BOOKS

Functions of Matrices, Theory and Computation
by Nicholas J. Higham

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General overview.
Since the time of Cayley and Sylvester, there has been great interest in the concept of matrix function. This interest, mainly motivated by the beauty and elegance of the underlying theory, has increased in the years due to the many applications that this concept has had in scientific computing and engineering. Nowadays, the interest is mainly addressed to the design and analysis of algorithms for computing matrix functions. The current research is very active and is continuously fed by the increasing demand from applications, to provide highly effective methods for solving diffcult computational problems.

The research monograph under review is the first treatment devoted to matrix functions and to algorithms for their computation. It constitutes a comprehensive, detailed and self-contained treatise on this subject. The author has collected an impressive amount of material, including theory, algorithms, applications, references and historical notes, reorganized and presented in an extraordinarily clear, neat and rigorous way.

The state of the art in research is reported with the description and analysis of algorithms for computing the most important matrix functions. Besides the matrix exponential, which is by far the most studied matrix function, the book deals with logarithm, matrix sign, square root, p^{th} root, polar decomposition, matrix cosine and matrix sine. Additional algorithms not treated in the book are addressed and commented with references to the current literature.

The reader is guided on a fascinating trip which touches nice pieces of elegant mathematics, including matrix theory and approximation theory, and where the analysis of numerical algorithms has a prominent role. Concepts and results are presented with great clarity and through illuminating examples. Throughout the book, intuition always has the priority with respect to technicalities in a clean mathematical framework. Plenty of historical notes enrich the presentation and give the subject great appeal.

Different aspects, like abstract theoretical properties and more concrete issues related to numerical analysis and algorithms, have a nice and effective interplay. For any function of interest, the fundamental steps of analyzing the conditioning of the problem and the algorithms for its solution, including accuracy of the approximation, convergence speed and the stability of the proposed iterations, together with the complexity of the related computations, are systematically pursued in a formally rigorous way. Besides the specific results of interest, the book implicitly teaches about the methodological approach that any numerical analyst should autonomously follow.

A reader interested in matrix analysis, numerical algorithms, and computational issues may find a lot of useful and amusing material whose interest goes beyond the subject of matrix functions. Here are just a few examples to give the flavor: the evaluation of a matrix polynomial by means of four different algorithms, including the Paterson-Stockmeyer method; the role played by the Fréchet derivative in the analysis of the sensitivity and conditioning of matrix functions and in the analysis of the stability of matrix iterations; the analysis of Newton's iteration, Padé approximation, and the role of scaling applied in different forms for different computational problems.

Each chapter contains a section with historical notes, references and other details concerning subjects of related interest not covered in the book. These notes are not limited just to simple citations but contain precise comments and descriptions of the different approaches encountered in the literature, including historical comments and the chronology of discoveries. Many of these comments have general interest and go beyond the specificity of the subject. Each chapter has also a list of exercises which range from simple problems addressed to graduate students, to more difficult exercises and advanced research problems which should stimulate scholars to work on this topic. At the end of the book, the solutions to all the exercises are reported.

The bibliography with its 625 references spans the period going from the early 1800's to nowadays. Honestly, it was a great pleasure bumping into quotations from Peano's work of the last decades of the 19^{th} century and ones from the two Italian mathematicians, Michele Cipolla and Giovanni Giorgi, who greatly contributed to the subject in the 1920's and 30's.

Outline of the chapters.
Chapter one contains a concise treatment of the theory of matrix functions, including definitions and properties valid for general functions. Three definitions of primary matrix function are given, via Jordan canonical form, Hermite interpolation and the Cauchy integral theorem. Their equivalence is proved. The concept of nonprimary matrix function is introduced.

Chapter 2 is addressed to applications, touching an unexpectedly large spectrum of cases including differential equations, Markov models, control theory, theoretical particle physics, nonlinear matrix equations, and matrix geometric means.

Chapter 3 concerns sensitivity analysis of matrix functions with respect to perturbations of the data, i.e., the analysis of conditioning. Properties of the Fréchet derivative are proved and used for providing estimates of the condition number. Several algorithms for computing the condition number are described and discussed.

Chapter 4 surveys the main techniques for computing or approximating a general matrix function. The main tools of this chapter are algorithms for matrix polynomial evaluation, properties of matrix Taylor series, rational approximation including Padé approximation and continuous actions, and diagonalization and triangularization techniques, including Schur decomposition and the Parlett recurrence. Functional iterations are analyzed in detail, including the related problem of convergence, termination criteria and stability. Preprocessing techniques and upper bounds to the norm of a function of a matrix are the last two topics of the chapter.

The following Chapters 5-8 are focused on the algorithmic core of the book. They deal with specific matrix functions, namely, matrix sign, matrix square root, matrix p^{th} root, and polar decomposition, with their properties and algorithms for their computation. Each chapter contains the fundamental steps of sensitivity analysis, and, concerning iterative algorithms, the analysis of convergence and stability of the iteration, together with terminating conditions. Here the major part is played by the Schur method, Newton's iteration and its variants, the Padé family of iterations, and the matrix sign iteration. The interplay between these matrix functions is pointed out.

Chapter 9 deals with a general purpose algorithm for computing a matrix function. The algorithm is based on the reordered Schur decomposition of the matrix, followed by the application of the Parlett recurrence in the block form.

Chapters 10, 11 and 12 focus on properties and algorithms of matrix exponential, matrix logarithm and matrix cosine and sine, respectively. Once again, the design and analysis of algorithms for their computation are the main issues of these chapters together with sensitivity analysis and results for estimating the condition numbers.

Computing the product of a matrix function and a vector is the core of Chapter 13. This subject is a very hot topic of increasing interest because of the need of dealing with very large and sparse input matrices. The approaches based on Krylov subspaces, quadrature and differential equations are outlined.

Chapter 14, the final and the shortest, considers the case where the input matrix has some specific structure. A list of relevant structured matrices associated with some scalar product is just reported, including Hamiltonian and symplectic matrices. The exponential decay properties of functions of band matrices are analyzed.

The Appendix is divided into five parts. Part A contains the notation list. Part B reports the main basic tools of matrix analysis and numerical analysis used throughout the book. Useful tables with the arithmetic cost of the main matrix computations are reported in Part C, while Part D contains the list of functions in the Matrix Function Toolbox of MATLAB, which contains MATLAB implementations of many of the algorithms described in the book. Part E offers solutions for the problems of each chapter.

Conclusions.
I enjoyed reading this book a lot and I am sure that anyone fond of matrices and linear algebra will be greatly delighted by it. Certainly, students and scholars interested in computational issues will find the book even more useful and stimulating.

The book is recommended to any scholar who has a theoretical or computational interest on functions of matrices, and to anyone playfully devoted to matrix theory, computations and algorithms.

The book is fundamental for a graduate course in functions of matrices, but it will also be of great support and interest in any course touching topics in computational linear algebra.