Review of Functions of Matrices: Theory and Computation, Nicholas J. Higham. SIAM, Philadelphia, PA (2008). xx+425 pp.

This is a marvellous book and I am going to praise it.

Recently I was introduced to the algebra $Pol(M(n))$ of polynomial functions on complex square matrices X, a typical member is $x_{12}x_{23}^2 x_{34}^3 - x_{55}^4$. (Why do algebraic geometers eschew the useful word functional?) Higham has nothing to say about such functionals, indeed he begins by listing the various uses of his title which he does not take up, another example being the elementwise operations on matrices such as the Hadamard (or Schur) product.

What this book is about are some of the best known elementary scalar functions of one variable that we overload by allowing the variable to be a square matrix, real or complex. The result is usually well defined but not always. For example, there is no square root of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ but even for matrices with lots of square roots (the identity matrix) it is a nontrivial task to classify them (see Section 1.6).

Many mathematical books both deserve and receive reviews of the following sort. 'The early chapters present the basic definitions and results in a well organized manner and later chapters develope the more advanced concepts in a careful and lucid way'. These assessments certainly apply to Higham's book but I want to stress that I have found intriguing results, new to me, in almost every section, particularly Chapter 1. Below I will mention some of them. The exercises at the end of each chapter vary from straightforward, through challenging, and on to research topics. I spent more time than I care to admit on some of them. Does $\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}$ have a solution? When does $\log B$ exist and yet $B^{1/2}$ not equal $\exp((\log B)/2)$?

When I was first introduced to functions of a matrix I was bothered by the fact that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ whose square is $-I_2$ is not $\sqrt{-I_2}$ according to any of the three (equivalent) definitions. This puzzle is quickly cleared up by introducing the notions of primary and nonprimary functions; but those words were never uttered when I was a student.

Let me now describe the contents. There is a long first chapter on Theory followed by a long chapter on Applications. Next comes Sensitivity and the Frechet derivative followed by methods for general functions. Then come nine chapters each devoted to a specific elementary function or a factorization, namely Schur and polar. Finally a chapter on $f(A)b$ and a fascinating Miscellany. I have also found the two appendices helpful in filling gaps in my knowledge.

In each chapter on a particular function there is a section on conditioning which gives expressions, as well as equations, to obtain the Frechet derivative in a given direction and hence obtain a computable a posteriori bound on the condition number. The next item is to see what simplifications occur when the matrix is put into Schur form. After that Higham examines the special form taken by Newton's method in each case and how scaling can can accelerate convergence. For the square root he examines six variations on the Newton recurrence and gives detailed numerical comparisons. All the approaches mentioned so far would give exact output in exact arithmetic and infinite time. There is no truncation error. Of course this is not realistic and so Pade approximations are serious contenders when computer arithmetic is invoked. One of the big contributions of this book is that it gives us, for the first time, a comprehensive look at all the aspects for each of the

functions of interest to users. There is an implicit restriction to matrices whose orders are small enough to permit factoring or inversion. In the first decade of the 21st century that limitation permits orders of a few hundreds but the constraint of the conditon number will be greater than that of size. I suspect that most applciations are to matrices of order less than 100.

As we might expect the chapters on the square root and the exponential are a bit longer than those on other functions. The subject to which this book is devoted is almost as old as matrix theory but it was the advent of sophisticated scientific computing, e.g. Matlab, that has put it on the front page. It was computation that lead to many of the results found in the first chapter: Theory. Just as the interplay of theory and experiment refined both facets of Science in the past, so now the interplay between theory, computation and experiment serves to advance all three branches of knowledge.

Higham does briefly consider large and sparse matrices in the last chapter on computing $f(A)b$. Here Krylov subspaces and contour integration play significant roles.

Let me now share a few of the nice things I learnt from the book.

1. If the given matrix has low rank and can be written as AB, with A tall and skinny then $f(tI + AB)$ may be written as $f(t)I + AMB$ for a suitable small matrix M that involves small BA. This takes us back to the Sherman-Morrison-Woodbury formulae.
2. For what class of matrices is it true that if P commutes with Q then P is a polynomial in Q?
3. In Chapter 5 we learn that the sign function (1 for scalars in the right half plane, -1 in the left, imaginary axis forbidden) plays a much more fundamental role than merely finding polygons that include eigenvalues. In general, if B^2 has no negative eigenvalues then sign(B) is a nonprimary square root of I but its norm may be arbitrarily large. Can you guess sign$\begin{pmatrix} O & B \\ I & O \end{pmatrix}$? The author gave the beautiful answer in 1997.
4. The sign function has deep connections with pth roots of a matrix and with the polar decomposition. It is shown that all iterations for computing sign(B) are backward stable.
5. I was not aware of the Paterson-Stockmeyer method for evaluating a polynomial. It minimizes the highest power needed (as low as sqrt(degree)) at the cost of saving those powers that are needed.
6. Higham presents four different variations on Newton's iteration for $B^{1/2}$

$$X_{k+1} = \frac{1}{2}(X_k + X_k^{-1}B), \quad X_1 = \frac{1}{2}(B + I).$$

7. It is interesting that some nonprimary square roots may have much smaller norm than the primary ones. How to find a minimal norm square root is an open question.

I am impressed by the huge amount of material that Higham has absorbed on our behalf and has regurgitated in a beautifully organized, clear, and friendly manner. The quotations at the end of each chapter give a nice historical perspective on the preceeding technical material and the notes indicate the size of his readng list.

This book, at this time, defines the field of matrix functions and its computational aspects.

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