Continuous functions nowhere differentiable.

In 1872 Weierstrass showed that there exist continuous functions that are nowhere differentiable. Such a function can be constructed out of infinitely many translates of $|x|$, a function which is not differentiable at the origin. Note that this example is not due to Weierstrass, for that, go to the end.

On $[0,1]$ let

$$g_0(x) = \begin{cases} 
  x & \text{if } 0 \leq x \leq 0.5, \\
  1 - x & \text{if } 0.5 \leq x \leq 1.
\end{cases}$$

This can be written as

$$g_0(x) = \frac{1}{2} - \left| x - \frac{1}{2} \right|.$$ 

Graph of $g_0$:

Next define $g_1$ on $\mathbb{R}$ by demanding that $g_1(x + m) = g_0(x)$ for all $x \in [0,1]$ and $m \in \mathbb{Z}$, so $g_1$ now has period 1. This can be written as

$$g_1(x) = \frac{1}{2} - \left| x - \lfloor x \rfloor - \frac{1}{2} \right|$$

where $\lfloor x \rfloor$ is the largest integer $n \leq x$. 

1
Then for any \( n \geq 1 \) let
\[
g_n(x) = g_1 \left( \frac{4^{n-1}x}{4^{n-1}} \right).
\]

Note that the period of \( g_n \) is \( 1/4^{n-1} \). Finally define
\[
g(x) = \sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} g_1 \left( \frac{4^{n-1}x}{4^{n-1}} \right)
\]
for \( x \in \mathbb{R} \).

Graph of \( g_1 + g_2 \) on \([0, 1]\):

\[
\begin{array}{c}
\text{Graph of } g_1: \\
\end{array}
\]
Graph of $g_1 + g_2 + g_3$ on $[0, 1]$:

Graph of $g_1 + g_2 + g_3 + g_4$ on $[0, 1]$:

Graph of $g_1 + g_2 + g_3 + g_4 + g_5$ on $[0, 1]$:
There is no point in drawing the sum with more terms, for the additional
terms would give a contribution smaller than the width of the line used to
draw these graphs. Thus the graph would look identical to the last one above.

Because $|g_n(x)| \leq 2/4^n$ for all $x$, a comparison test will give that $g(x)$ is
defined for all $x$. But we have to do more if we hope to say that “the infinite
sum of continuous functions is continuous”. For this you have to use the
Weierstrass M-test and I leave it to the interested student to look this up.

One interesting property of $g(x)$ is that it is nowhere monotonic. That
is, there is no point $a \in \mathbb{R}$ which is contained in some interval $(a - \delta, a + \delta)$
on which $g$ is monotonic. In other words, for all $a \in \mathbb{R}$ and all $\delta > 0$ there
exist $x, y \in (a - \delta, a + \delta)$ with $x > y$ and $g(x) > g(y)$ and there exist
$x', y' \in (a - \delta, a + \delta)$ with $x' > y'$ and $g(x') < g(y')$. We do not prove this
here. We do, instead, prove

**Theorem**

The function $g$ is nowhere differentiable on $\mathbb{R}$.

**Proof** Let $a \in \mathbb{R}$ be given. Let $n \geq 1$.

**Claim** There exists $h_n$ (depending on $a$ as well as $n$) such that

$$|h_n| = \frac{1}{4^n} \quad \text{and} \quad |g_n(a + h_n) - g_n(a)| = |h_n|.$$  

**Proof of Claim** Since we require $|h_n| = 1/4^n$, we must have $h_n = \pm 1/4^n$
with the sign to be chosen. Write $4^{n-1}a = q + r$. Then

$$4^{n-1}(a + h_n) = 4^{n-1}(a \pm \frac{1}{4^n}) = q + r \pm \frac{1}{4}$$

where the sign in $\pm$ is to be chosen.

If $0 \leq r < \frac{1}{4}$ choose $h_n = \frac{1}{4^n}$, so $4^{n-1}(a + h_n) = q + r + \frac{1}{4}$
and, since $0 < r + \frac{1}{4} < \frac{1}{2}$, we have $g_1\left(4^{n-1}(a + h_n)\right) = r + \frac{1}{4}$,

if $\frac{1}{4} \leq r < \frac{1}{2}$ choose $h_n = -\frac{1}{4^n}$, so $4^{n-1}(a + h_n) = q + r - \frac{1}{4}$
and, since $0 < r - \frac{1}{4} < \frac{1}{2}$, we have $g_1\left(4^{n-1}(a + h_n)\right) = r - \frac{1}{4}$,

if $\frac{1}{2} \leq r < \frac{3}{4}$ choose $h_n = \frac{1}{4^n}$, so $4^{n-1}(a + h_n) = q + r + \frac{1}{4}$
and, since $\frac{1}{2} < r + \frac{1}{4} < 1$, we have $g_1\left(4^{n-1}(a + h_n)\right) = 1 - \left(r + \frac{1}{4}\right)$,

if $\frac{3}{4} \leq r < 1$ choose $h_n = -\frac{1}{4^n}$, so $4^{n-1}(a + h_n) = q + r - \frac{1}{4}$
and, since $\frac{1}{2} < r - \frac{1}{4} < 1$, we have $g_1\left(4^{n-1}(a + h_n)\right) = 1 - \left(r - \frac{1}{4}\right)$,
Thus, with these choices of $h_n$, we have

If $0 \leq r < \frac{1}{4}$ then $g_1(4^{n-1}a) = r$ so $g_1(4^{n-1}(a + h_n)) - g_1(4^{n-1}a) = \frac{1}{4}$,

if $\frac{1}{4} \leq r < \frac{1}{2}$ then $g_1(4^{n-1}a) = r$ so $g_1(4^{n-1}(a + h_n)) - g_1(4^{n-1}a) = -\frac{1}{4}$,

if $\frac{1}{2} \leq r < \frac{3}{4}$ then $g_1(4^{n-1}a) = 1 - r$ so $g_1(4^{n-1}(a + h_n)) - g_1(4^{n-1}a) = -\frac{1}{4}$,

if $\frac{3}{4} \leq r < 1$ then $g_1(4^{n-1}a) = 1 - r$ so $g_1(4^{n-1}(a + h_n)) - g_1(4^{n-1}a) = \frac{1}{4}$.

Thus in all cases we have

$$|g_1(4^{n-1}(a + h_n)) - g_1(4^{n-1}a)| = \frac{1}{4}.$$  

Hence

$$|g_n(a + h_n) - g_n(a)| = \frac{1}{4^{n-1}} |g_1(4^{n-1}(a + h_n)) - g_1(4^{n-1}a)|$$

$$= \frac{1}{4^{n-1}} \times \frac{1}{4} = \frac{1}{4^n} = |h_n|$$

as required. □

**Note** that if above we choose any $h'_n$ with $|h'_n| \leq |h_n|$ but *with the same sign* then the proof above shows that

$$|g_1(4^{n-1}(a + h'_n)) - g_1(4^{n-1}a)| = 4^{n-1}|h'_n|$$

and thus

$$|g_n(a + h'_n) - g_n(a)| = |h'_n|. \quad (1)$$

**Claim** With the choice of $h_n$ above we have

$$|g_m(a + h_n) - g_m(a)| = \begin{cases} |h_n| & \text{if } m \leq n \\ 0 & \text{if } m > n. \end{cases}$$

**Proof of Claim** Assume $m > n$. Then

$$g_m(a + h_n) = \frac{1}{4^{m-1}} g_1(4^{m-1}(a + h_n))$$

$$= \frac{1}{4^{m-1}} g_1(4^{m-1}(a \pm \frac{1}{4^n}))$$

since $h_n = \pm 1/4^n$ for some choice of the sign,

$$= \frac{1}{4^{m-1}} g_1(4^{m-1}a \pm 4^{m-n-1})$$

$$= \frac{1}{4^{m-1}} g_1(4^{m-1}a)$$

since $4^{m-n-1} \in \mathbb{Z}$ and $g_1$ is of period 1

$$= g_m(a).$$
Hence
\[ g_m (a + h_n) - g_m (a) = 0. \]

Assume \( m < n \). Then
\[ g_m (a) = \frac{1}{4^{n-1}} g_1 (4^{n-1} a) = \frac{4^{n-m}}{4^{n-1}} g_1 \left( 4^{n-1} \left( \frac{a}{4^{n-m}} \right) \right) = 4^{n-m} g_n \left( \frac{a}{4^{n-m}} \right). \]

Thus
\[ g_m (a + h_n) - g_m (a) = 4^{n-m} \left( g_n \left( \frac{a}{4^{n-m}} + \frac{h_n}{4^{n-m}} \right) - g_n \left( \frac{a}{4^{n-m}} \right) \right). \]

Use the note above, as in (1) with \( h'_n = h_n / 4^{n-m} \) to say
\[ |g_m (a + h_n) - g_m (a)| = 4^{n-m} \left( \left| g_n \left( \frac{a}{4^{n-m}} + \frac{h_n}{4^{n-m}} \right) - g_n \left( \frac{a}{4^{n-m}} \right) \right| \right) \]
\[ = 4^{n-m} \times \left| \frac{h_n}{4^{n-m}} \right| = |h_n| \]
as required.

We use this claim to say that
\[ g (a + h_n) - g (a) = \sum_{m=1}^{\infty} (g_m (a + h_n) - g_m (a)) = \sum_{m=1}^{n} (g_m (a + h_n) - g_m (a)), \]
because the terms are zero when \( m > n \). Then
\[ \frac{g (a + h_n) - g (a)}{h_n} = \sum_{m=1}^{n} \frac{g_m (a + h_n) - g_m (a)}{h_n}. \]  \hspace{1cm} (2)

Since \( |g_m (a + h_n) - g_m (a)| = |h_n| \) we have that, for each term in the sum,
\[ \frac{g_m (a + h_n) - g_m (a)}{h_n} = +1 \text{ or } -1. \]

Thus \( (g (a + h_n) - g (a)) / h_n \) is a finite sum of +1’s and −1’s.

First, this means that for every \( n \geq 1 \), the sum \( (g (a + h_n) - g (a)) / h_n \) in (??) is an integer.

But further, \( (g (a + h_n) - g (a)) / h_n \) is a sum of \( n \) terms, each either +1 or −1. Let \( r \) be the number of terms that are +1, and \( s \) the number that are −1. Then the number of terms in the sum \( n = r + s \) while the value of the sum is \( r - s \). Note that if \( n \) is odd then \( r \) and \( s \) have different parity (i.e.
one is odd, the other even) and so \( r - s \), the value of the sum, is odd. If \( n \) is even then \( r \) and \( s \) have the same parity, in which case \( r - s \), the value of the sum, is even.

Hence the sequence

\[
\left\{ \frac{g(a + h_n) - g(a)}{h_n} \right\}_{n \geq 1}
\]

is of integers which are odd when \( n \) is odd and even when \( n \) is even. Hence

\[
\lim_{n \to \infty} \frac{g(a + h_n) - g(a)}{h_n}
\]

(3)
cannot exist with finite limit. If \( g \) were differentiable at \( a \) then

\[
\lim_{h \to 0} \frac{g(a + h) - g(a)}{h}
\]

(4)
would exist. Since the sequence \( \{h_n\}_{n \geq 1} \) chosen above satisfies \( \lim_{n \to \infty} h_n = 0 \), the existence of (3) would imply the existence of (2) with the same value. Since the limit in (2) does not exist the function cannot be differentiable at \( a \).

Note you may think that this example works because the function \( g_1 \) is non-differentiable at a point. This is not the case. It works because we take a sequence of periodic functions \( g_n \) with ever decreasing periods (i.e., ever higher frequencies of oscillation). We have to dampen the magnitude of the oscillations (by the factor \( 1/4^{n-1} \) in the sum) so we can add them together.

We can get these oscillations from trigonometric functions. And this was the example of Weierstrass’s.

**Theorem** Let \( 0 < a < 1 \), and choose a positive odd integer \( b \) large enough that \( \frac{\pi}{ab - 1} < \frac{2}{3} \). Define the function \( W : \mathbb{R} \to \mathbb{R} \),

\[
W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x).
\]

Then \( W \) is continuous on \( \mathbb{R} \) but differentiable nowhere.

**Note**, that this shows that an infinite sum of differentiable functions need not be differentiable anywhere!

**Reference:**