Revision Notes On Series

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Convergence Of Series

A sequence of numbers is an indexed list of the form \( u_1, u_2, u_3, \ldots, u_n, \ldots \). Sequences typically arise as the solution to some recurrence relation. It is of interest to know if the \( u_n \) tend to a definite limit \( \ell \) as \( n \) tends to infinity. If so, then we write \( \lim_{n \to \infty} u_n = \ell \) or, briefly, just \( u_n \to \ell \). It matters little if the sequence is numbered from \( k = 0 \) or \( k = 1 \); what is usually important is the behaviour for large values of \( k \).

If we add up the first \( n \) terms in a sequence we get a series such as

\[
\text{Sum of first } n \text{ terms } = \sum_{k=1}^{n} u_k = u_1 + u_2 + \cdots + u_n
\]

and then we need to know if this sum is finite when we let the number of terms tend to infinity; if the sum is finite, then we say the series is convergent otherwise it is divergent. Three important series should be remembered and are often used in testing others (the first one is surprisingly divergent):

- \( u_n = \frac{1}{n} \) (Harmonic Series): \( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \) Divergent to \(+\infty\)
- \( u_n = \frac{1}{n^2} \) (Quadratic Series): \( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots \) Convergent to \( \frac{\pi^2}{6} \)
- \( u_n = \frac{1}{n!} \) (Exponential Series): \( \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots \) Convergent to \( e \)

Given an infinite series \( u_1 + u_2 + \cdots + u_n + \cdots \), here is an outline procedure to test its convergence:

1. If \( \lim_{n \to \infty} u_n \neq 0 \) then series divergent. If \( \lim_{n \to \infty} u_n = 0 \), go to 2.
2. If \( \lim_{n \to \infty} \frac{|u_{n+1}|}{|u_n|} = \ell \), then series is: convergent if \( \ell < 1 \); divergent if \( \ell > 1 \); if \( \ell = 1 \) then go to 3.
3. If \( \lim_{n \to \infty} \frac{|u_{n+1}|}{|u_n|} = \ell \), then the result is indeterminate, except for the special case of alternating series:
   - If for all \( n \), \( u_n \) and \( u_{n+1} \) are alternating in sign and \( \lim_{n \to \infty} |u_n| = 0 \), then series convergent. Example:
     \[
     u_n = \frac{(-1)^n}{n} \] (Alternating Series): \(- \frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^n}{n} + \cdots \) Convergent to \(- \log_e 2 \)

In general, it is quite difficult to find a simple expression for the limit of a convergent series.

Power Series

Generalizing polynomials, a large class of functions can be expressed as a convergent series of the form

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots
\]

for some constant coefficients \( a_n \) and some range of values \(-R < x < R\). Important example:

**Exponential Function**: \( e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \) Convergent for all real \( x \).

There is a special case for functions that have derivatives of all orders. Then, using \( f^{(n)}(a) \) to represent the \( n^{th} \) derivative of \( f \) evaluated at \( a \) we have

**Taylor Series**: \( f(x + a) = f(a) + f'(a) x + \frac{f''(a) x^2}{2!} + \cdots + \frac{f^{(n)}(a) x^n}{n!} + \cdots \)

By the procedure given above, the Taylor series is convergent for all \( |x| < R \) where \( R = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|} \).

The values \( x = \pm R \) have to be tested individually for convergence; \( R \) is the **Radius of Convergence**.

In the case that \( a = 0 \), then a Taylor Series is called a **MacLaurin Series**.

Note that some functions cannot be expressed as a power series. For example, \( f(x) = |x| \) is defined for all real \( x \) but does not have a power series expansion for this range of \( x \). It is clear why it cannot have a MacLaurin series: it is not differentiable at \( x = 0 \) because there is a corner in the graph there.