Example slides from a few Prosper-based lectures

Matthias Heil & Andrew L. Hazel

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M.Heil@maths.man.ac.uk & ahazel@maths.man.ac.uk

http://www.maths.man.ac.uk/~mheil

Department of Mathematics

University of Manchester

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- For simplicity assume $\mathbf{x}_0 = \mathbf{0}$, then:

$$\mathbf{x}_{n+1} = \sum_{i=0}^{n} \mathbf{r}_i \quad \text{where } \mathbf{r}_0 = \mathbf{b}.$$

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$$\mathbf{x}_{n+1} = (\mathbf{I} - \mathbf{A})\mathbf{x}_n + \mathbf{b} \quad (1) \qquad \mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{r}_n \quad (2) \qquad \mathbf{x}_{n+1} = \sum_{i=0}^{n} \mathbf{r}_i \quad (3)$$

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$$\mathbf{r}_{n+1} = (\mathbf{I} - \mathbf{A})^{n+1}\mathbf{r}_0$$

Note that this shows that the iteration only converges if $\rho(\mathbf{I} - \mathbf{A}) < 1$.

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- Insert (4) into (3):

$$\mathbf{x}_{n+1} = \sum_{i=0}^{n} (\mathbf{I} - \mathbf{A})^i \mathbf{r}_0 = \sum_{i=0}^{n} (\mathbf{I} - \mathbf{A})^i \mathbf{b}$$

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- So while we're carrying out the iteration

$$\mathbf{x}_n = (\mathbf{I} - \mathbf{A})\mathbf{x}_{n-1} + \mathbf{b}$$

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- In other words:

$$\mathbf{x}_n \in span(\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, ..., \mathbf{A}^{n-1}\mathbf{b}) =: \mathcal{K}_n(\mathbf{A}, \mathbf{b})$$

where $\mathcal{K}_n(\mathbf{A}, \mathbf{b})$ is the Krylov subspace.

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- Our (well, Richardson's...) method produces:

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- Yes, but it depends how you define 'better', or indeed 'best' as we're trying to find the 'best' approximation in a given Krylov subspace.
- Various approaches and the resulting schemes:
  - Minimise the residual (MINRES, GMRES)
  - Minimise the error (GMERR)
  - Make the residual orthogonal to the Krylov subspace (Ritz/Galerkin: CG, CGS)
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  - ...
- An important technical question is how to construct the basis for the Krylov subspace if we abandon our simple iterative scheme.

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- Idea: Solve a 'preconditioned' problem

$$\mathcal{P}\mathbf{Ax} = \mathcal{P}\mathbf{b}$$

where $\mathcal{P}$ is chosen such that the iterative method applied to $\mathcal{P}\mathbf{A}$ converges more quickly.

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- Some popular choices:
  - Diagonal preconditioner (also in block form):

$$\mathcal{P} = (diag\mathbf{A})^{-1}$$

  - Incomplete LU factorisation:

$$\mathcal{P} = (\mathbf{L}_{inc}\mathbf{U}_{inc})^{-1}$$

- In general, the choice of preconditioner is very problem dependent.

Assessing the convergence: The minimal polynomial

- We have already seen that the convergence is related to the eigenvalues of the iteration matrix; $\rho(\mathbf{I} - \mathbf{A}) < 1$ for the Richardson iteration.

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for some coefficients β_j.

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- Applying this to $q(\mathbf{B}) = 0$ gives:

$$0 = \beta_0 \mathbf{I} + \beta_1 \mathbf{B} + ... + \beta_N \mathbf{B}^m$$

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- This shows that

$$\mathbf{x} \in \mathcal{K}_m(\mathbf{B}, \mathbf{b}).$$

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- This characterises the solution to $\mathbf{Bx} = \mathbf{b}$ as a member of a Krylov subspace

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- Recall that during the n-th iteration, a Krylov subspace method will determine the 'best' approximate solution in the Krylov subspace $\mathcal{K}_n(\mathbf{B}, \mathbf{b})$.

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- $\Longrightarrow$ The solution will be found in at most m iterations!
- This provides an alternative characterisation of a good preconditioner:

> 'A sufficient condition for a good preconditioner is that the preconditioned matrix $\mathbf{B} = \mathcal{P}\mathbf{A}$ has a low degree minimal polynomial.' (Murphy *et al.* 2000)