Decay properties for functions of matrices over $C^*$-algebras

Michele Benzi
(joint work with Paola Boito, Université de Limoges, France)

Decay properties of inverses, exponentials and other functions of band or sparse matrices over $\mathbb{R}$ or $\mathbb{C}$ have been investigated by several authors in recent years [3, 4, 13, 18, 19, 22]. Such properties play an important role in various applications including electronic structure computations in quantum chemistry [2, 5], quantum information theory [8, 9, 15, 25], high-dimensional statistics [1], solid state physics [23] and numerical analysis [27], to name a few.

Following a suggestion by P.-L. Giscard and coworkers [16], we consider generalizations of existing decay estimates to functions of matrices with entries in more general algebraic structures than the familiar fields $\mathbb{R}$ or $\mathbb{C}$. In particular, we propose extensions to functions of matrices with entries from the following classes of normed algebras over $\mathbb{C}$:

1. Commutative algebras of continuous functions;
2. Non-commutative algebras of bounded operators on a Hilbert space.

The theory of complex $C^*$-algebras provides the natural abstract setting for the desired extensions [21, 24]. A $C^*$-algebra $\mathcal{A}$ is a Banach algebra endowed with an involution $^* : \mathcal{A} \to \mathcal{A}$ which satisfies $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$. Matrices over such algebras arise naturally in various application areas, including parametrized linear systems and eigenproblems [7, 26], differential equations [12], control theory [6, 10, 11], and quantum physics [14]. The study of matrices over $C^*$-algebras is also of independent mathematical interest; see, e.g., [17, 20].

Using the holomorphic functional calculus, we establish exponential off-diagonal decay results for analytic functions of banded $n \times n$ Hermitian matrices over $C^*$-algebras, both commutative and non-commutative. Our decay estimates are expressed in the form of computable bounds on the norms of the entries of $f(A)$ where $A = [a_{ij}]$ is an $n \times n$ matrix with entries $a_{ij} = a_{ji}^*$ in a $C^*$-algebra $\mathcal{A}$ and $f$ is an analytic function defined on an open subset of $\mathbb{C}$ containing the spectrum of $A$, viewed as an element of the $C^*$-algebra $\mathcal{A}^{n \times n}$. The interesting case is when the constants in the bounds do not depend on $n$. Functions of general sparse matrices over $\mathcal{A}$ will also be discussed.

Although this work is primarily theoretical, we plan to show the results of some actual computations. If time allows, we will also briefly address the non-Hermitian case.

References


