timescale \( t^* \), with frequency of oscillation \( \pi/(2T^*) \) and period \( 4T^* \). The amplitude \( T^*/z_s^* \) is chosen to ensure that order \( T^* \) changes in temperature occur on the natural snow conduction length scale \( z_s^* \) defined in (11). Note that the vertical heat flux \( q_z = -K^s \partial T_s/\partial z \) is dependent on the effective thermal conductivity. The same vertical heat flux \( q_z \) will therefore produce larger temperature changes \( T^* \) in lower density snow. Deep within the rock domain the vertical temperature gradient tends to zero

\[ z \to -\infty : \quad \frac{\partial T_r}{\partial z} \to 0 \]  \tag{9}

as the geothermal heat flux is negligible [Gray and Molland, 1994]. All the problems considered here are long time oscillatory solutions, in either one or two dimensions, where the transients imposed by initial conditions are negligible. General problems without periodic boundary conditions of the form (8) require the prescription of the initial temperature distribution.

### Nondimensional Variables

The timescale \( t^* \) is imposed by the atmospheric forcing at the surface of the snowpack. For most problems of interest this is usually the diurnal timescale \( t_d = 2 \times 10^4 \text{ s} \). However, the seasonal timescale \( t_s = 10^7 \text{ s} \) may also be applicable for permanent snow covers in high latitudes. The passage of a storm on a timescale \( t^* = 10^5 - 10^6 \text{ s} \) is an important intermediate situation, which is bracketed between the diurnal and seasonal forcing scenarios investigated here. The temperature is determined by the external forcing and is decomposed into a mean background level \( T_m \) and an amplitude of fluctuation \( T^* \). The time and the temperature are scaled as

\[ t = t^* \tilde{t}, \quad T = T_m + T^* \tilde{T}, \]  \tag{10}

where the tilde indicates a nondimensional variable. Changes in temperature are transmitted through the snow and rock by conduction. A balance between the local heating terms and the vertical conduction term, in (1) and (3), determines the natural length scales (skin depths) over which order unity changes in the vertical temperature occur in timescale \( t^* \),

\[ z_s^* = \sqrt{\frac{K^s t^*}{\rho C_p^s}}, \quad z_r^* = \sqrt{\frac{K^r t^*}{\rho C_p^r}}, \]  \tag{11}

in the snow and rock, respectively, where \( K^s \) is the magnitude of the effective thermal conductivity. Recalling the effective thermal conductivity values implied by the quadratic relation (2), it follows that \( z_s^* \) for low-density snow is approximately 6.6 cm, rising to 10 cm for higher density snow for forcing on diurnal timescales. For seasonal forcing the thermal length scale, \( z_r^* \), increases to 1.48 m for low-density snow and 2.26 m for high-density snow. This is a measure of how far order unity changes in temperature penetrate into the snowpack on a given timescale and can be regarded as the skin depth of the temperature wave oscillations. If the snow is a lot deeper than the skin depth the temperature is unaltered by the presence of the bump, as no heat ever reaches it. The thermal length scale, \( z_r^* \), in the rock is longer for both diurnal (16 cm) and seasonal (3.58 m) forcing.

The magnitude of the vertical temperature gradient \( q_z^* \) determines the size of the temperature fluctuations \( T^* \) by (8), which are of order

\[ T^* - z_s^* q_z^*/K^s. \]  \tag{12}

Changes in heat flux of magnitude \( q_z^* = 5 \text{ W m}^{-2} \) will produce temperature changes of 1.2 K and 26 K in high-density snow on diurnal and seasonal timescales. In low-density snow, diurnal temperature changes of 7.3 K are produced for the same heat flux magnitude.

There are five length scales in the problem; three are imposed by the geometry (\( d^*, h^*, l^* \)) and two arise naturally from the differential equations (\( z_s^*, z_r^* \)). Physically,
the most important of these is the snow depth \( d^* \), and this is therefore used to nondimensionalize both the \( z \) and \( z \) coordinates:

\[
(x, z) = d^*(\tilde{x}, \tilde{z}).
\]  

(13)

The scalings (10) and (13) are now substituted into the energy balances (1) and (3) to obtain the nondimensional heat conduction equations,

\[
\frac{\partial \tilde{T}_s}{\partial \tilde{t}} = \mu_s \frac{\partial}{\partial \tilde{x}} \left( K_s^e \frac{\partial \tilde{T}_s}{\partial \tilde{x}} \right) + \mu_s \frac{\partial}{\partial \tilde{z}} \left( K_s^e \frac{\partial \tilde{T}_s}{\partial \tilde{z}} \right),
\]

(14)

\[
\frac{\partial \tilde{T}_r}{\partial \tilde{t}} = \mu_r \left( \frac{\partial^2 \tilde{T}_r}{\partial \tilde{z}^2} + \frac{\partial^2 \tilde{T}_r}{\partial \tilde{\eta}^2} \right),
\]

(15)

where a nondimensional effective thermal conductivity \( K^e = K^*/K_s^e \). The dimensionless numbers \( \mu_s, \mu_r \) equal the squares of ratio of the vertical conduction lengths to the snow depth

\[
\mu_s = (z_s^*/d^*)^2, \quad \mu_r = (z_r^*/d^*)^2.
\]

(16)

where the nondimensional thermal length scales are defined as \( z_s = z_s^*/d^* \) and \( z_r = z_r^*/d^* \). If the thermal length scales are much larger than the depth of the snow, then the time derivative in (14) and (15) is of minor importance and a steady state temperature distribution develops. For diurnal surface forcing, snow depths of 0.6 cm or less are required for the steady state to hold, while seasonal forcing needs snow depths of less than 14 cm. Arguing from another perspective, a steady state temperature distribution would only develop in a 1-m snowpack if the external forcing was sustained for several years. Clearly, the steady state temperature distribution never has sufficient time to develop in all but the thinnest of snow coverings, and the complete time dependent heat conduction problem must in general be solved.

The energy balances (14) and (15) are driven by the conditions imposed along the external boundaries. The sidewalls of the domain act as insulators in all three cases, either by virtue of the symmetry condition or since the horizontal gradients decay in the far-field. Deep within the rock there is also very little of interest taking place, as the amplitude of the temperature fluctuation decays rapidly. The dominant forcing is at the snow surface where sinusoidal temperature gradient variations approximate the heating and cooling phases of the atmosphere over diurnal or seasonal cycles. The boundary conditions (8), (9), and (7) become

\[
\tilde{z} = 1 : \frac{\partial \tilde{T}_s}{\partial \tilde{z}} = \frac{1}{z_s} \cos \left( \frac{\pi \tilde{x}}{2} \right),
\]

\[
\tilde{z} \gg \infty : \frac{\partial \tilde{T}_s}{\partial \tilde{z}} > 0,
\]

\[
\text{sidewalls : } \frac{\partial \tilde{T}_r}{\partial \tilde{z}} = 0,
\]

(17)

in nondimensional variables. At the interface between the rock and snow the continuity in temperature (6) and the energy jump (5) conditions become

\[
\tilde{z} = \tilde{g} : \tilde{T}_s = \tilde{T}_r, \quad K^e \frac{\partial \tilde{T}_s}{\partial \tilde{z}} = \frac{\partial \tilde{T}_r}{\partial \tilde{\eta}}, \quad (18)
\]

where the ratio of thermal conductivities \( K = K^*/K_s^e \) \( \approx 0.016 \) or 0.15 for low- and high-density snow, respectively. The nondimensional bump profiles transform to

Case 1 \( \tilde{g}(\tilde{z}) = \frac{h}{2} \left( 1 + \tanh(\pi \tilde{x}/l) \right) \), \( -\infty < \tilde{z} < \infty \)  

Case 2 \( \tilde{g}(\tilde{z}) = \frac{h}{2(\pi \tilde{z}^2)} \), \( -\infty < \tilde{z} < 0 \)  

Case 3 \( \tilde{g}(\tilde{z}) = \frac{h}{2} \left( 1 - \cos(\pi \tilde{z}/l) \right) \), \( 0 \leq \tilde{z} \leq l \)

(19)

where the nondimensional bump height \( h \) and bump width \( l \) are defined as \( h = h^*/d^* \), \( l = l^*/d^* \), which follow directly from the coordinate scaling (13).

One-Dimensional Periodic Solution

The conduction problem reduces to one dimension when the interface is flat and parallel to the surface, \( \tilde{g} = h \), and there are no lateral gradients of the thermal conductivity or the boundary conditions. In the two-dimensional model there are three geometrical length scales, two of which are redundant in one dimension. The length \( d^* \) is therefore assumed equal to the rock's thermal length \( z_r^* \), which implies \( \mu_r = 1 \), and a new snow depth magnitude \( \eta^* = d^* - h^* \) is defined such that the nondimensional snow depth \( \eta = \eta^*/d^* = 1 - h \).

A single nondimensional spatial unit corresponds to a physical length of 16 cm for diurnal forcing and 3.58 m for seasonal forcing. The solution is more easily found by making the coordinate transformation \( \zeta = \tilde{z} - h \), so that the origin lies on the interface between the rock and snow domains and the snow surface now lies at \( \zeta = \eta - 1 \). Assuming that the snow has uniform thermal conductivity \( K^* = K^*_s (K^e = 1) \), there is a long time oscillatory temperature solution in the snow

\[
\frac{T_s}{a} = (1 + \Omega^2) e^{\lambda_r (\zeta + \eta)} \cos(\omega t' + \lambda_s (\zeta - \eta))
\]

\[
-\frac{1 - \Omega^2}{2} \frac{e^{\lambda_r (\zeta - \eta)}}{\cos(\omega t' - \lambda_s (\zeta + \eta))} \cos(\omega t' - \lambda_s (\zeta - \eta)),
\]

and in the rock

\[
\frac{T_r}{2a} = (1 + \Omega^2) e^{\lambda_r (\zeta + \eta)} \cos(\omega t' + \lambda_r \zeta - \lambda_s \eta)
\]

\[
-\frac{1 - \Omega^2}{2} \frac{e^{\lambda_r (\zeta - \eta)}}{\cos(\omega t' + \lambda_r \zeta + \lambda_s \eta)},
\]

where

with frequency \( \omega = \pi/2 \), as \( \omega t' = \omega t - \pi/4 \), and the period \( 2\pi/\omega = 4 \) time units. The wave numbers for the snow and the rock are \( \lambda_s = (\omega/2\mu_s)^{1/2} \) and \( \lambda_r = (\omega/2\mu_r)^{1/2} \), respectively. The normalisation factor \( a = (z_r^*/(\sqrt{2}\mu_s^*\psi \lambda_s)) \)
\[ \psi = (1 + \Omega^2) e^\theta + (1 - \Omega^2) e^{-\theta} - 2(1 - \Omega^2) \cos \theta, \] (22)

\[ \theta = 2\lambda_1\eta \text{ and } \Omega = \lambda_1/\lambda \] equals 25.4 for low-density snow and 4.15 for high-density snow. This solution explains some of the main features of the thermal regime and serves as a useful check on more general numerical algorithms.

In Figure 2 the periodic one-dimensional solution for low-density snow \((\rho^* = 100 \text{ kg m}^{-3})\) is plotted in the form of a contour plot in space \(\tilde{z}\) and time \(\tilde{t}\), over a complete cycle (4 time units). The solid lines show where the isotherms lie when the snow/rock interface lies at \(\eta = 1/2\), and the dotted lines represent where the equivalent isotherms would lie if the interface lay “deep” within the snowpack. In Figure 3 a similar plot is presented for high-density snow \((\rho^* = 400 \text{ kg m}^{-3})\). The physical depth of the shallow snow \(\eta^* = z^*/2\) corresponds to 8 cm for diurnal forcing and 1.79 m for seasonal forcing.

When the snow is “deep,” that is, \(\eta^* \gg z^*_d\), the temperature waves never penetrate far enough into the snow to feel the effect of the rock, and the solution is almost identical to a periodic forcing in a semi infinite domain [Gray and Morland, 1994]. In deep snow the temperature changes decay exponentially with a decay rate \(\lambda_1\) that is larger in the low-density snow \((\lambda_1 = (\pi^2/\tilde{z}^*_d)^2/21.14)\) than in high-density snow \((\lambda_1 = 1.4)\) or the rock \((\lambda_1 = \sqrt{\tilde{z}^*_d}/2 = 0.886)\). The temperature fluctuations are propagated into the snow with velocity \(v_1 = \omega/\lambda_1 = \sqrt{2\mu/\lambda_1}\), which is less for low-density snow, \(v_1 = 0.73\), than for high-density snow, \(v_1 = 1.12\), or the rock \(v_1 = \sqrt{2\mu/\lambda_u} = 1.77\). Recall that the surface temperature gradient varies sinusoidally in time starting from its maximum at \(\tilde{t} = 0\). For deep snow the surface temperature lags behind by exactly \(\eta^*/(4\omega) = 1/2\) a time unit.

Many of the features of the deep snow solution are also present in the shallow snow. The most striking new feature is the discontinuity in the temperature gradient at the interface between the rock and snow domains. The interface conditions (18) require temperature gradients in the snow of order \(1/K = 66\) and 0.6 (for low- and high-density snow) before order unity temperature changes are transmitted into the underlying rock. The isotherms are therefore all squeezed into the shallow snow depth in Figure 2 as low-density snow is an exceptionally good insulator. However, the temperature gradients in high-density snow (Figure 3) are strong enough to force heat through.

The surface temperature maxima and minima are attained at times

\[ \tilde{t} = \frac{1}{\omega} \left( \frac{\pi}{4} + n\pi \right) + \tan^{-1} \left( \frac{2(1 - \Omega^2) \sin \theta}{(1 + \Omega^2) e^\theta - (1 - \Omega^2) e^{-\theta}} \right), \] (23)

which are affected by the presence of the interface. For instance, the maximum shallow snow temperature occurs at \(\tilde{t} = 0.383\) and 0.313 time units in Figures 2 and 3, respectively, which both precede the deep snow maximum at \(\tilde{t} = 0.5\). However, if an interface were placed at \(\eta = 1\) in the low-density snow, the maximum would occur after the deep snow maximum at \(\tilde{t} = 0.515\).

**Change of Variables**

The two-dimensional model is complicated by the presence of a nonplanar interface between the rock and
Snow domains, over which jump conditions (18) are applied. Approximate analytic solutions can be found when the bump is small in amplitude or the thermal length scales are very long or very short, but for most problems of interest numerical methods are the only means of determining the solution. The numerical representation of the jump conditions presents some computational problems, as regularly spaced grid nodes do not necessarily coincide with positions lying on the interface, so the normal derivatives are hard to evaluate.

Three different interfaces are considered here, recall (19), for which solutions may be constructed on restricted domains in which \( \tilde{g}(\tilde{z}) \) is monotonically increasing. This allows the replacement of \( \tilde{z} \) by the coordinate transformation

\[
\xi - \tilde{g}(\tilde{z}),
\]

which maps the interface \( \tilde{z} = \tilde{g} \) onto the 45° line \( \tilde{z} = \xi \) in \((\xi, \tilde{z})\) coordinates. Figure 4 illustrates the mapped domains for the three cases. In the new variables, uniformly spaced grid points fall exactly on the interface. The energy balances (14) and (15) become

\[
\begin{align*}
1 \frac{\partial \tilde{T}_s}{\partial \tilde{t}} &= \gamma' \frac{\partial}{\partial \xi} \left( K_s \frac{\partial \tilde{T}_s}{\partial \xi} \right) + \gamma'' K_s \frac{\partial \tilde{T}_s}{\partial \xi} \frac{\partial}{\partial \xi} \left( K_s \frac{\partial \tilde{T}_s}{\partial \xi} \right), \\
1 \frac{\partial \tilde{T}_r}{\partial \tilde{t}} &= \gamma' \frac{\partial^2 \tilde{T}_r}{\partial \tilde{z}^2} + \gamma'' \frac{\partial \tilde{T}_r}{\partial \tilde{z}} + \frac{\partial^2 \tilde{T}_r}{\partial \tilde{z}^2},
\end{align*}
\]

where the dash denotes differentiation with respect to \( \xi \) and the interface slope is

\[
\gamma(\xi) = \frac{\partial \xi}{\partial \tilde{z}}.
\]

Note that the temperature denoted by \( \tilde{T} \) in \((\tilde{x}, \tilde{z}, \tilde{t})\) is now represented by \( T \) in the mapped coordinates \((\xi, \tilde{z}, \tilde{t})\). The slope \( \gamma \) and curvature factor \( \gamma'' \) for each of the coordinate transformations in (19) are catalogued in Table 1.

The system is subject to the external boundary conditions

\[
\tilde{z} = 1: \frac{\partial \tilde{T}_s}{\partial \tilde{z}} = \frac{1}{z_s} \cos \left( \frac{\pi s}{2} \right),
\]

\[
\tilde{z} \to -\infty: \frac{\partial \tilde{T}_r}{\partial \tilde{z}} \to 0,
\]

\[
\xi = 0, h: \frac{\partial \tilde{T}}{\partial \xi} = 0.
\]

However, the sidewall conditions, the third equation in (28), are trivially satisfied, as the slope \( \gamma = 0 \) at

<table>
<thead>
<tr>
<th>Function</th>
<th>(1) Step</th>
<th>(2) Isolated Bump</th>
<th>(3) Repeated Bumps</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi )</td>
<td>( h \left{ \frac{1 + \tanh(\pi \xi / l)}{1 + (\pi \xi / 2l)^2} \right} )</td>
<td>( h \left{ \frac{1 + \tanh(\pi \xi / l)}{1 + (\pi \xi / 2l)^2} \right} )</td>
<td>( h \left{ \frac{1 - \cos(\pi \xi / l)}{1 - (\pi \xi / 2l)^2} \right} )</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>( \frac{2 \pi h \xi \xi - \xi^3}{h \xi} )</td>
<td>( \frac{\pi h \xi}{h \xi} \sqrt{h \xi^2 - \xi^3} )</td>
<td>( \frac{\pi}{\xi} \sqrt{h \xi^2 - \xi^3} )</td>
</tr>
<tr>
<td>( \gamma'' )</td>
<td>( \frac{4(\pi^2 - h - 2 \xi)(h \xi - \xi^2)}{4(h \xi^2 - 8 \xi^3)} )</td>
<td>( \frac{\pi^2 h}{2 l^2} )</td>
<td>( \frac{\pi^2 h}{2 l^2} )</td>
</tr>
<tr>
<td>( \gamma(0) \gamma'(0) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \gamma(h) \gamma'(h) )</td>
<td>0</td>
<td>( -\frac{\pi^2 h}{2 l^2} )</td>
<td>( -\frac{\pi^2 h}{2 l^2} )</td>
</tr>
</tbody>
</table>
\[ \frac{\partial \hat{T}}{\partial \hat{n}} = \frac{1}{\Delta} \left( -\gamma \frac{\partial \hat{T}}{\partial \hat{\xi}} + \frac{\partial \hat{T}}{\partial \hat{\eta}} \right), \quad \Delta = \left( 1 + \gamma^2 \right)^{\frac{1}{2}}, \quad (32) \]

and the normalization factor \( \Delta \) cancels out on either side of the interface condition (31). The temperature gradients within the snow are critical in determining the rate and type of recrystallization that occurs. It is therefore useful to express the magnitude of the dimensionless temperature gradient in the mapped coordinates:

\[ |\nabla \hat{T}| = \sqrt{\gamma^2 \left( \frac{\partial \hat{T}}{\partial \hat{\xi}} \right)^2 + \left( \frac{\partial \hat{T}}{\partial \hat{\eta}} \right)^2}. \quad (33) \]

The mapped domain is illustrated in Figure 4. The flat surface of the snowpack lies at \( \hat{z} = 1 \) and the interface between the snow domain \( R^s \) and rock domain \( R^r \) is the \( 4\pi \) line between the origin \( O \) at \( (0, 0) \) and \((h, h)\). The width of the transformed domain is \( h \). The vertical far-field condition, the second equation in (28), is applied at a finite depth for numerical purposes, but this is chosen far enough down to ensure that it does not affect the solution. The fine vertical and horizontal lines in Figure 4 represent evenly spaced grid lines, and the positions where they intersect are the grid nodes used in the finite difference procedure. Note that the grid nodes lie precisely on the transformed interface.

Equations (25) and (26) are solved subject to the first two boundary conditions in (28) and the regularity conditions at the boundaries.

If the curvature \( \gamma' \gamma \) is also zero at \( \xi = 0 \) or \( h \) (as is the case for both with the step, and for \( \xi = 0 \) \((z \rightarrow -\infty)\) for the isolated bump), then (29) and (30) reduce to the one-dimensional heat conduction equations and the temperature distribution on such a sidewall is independent of the geometry. This one-dimensional temperature solution is then fed in as a known boundary condition for the interior solution. In general, however, the complete regularity conditions must be solved as part of the system.

At the internal interface the jump conditions (17) become

\[ \hat{z} = \xi: \quad \hat{T}_s = \hat{\xi}, \quad K \hat{K} \frac{\partial \hat{T}_s}{\partial \hat{n}} = \frac{\partial \hat{T}_s}{\partial \hat{n}}, \quad (31) \]

where

\[ \frac{\partial \hat{T}}{\partial \hat{n}} = \frac{1}{\Delta} \left( -\gamma \frac{\partial \hat{T}}{\partial \hat{\xi}} + \frac{\partial \hat{T}}{\partial \hat{\eta}} \right), \quad \Delta = \left( 1 + \gamma^2 \right)^{\frac{1}{2}}, \quad (32) \]

and the normalization factor \( \Delta \) cancels out on either side of the interface condition (31). The temperature gradients within the snow are critical in determining the rate and type of recrystallization that occurs. It is therefore useful to express the magnitude of the dimensionless temperature gradient in the mapped coordinates:

\[ |\nabla \hat{T}| = \sqrt{\gamma^2 \left( \frac{\partial \hat{T}}{\partial \hat{\xi}} \right)^2 + \left( \frac{\partial \hat{T}}{\partial \hat{\eta}} \right)^2}. \quad (33) \]

The mapped domain is illustrated in Figure 4. The flat surface of the snowpack lies at \( \hat{z} = 1 \) and the interface between the snow domain \( R^s \) and rock domain \( R^r \) is the \( 4\pi \) line between the origin \( O \) at \( (0, 0) \) and \((h, h)\). The width of the transformed domain is \( h \). The vertical far-field condition, the second equation in (28), is applied at a finite depth for numerical purposes, but this is chosen far enough down to ensure that it does not affect the solution. The fine vertical and horizontal lines in Figure 4 represent evenly spaced grid lines, and the positions where they intersect are the grid nodes used in the finite difference procedure. Note that the grid nodes lie precisely on the transformed interface.

Equations (25) and (26) are solved subject to the first two boundary conditions in (28) and the regularity conditions at the boundaries.

If the curvature \( \gamma' \gamma \) is also zero at \( \xi = 0 \) or \( h \) (as is the case for both with the step, and for \( \xi = 0 \) \((z \rightarrow -\infty)\) for the isolated bump), then (29) and (30) reduce to the one-dimensional heat conduction equations and the temperature distribution on such a sidewall is independent of the geometry. This one-dimensional temperature solution is then fed in as a known boundary condition for the interior solution. In general, however, the complete regularity conditions must be solved as part of the system.

At the internal interface the jump conditions (17) become

\[ \hat{z} = \xi: \quad \hat{T}_s = \hat{\xi}, \quad K \hat{K} \frac{\partial \hat{T}_s}{\partial \hat{n}} = \frac{\partial \hat{T}_s}{\partial \hat{n}}, \quad (31) \]
conditions (29) and (30) from an initially isothermal state by an alternating direction implicit scheme [Peaceman and Rachford, 1962]. This is a time-splitting method [e.g., Press et al., 1986] which requires only the solution of simple tridiagonal systems. Asymptotic expansions for small $\xi$ and $\zeta$ are required to advance the solution past the corners at $(0,0)$ and $(h,h)$ between the rock and snow domains. The algorithm has been tested by considering two special cases. First, if the thermal properties in the rock and snow domains are identical the problem reduces to a one-dimensional heat conduction solution in a uniform semi infinite medium, regardless of the interface shape. This tests whether the internal jump conditions (31) are correct. Second, with certain mappings the temperature distribution reduces to the one-dimensional solution (20) and (21) at certain known locations, so a general algorithm can be checked for accuracy at these positions. This is particularly useful for ensuring that the far-field boundary condition, the second equation in (28), has been taken deep enough into the rock and that the surface heating, the first equation in (28), has been applied for a sufficiently long time for the transients to have decayed away to leave a periodic solution.

Illustrations

The snow temperature around a buried rock was measured [Logan, 1992] on a 10 x 10 cm vertical grid on December 9, 1989, near Webster Pass in central Colorado. The data provide a two-dimensional snow temperature field at a single instant in time. The snow surface temperature directly above the bump was warmer than in the surrounding snowpack and the temperature gradients were locally intensified by the presence of the rock. In response to this influence very coarse depth hoar crystals were found up to 20 cm from the rock.

Logan’s [1992] data are in qualitative agreement with a typical two-dimensional periodic temperature solution for an isolated bump (height $h = 0.5$ and width $l = 1$) buried to a maximum depth $d^* = z^*_c$ by high-density snow ($\rho = 400 \text{ kg m}^{-3}$). This shows that in the horizontal far-field (as $\zeta \to -\infty$) the curvature of the bump tends to zero ($\gamma' \to 0$) and the regularity conditions (29) and (30) reduce to one-dimensional heat conduction equations. The temperature therefore reduces to a one-dimensional periodic solution as $\zeta \to -\infty$, $\zeta = 0$). Halfway up the bump at $\zeta = -2/\pi$ ($\xi = h/2$) lateral diffusion moderates the amplitude of the temperature oscillations (Figure 5) and the two-dimensional periodic (solid) isotherms lie slightly inside those of the equivalent one-dimensional analytic solution (dashed isotherms). At the apex of the bump $\zeta = 0$ lateral diffusion intensifies the strength of the temperature oscillations (Figure 6), as the one-dimensional (dashed) isotherms lie inside the two-dimensional (solid) isotherms. Note, that the amplitude of the temperature oscillations over the bump (Figure 6) is lower than halfway down the bump (Figure 5) and in the far-field. This is because the presence of an interface intensifies the local temperature gradients, so if the same number of isotherms are to be squeezed into a reduced snow depth, as in Figure 3, a larger surface heat flux is required. As the surface heat flux is spatially uniform in this case, the surface temperature gradient is uniform and the boundary condition can only be satisfied by reducing the amplitude of the temperature oscillations.

Figure 7 shows a cross section of the isotherms (solid lines) at time $t = 0$ when the surface heat flux is at a

![Figure 8](image-url)
maximum. The dashed lines show a sequence of adjacent one-dimensional temperature profiles with the interface positioned at $\tilde{z} = \tilde{g}$, which is equivalent to the case when there is no lateral heat diffusion. The two-dimensional solution is still dominated by vertical conduction; however, the lateral temperature gradients close to the bump are moderated by increasing the temperature over the apex and reducing the temperature slightly away from the bump, so that the isotherms describe a straighter path. In the far-field, which is in practice several order unity length scales away from the bump, the two temperature fields are identical. Once heat has managed to penetrate into the rock it travels 50% faster than in the snow, which accounts for the deeper penetration of the $T = 0$ isotherm in Figure 7. The modulus of the temperature gradient $|\nabla T|$ is plotted in Figure 8 at time $\tilde{t} = 0$. Greatly intensified temperature gradients form a lens over the top of the bump; these are in part due to the natural one-dimensional squeezing of the isotherms and are also intensified by the moderation of the lateral gradients. Further down the bump $|\nabla T|$ is less than the values suggested by the one-dimensional solution, because lateral diffusion has moderated the vertical gradients.

A cross section of the isotherms during a period of surface cooling, corresponding to Logan's [1992] observations, occurs at $\tilde{t} = 2$. These are identical to the isotherms at $\tilde{t} = 0$ except that the temperatures have the opposite sign. During cooling the temperatures over the rock will be warmer than the surroundings in agreement with Logan's [1992] observations and during warming the situation is reversed. The modulus of the temperature gradient at time $\tilde{t} = 2$ is identical to that at $\tilde{t} = 0$, plotted in Figure 8, but the flow of heat is in the opposite direction. Recall that crystal growth increases linearly with the temperature gradient [Colbeck, 1983] and nonlinearly with temperature [Lamb and Hobbs, 1971]. It follows that during cooling the bump provides very favorable conditions for crystal growth as the temperatures are warmer and the temperature gradients are larger than in the surrounding snow.

The effect of varying the parameters $\tilde{h}$, $l$, $\rho'$, and $z_s$ is investigated by measuring the maximum temperature and the maximum temperature gradient at four locations $P_1$, $P_2$, $P_3$, and $P_4$ in the snow. The first point $P_1$ is positioned at the snow surface directly above the bump at $(0,1)$, point $P_2$ is at the apex of the bump at $(0, \tilde{h})$, $P_3$ lies midway down the bump at $(-2l/\pi, \tilde{h}/2)$ and $P_4$ is at the base of the bump at $(-\infty, 0)$. Note that the points $P_1, P_4$ are fixed spatially, but positions $P_2, P_3$ vary with $\tilde{h}$ and $l$. Each of the parameters is then varied in turn away from the those used in the basic simulation: $\tilde{h} = 1/2$, $l = 1$, $\rho' = 400 \text{ kg m}^{-3}$ and $z_s = 1$.

Figure 9 illustrates how the temperature distribution changes as a function of $\tilde{h}$. When $\tilde{h} = 0$ the interface between the rock and the snow domains is flat and the system reduces to the one-dimensional periodic solution. As all the interface points $(P_2, P_3, P_4)$ lie at the same level, the maximum temperatures all equal the
Figure 10. The effect of changing the bump length $l$ on the maximum temperature (solid lines) and maximum modulus of the temperature gradient (dashed lines) attained at positions $P_1$, $P_2$, $P_3$, and $P_4$. The remaining parameters are held constant at $h = 1/2$, $\rho' = 400 \text{ kg m}^{-3}$ and $z_s = 1$.

Figure 11. The effect of changing the snow density $\rho^s$ on the maximum temperature (solid lines) and maximum modulus of the temperature gradient (dashed lines) attained at positions $P_1$, $P_2$, $P_3$, and $P_4$. The remaining parameters are held constant at $h = 1/2$, $l = 1$ and $z_s = 1$. 
analytic value, \( \max T = 0.129 \). The far-field point \( P_6 \) is independent of the curvature and therefore remains fixed at this value regardless of the variation of \( h \). There is a slight increase in the temperature and temperature gradient at the apex and midway up the bump; however, for all practical purposes the bump pushes the isotherms into a smaller space without changing the temperature too much. The temperature gradient at the midpoint point rises only slightly, but the temperature gradients over the bump increase dramatically as the bump height increases. The surface temperature variations at \( P_1 \) decrease with increasing bump height.

Lateral diffusion decreases as the length of the bump increases and the temperature and temperature gradient tend to those of a sequence of one-dimensional solutions with an interface positioned at \( \hat{z} = \hat{g} \). That is, the contours tend to the dashed lines in Figures 5, 6, 7, and 8. If the length of the bump exceeds several snow thermal conduction scales \( z_s \), then the maximum temperatures and gradients at \( P_1 \), \( P_2 \), \( P_3 \), and \( P_4 \) are simply those for no lateral diffusion. When the bump length is smaller than \( z_s \), then lateral diffusion dramatically increases \( |\nabla \hat{T}| \) over the bump (Figure 10) and slightly increases the surface temperatures as well.

Decreases in density do not greatly affect the nondimensional temperatures, the main effect of lowering the density is to steadily increase \( |\nabla \hat{T}| \) over the bump (Figure 11). However, the physical temperature magnitudes \( T^* \) in (12) are much larger for low-density snow than they are for high density snow.

When the skin depth becomes smaller (Figure 12), the effect of the bump is reduced and the snow surface temperature is spatially uniform. That is, when the thermal waves do not penetrate far enough into the pack the temperature behavior is exactly the same as for a deep snowpack [Gray and Morland, 1994]. For shallow snowpacks, when the depth is the same order or less than the skin depth, maximum surface temperatures can be dramatically moderated over the bump, and temperature gradients rise sharply throughout the snow.

Changing the geometry from a bump to a single step and a series of bumps changes the spatial distribution of the isotherms. Once again they lie fairly close to the sequence of adjacent one-dimensional temperature profiles with the interface positioned at \( \hat{z} = \hat{g} \). However, the precise form of the regularity conditions (29) and (30) has an important influence on the interior solution. To make comparisons between the three geometries a cross section of the periodic two-dimensional temperature at \( \hat{z} = 0 \) is illustrated in Figures 13 and 14 for the step and multiple bumps, respectively. These Figures are directly comparable with Figure 7 for the single bump.

The curvature terms \( \gamma \gamma' \) at \( \xi = 0, 1 \) are evaluated for each of the geometries in Table 1. For the step
\( \tilde{z} \to -\infty \) for the isolated bump in Figure 7, and indeed for any bump whose slope and curvature tend to zero in the far-field or at a symmetry line. The isotherms in Figure 14 do not tend to the one-dimensional periodic dotted isotherms at either of the symmetry lines, since \( \gamma \gamma' \) is nonzero. The full regularity conditions therefore give greater scope for lateral heat diffusion to moderate the temperature at the base of the bump and intensify the temperature at the apex of the bump.

**Conclusions**

The thermal regime in the vicinity of a snow-buried rock outcrop is altered because of the difference in thermal properties of the snow and the rock. The most important factor which determines the temperature field is how far below the surface of the snow the bump is buried. If the entire rock is buried deeper than the skin depth the snow surface temperature is spatially uniform and there is no significant distortion of the isotherms. The snow’s skin depth is determined by the first scaling in (11) and is a function of the snow density and the timescale of the surface forcing. Low-density snow has a skin depth of 6.6 cm for diurnal forcing and 1.48 m for seasonal forcing, whilst high-density snow has skin depths of 10 cm and 2.26 m, respectively. The passage of a storm on an intermediate timescale can prove to be a significant event if it increases the diurnal skin depth sufficiently to reach the rocks.

Lateral heat diffusion is important if the snow depth changes significantly on a length scale comparable with the skin depth. The temperature distribution around elongated bumps, where \( l \gg z_* \), is approximated by a sequence of adjacent one-dimensional solutions with an interface placed at \( z = g(x) \). When lateral diffusion becomes important, the temperature variations near the apex of the bump are enhanced above those of the one-dimensional solution and moderated lower down the bump. This reduces the lateral temperature gradients and straightens the isotherms. However, the temperature gradients near the apex of the bump can be dramatically enhanced. This lens of enhanced temperature gradients increases and pushes down the sides of the bump if \( l \) is reduced and \( h \) is increased.

During cooling the bump provides very favorable conditions for crystal growth as the temperatures are warmer and the temperature gradients are larger than in the surrounding snow, both of which increase the crystal growth rate [Colbeck, 1983; Lamb and Hobbs, 1971]. Sufficiently large temperature gradients can produce faceted cohesionless depth hoar crystals which reduce the strength of the snow and increase the potential risk of a slab avalanche release.

**Acknowledgments.** This research has been supported by the Natural Environmental Research Council and the In-House Laboratory Independent Research funding at CRREL.
References


S. C. Colbeck, Cold Regions Research and Engineering Laboratory, 72 Lyme Road, Hanover, NH 03755-1290.
J. M. N. T. Gray, Institut für Mechanik, Technische Hochschule Darmstadt, 64280 Darmstadt, Germany. (e-mail: gray@mechanik.th-darmstadt.de)
L. W. Morland, School of Mathematics, University of East Anglia, Norwich, NR4 7JF, England.

(Received September 12, 1994; revised March 17, 1995; accepted April 11, 1995.)