Recent Developments in Symmetric Quadratic Eigenvalue Problems

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Quadratic Eigenvalue Problems

$n \times n$ quadratic matrix polynomial

$$Q(\lambda) = \lambda^2 M + \lambda C + K.$$ 

Find scalars $\lambda$ and nonzero vectors $x, y$ s.t.

$$Q(\lambda)x = 0, \quad y^*Q(\lambda) = 0, \quad x, y \neq 0.$$ 

A nonsingular implies $Q$ has $2n$ finite eigenvalues: the roots of $\det(Q(\lambda)) = 0$.

When $A$ is singular, the degree of $\det(Q(\lambda)) = r < 2n$ and $Q$ has $r$ finite eigenvalues and $2n - r$ infinite eigenvalues.
Example

\[ M = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -6 & 0 \\ 2 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K = I. \]

\[ Q(\lambda) = \lambda^2 M + \lambda C + K \text{ is regular: } \det Q(\lambda) \neq 0. \]

Six eigenpairs \((\lambda_k, x_k), \ k = 1:6, \) given by

<table>
<thead>
<tr>
<th>(k)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_k)</td>
<td>1/3</td>
<td>1/2</td>
<td>1</td>
<td>(i)</td>
<td>(-i)</td>
<td>(\infty)</td>
</tr>
<tr>
<td>(x_k)</td>
<td>[1]</td>
<td>[1]</td>
<td>[0]</td>
<td>[0]</td>
<td>[0]</td>
<td>[1]</td>
</tr>
</tbody>
</table>
<pre><code>| \[1\] | \[1\] | \[1\] | \[0\] | \[0\] | \[0\] |
| \[0\] | \[0\] | \[0\] | \[1\] | \[1\] | \[0\] |
</code></pre>
# Classes of QEPs

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Eigenvalue</th>
<th>Eigenvector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$ nonsingular</td>
<td>$2n$ finite $\lambda$'s</td>
<td></td>
</tr>
<tr>
<td>$M$ singular</td>
<td>finite, $\infty$ $\lambda$'s</td>
<td></td>
</tr>
<tr>
<td>$M, C, K$ real</td>
<td>$\lambda$'s real or come in pairs $(\lambda, \bar{\lambda})$</td>
<td>$x$ right ei'vec of $\lambda$, right ei'vec of $\bar{\lambda}$</td>
</tr>
<tr>
<td>$M, C, K$ Hermitian</td>
<td>$\lambda$'s real or come in pairs $(\lambda, \bar{\lambda})$</td>
<td>$x$ right ei'vec of $\lambda$, $x$ left ei'vec of $\bar{\lambda}$</td>
</tr>
<tr>
<td>$M &gt; 0, C, K \geq 0$</td>
<td>$\Re(\lambda) \leq 0$</td>
<td></td>
</tr>
<tr>
<td>$M, C &gt; 0, K \geq 0$, $\gamma(M, C, K) &gt; 0$</td>
<td>$\lambda$'s real, gap between $n$ large/small $\lambda$.</td>
<td>$n$ lin. ind. ei'vecs corr. to $n$ largest $\lambda$'s.</td>
</tr>
<tr>
<td>$M, K$ Hermitian, $M &gt; 0$, $C = -C^*$</td>
<td>$\lambda$'s purely imaginary or come in pairs $(\lambda, -\bar{\lambda})$</td>
<td>$x$ right ei'vec of $\lambda$, left ei'vec of $\bar{\lambda}$</td>
</tr>
<tr>
<td>$M, K &gt; 0$ real, $C = -C^T$</td>
<td>$\lambda$'s purely imaginary</td>
<td></td>
</tr>
</tbody>
</table>

$$\gamma(M, C, K) = \min_{||x||_2 = 1} \left[ (x^* C x)^2 - 4(x^* M x)(x^* K x) \right]$$
Outline

- Applications

- Hyperbolic and elliptic QEPs
  - Testing for hyperbolicity
  - Distance to non hyperbolic/elliptic QEP

- Numerical methods for symmetric QEPs
  - Reduction to simple forms
  - Eigensolvers for tridiagonal-diagonal symm. pairs
  - Structure preserving algorithms
Vibration Analysis of Structural Systems

Undamped structural eigenproblem in engineering

\[ \lambda^2 M x + K x = 0. \]

In practice systems are subject to damping

\[ \lambda^2 M x + \lambda C x + K x = 0. \]

Proportional or Rayleigh damping: \( C = \alpha M + \beta K \).
Damped QEP transformed to \( n \) decoupled eq.

Modern structures lead to non-classically damped eigenproblems

\[ \lambda^2 M x + \lambda C x + K x = 0. \]
Some More Applications

- Constrained least squares problem

\[
\min_{x^T x = \alpha^2} \left\{ x^T A x - 2 b^T x \right\},
\]

where \( A = A^T \in \mathbb{R}^{n \times n} \), \( b \in \mathbb{R}^n \).

The solution is \( x = (A - \lambda I)^{-1} b \), where \( \lambda \) is the smallest eigenvalue of

\[
\left( \lambda^2 I - 2 \lambda A + (A^2 - \alpha^{-2} bb^T) \right) y = 0.
\]

- Signal Processing (Davila’s subspace approach).
Hyperbolic and Overdamped Systems

\[ Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0. \]

The QEP is Hermitian if \( M, C, K \) are Hermitian.

**Definition 1**  The QEP is hyperbolic if it is Hermitian with \( M > 0 \) and \( (x^*Cx)^2 > 4(x^*Mx)(x^*Kx) \) for all \( x \neq 0 \).

For any eigenpair \((x, \lambda)\), \( \lambda^2 x^*Mx + \lambda x^*Cx + x^*Kx = 0 \), with solutions

\[ \lambda = \left( -x^*Cx \pm \sqrt{(x^*Cx)^2 - 4(x^*Mx)(x^*Kx)} \right) / (2x^*Mx). \]

**Definition 2**  The QEP is overdamped if it is hyperbolic with \( C > 0 \) and \( K \geq 0 \).

For overdamped problems, \( \lambda_i \leq 0 \).
Elliptic Systems

**Definition 3** The QEP is **elliptic** if it is Hermitian with $M > 0$ and

$$ (x^* C x)^2 < 4(x^* M x)(x^* K x) \quad \text{for all } x \neq 0. $$

Elliptic QEPs have nonreal eigenvalues, and, necessarily, $K$ is pos. def.

**Theorem 1** Ellipticity of an Hermitian QEP with $M > 0$ is equivalent to either of the conditions

1. $Q(\mu) > 0$ for all $\mu \in \mathbb{R}$,

2. $(x^* C x)^2 < 4(x^* M x)(x^* K x)$ for all eigenvectors $x$ of the QEP.
Testing for Hyperbolicity

**Theorem 2**  *Hermitian QEP with $M > 0$ is hyperbolic iff $Q(\mu) < 0$ for some $\mu \in \mathbb{R}$.*

**Theorem 3**  *The Hermitian QEP with $M > 0$ is hyperbolic iff the pair $(A, B)$ is definite, where*

$$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \quad B = -\begin{bmatrix} C & M \\ M & 0 \end{bmatrix}.$$

**Proof.** $(A, B)$ definite iff $\alpha A + \beta B > 0$ for some $\alpha, \beta$. Assume $\alpha \neq 0$,

$$\alpha A + \beta B = \begin{bmatrix} I & -\frac{\beta}{\alpha} I \\ 0 & I \end{bmatrix} \begin{bmatrix} -\alpha K - \beta C - \frac{\beta^2}{\alpha} M & 0 \\ 0 & \alpha M \end{bmatrix} \begin{bmatrix} I & 0 \\ -\frac{\beta}{\alpha} I & I \end{bmatrix}.$$

So $\alpha A + \beta B$ is congruent to $\alpha \text{diag}(-(\mu^2 M + \mu C + K), M)$, where $\mu = \beta/\alpha$. The result follows from Theorem 2, since $M > 0$.  \[\square\]
Distance to non-Hyperbolic/Elliptic QEP

Let \( W(x, M, C, K) = \begin{bmatrix} 2x^*Mx & x^*Cx \\ x^*Cx & 2x^*Kx \end{bmatrix} \).

Note that \( \det(W) \begin{cases} < 0 & \forall x \neq 0 \text{ if QEP hyperbolic,} \\ > 0 & \forall x \neq 0 \text{ if QEP elliptic.} \end{cases} \)

\[ d(M, C, K) = \min \{ f(\Delta M, \Delta C, \Delta K) : \det(W(x, M + \Delta M, C + \Delta C, K + \Delta K)) = 0 \text{ for some } x \neq 0 \} \]

We have

\[ d(M, C, K) = \min_{\|x\|_2=1} g(x), \]

where \( g(x) = |\lambda_{opt}(W(x, M, C, K))| \) and \( \lambda_{opt} \) ei’values smallest modulus.

For hyperbolic problems, \( g(x) \) is differentiable for all nonzero \( x \).
Mass-Spring System

QEP is overdamped.

- Bisection alg applied to \((A, B)\) diagnoses pair definite, hence QEP hyperbolic, after 1 iteration.

- Computing \(d(M, C, K)\) via unconstrained minimization of \(\min_x g(x/\|x\|_2)\) shows \(d(M, C, K) = 2.0\), with optimal perturbations

\[
\Delta A = 0.125ee^T, \quad \Delta B = -0.25ee^T, \quad \Delta C = 0.125ee^T.
\]
Wave Equation

\[
\begin{cases}
    u_{tt} + \epsilon a(x) u_t = \Delta u, & x \in [0, \pi], \quad \epsilon > 0, \\
    u(t, 0) = u(t, \pi) = 0.
\end{cases}
\]

Approximating \( u(x, t) = \sum_{k=1}^{n} q_k(t) \sin(k\pi x) \) and applying Galerkin method leads to

\[
M \ddot{q}(t) + C \dot{q}(t) + K q(t) = 0,
\]

where \( q(t) = [q_1(t), \ldots, q_n(t)]^T \), \( M = (\pi/2)I_n \), \( K = (\pi/2)\text{diag}(j^2) \), and \( C = (c_{kj}) \), \( c_{kj} = \int_0^\pi \epsilon a(x) \sin(kx) \sin(jx) \, dx \).

We took \( n = 9 \), \( a(x) = x^2(\pi - x)^2 - \delta \), \( \delta = 201 \), \( \epsilon = 0.1 \).

The QEP is elliptic.

We found \( d(M, C, K) = 0.06 \).
Spectrum for Wave Equation

Ei’vals of original elliptic QEP marked “○” and those of perturbed QEP “+”
Second Order Systems

Undamped multi-degree of freedom second-order systems

\[ M\ddot{q} + C\dot{q} + Kq = Pf, \quad r = P^T q \]

leads to first order systems

\[ Ap + B\dot{p} = Gf, \quad s = G^T p, \quad p = [q^T, \dot{q}^T]^T. \]

Three types of analysis

- Computation of transient response \( p(t) \),
  \[ \ddot{p}(t) = B^{-1}(Gf(t) - Ap(t)). \]

- Computation of steady-state frequency response
  \[ r(\omega) = G^T(A - \omega^2 B)^{-1}Gf(\omega). \]

- Computation of natural frequencies \( \omega_n \) (determined from ei’vals.).
Linearizing Symmetric QEPS

\[ Q(\lambda) = \lambda^2 M + \lambda C + K. \]

First companion form:

\[
A - \lambda B = \begin{bmatrix} 0 & W \\ -K & -C \end{bmatrix} - \lambda \begin{bmatrix} W & 0 \\ 0 & M \end{bmatrix}.
\]

Second companion form:

\[
A - \lambda B = \begin{bmatrix} -K & 0 \\ 0 & W \end{bmatrix} - \lambda \begin{bmatrix} C & M \\ W & 0 \end{bmatrix},
\]

\(W\) can be any nonsingular \(n \times n\) matrix.

Taking \(W = -K\) in the first companion form \(W = M\) in the second companion form yield two symmetric linearizations.
Reduction to Simple Forms

$A, B: n \times n$ symmetric (indefinite) matrices.

Aim: Find $W$ nonsingular such that $W^TAW$ and $W^T BW$ are in reduced form (tridiagonal, diagonal . . .).

Two reductions requiring a finite number of steps:

- Simultaneous tridiagonalization of $(A, B)$. 
  *Useful for transient response and steady state computation.*

- Reduction to tridiagonal-diagonal form. 
  *First step in most eigensystem computation.*
  *Most compact form.*
Simultaneous Tridiagonalization


No hyp. made on nonsingularity or definiteness of $A, B$.

**Basic idea:** Assume there exists $G_1$ nonsingular such that

$$A_1 = G_1^T A G_1 = \begin{bmatrix} \kappa_1 & \tau_1 e_1^T \\ \tau_1 e_1 & \widetilde{A}_1 \end{bmatrix}, \quad B_1 = G_1^T B G_1 = \begin{bmatrix} \mu_1 & \sigma_1 e_1^T \\ \sigma_1 e_1 & \widetilde{B}_1 \end{bmatrix},$$

where $e_1 = [1, 0, \ldots, 0]^T$, $\kappa_1, \tau_1, \mu_1, \sigma_1 \in \mathbb{R}$.

Same idea applies to $\widetilde{A}_1$, $\widetilde{B}_1$. Then

$$Q = G_1 G_2 \cdots G_{n-2}$$

simultaneously tridiagonalizes $(A, B)$. 
Constructing $G$

How to construct $G \in \mathbb{R}^{\ell \times \ell}$ such that

\[
G^T e_1 = e_1, \quad G^T A G = \begin{bmatrix} \kappa & \tau e_1^T \\ \tau e_1 & \tilde{A} \end{bmatrix}, \quad G^T B G = \begin{bmatrix} \mu & \sigma e_1^T \\ \sigma e_1 & \tilde{B} \end{bmatrix} \quad (\star)
\]

Let $L = I + xy^T$ such that

\[
e_1^T L = e_1^T, \quad V^T (L^T K L) e_1 = \tilde{\kappa} w, \quad V^T (L^T M L) e_1 = \tilde{\mu} w,
\]

where $V = [e_2, e_3, \ldots, e_\ell] \in \mathbb{R}^{\ell \times (\ell - 1)}$, $\tilde{\kappa}, \tilde{\mu} \in \mathbb{R}$.

$w \in \mathbb{R}^{\ell - 1}$: free parameter vector.

Then $G = L \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix}$, $H$: Householder reflector, satisfies $(\star)$.
Numerical Experiments

- `-x-`: tridiagonalization using $G = L$ with $L = I + xy^T$.
- `-o-`: tridiagonalization using $G = LH$ with $H$: Householder matrix and $\kappa(L)$ minimal.

\[
R_A = \frac{\|Q^T AQ - T\|_2}{\|A\|_2 \|Q\|_2}
\]

\[
\kappa(G) = \max_k \kappa(G_k)
\]
Reduction to Tridiagonal-Diagonal Form

- First step in most eigensystem computations.
- Most compact form obtainable in a finite number of steps.

Consists of two stages:
- Reduction to symmetric-diagonal form,
  \[ M^T (A - \lambda B) M = C - \lambda J. \]
- Reduction to tridiagonal-diagonal form,
  \[ Q^T (C - \lambda J) Q = T - \lambda \tilde{J}. \]

\( J, \tilde{J} \): signature matrices with \( \pm 1 \) on the diagonal.
Symmetric-Diagonal Reduction

- **Block LDL^T factorization of** \( B \)

\[
P^TBP = LDL^T.
\]

\( P \): permutation matrix, \( L \) unit lower triangular.
\( D \): diagonal with \( 1 \times 1 \) or \( 2 \times 2 \) blocks.

- **Eigendecomposition of** \( D \):

\[
D = X|\Lambda|^{1/2}J|\Lambda|^{1/2}X^T, \quad J \in \text{diag}_q(\pm 1).
\]

\( X \): orthogonal, \( \Lambda \): diag. matrix of eigenvalues.

\[
M^{-1}(A - \lambda B)M^{-T} = C - \lambda J,
\]
\[
M = PLX|\Lambda|^{1/2}.
\]
Tridiagonal-Diagonal Reduction

\[ Q^T C Q = T, \quad Q^T J Q = \tilde{J}. \]

- Reduction by unified rotations.
- Reduction by unified Householder reflectors.
- Reduction by mix of Householder reflectors & hyperbolic rotations.
Unified Rotation

\[
G = \begin{bmatrix}
  c & \gamma s \\
-\gamma^*s & c
\end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad c^2 + \gamma s^2 = \pm 1, \quad \gamma = J_{11}/J_{22},
\]

\[
G^T J G = \hat{J}, \quad Gx = \alpha e_1.
\]

- \( J = \pm I \): \( G \) is a Givens rotation.
- \( J \neq \pm I \): \( G \) is a hyperbolic rotation.

For hyperbolic rotations, \( \kappa_2(G') = \frac{|c| + |s|}{||c| - |s||} \).

Condition number can be arbitrarily large.
**Reduction by Unified Rotation**

\[ C' = \begin{bmatrix}
\times & \times & 0 & 0 & 0 \\
\times & \times & \times & \times & \times \\
0 & \times & \times & \times & \times \\
0 & \times & \times & \times & \times \\
0 & \times & \times & \times & \times \\
0 & \times & \times & \times & \times \\
0 & \times & \times & \times & \times \\
0 & \times & \times & \times & \times \\
\end{bmatrix}, \quad J = \begin{bmatrix}
-1 \\
-1 \\
1 \\
1 \\
1 \\
-1 \\
\end{bmatrix}. \]

- Choose plane of rotation so that \( \kappa(G) \) is minimized.
  - Reduce # hyp. rots used in the reduction.
  - Reduce risk of 2 hyp. rots acting in same plane.
- Apply unified rotations implicitly.
Unified Householder Reflectors

\[ H = P \left( J - \frac{2vv^T}{v^T Jv} \right), \quad v^T Jv \neq 0. \]

For any \( x \) s.t. \( x^T Jx \neq 0 \), permutation \( P \) and \( v \) chosen s.t.

\[ Hx = \alpha e_1. \]

\( H \) is \( (\tilde{J}, J) \)-orthogonal, \( \tilde{J} = PJP \).

\[ \sigma^{-1}_{\text{min}}(H) = \sigma_{\text{max}}(H) = \frac{v^Tv}{|v^T Jv|} + \sqrt{\left( \frac{v^Tv}{v^T Jv} \right)^2} - 1. \]

Condition number can be arbitrarily large.
Reduction by Unified Householder Reflectors

Similar to Householder tridiagonalization of symmetric matrix.

- Choose $P$ s.t. $\kappa_2(H)$ is minimized.
- Product $Hy$ performed implicitly:

$$Hy = P \left( Jy - \frac{2v^T y}{v^T Jv} v \right).$$

- Twice less expensive than reduction via unified rotations.
- Not clear which one has better numerical properties.
**Mix Householder Reflectors/Hyperbolic Rotations**

Assume $J = \text{diag}(I_p, -I_q)$ and let $x = (x_p^T, x_q^T)^T$.

- $H_p, H_q$: householder reflectors s.t.
  \[ H_p x_p = \alpha_p e_1, \quad H_q x_q = \alpha_q e_1. \]

- $G$: hyperbolic rotation s.t. $G \begin{bmatrix} \alpha_p \\ \alpha_q \end{bmatrix} = \alpha e_1$.

Then
\[ S = G \begin{bmatrix} H_p & 0 \\ 0 & H_q \end{bmatrix}, \quad Sx = \alpha e_1. \]

Can show that
\[ \kappa_2(S) \leq \kappa_2(H). \]
Numerical Experiments

• \texttt{trd\_ur}: tridiagonalization by unified rotations.
• \texttt{trd\_BG}: Brebner & Grad’s pseudosymmetric Givens reduction.

\begin{equation}
C = \text{randn}(n); \quad C = C + C'; \quad J = \text{sign} \left( \text{randn}(n) \right);
\end{equation}

\begin{equation}
\mathcal{R} = \frac{\|Q^T C Q - T\|_2}{\|C\|_2 \|Q\|_2}
\end{equation}
Direct/Implicit Application of Rotations

How hyperbolic rotation are applied to a matrix can be crucial to the stability of the reduction.

$5 \times 5$ pair $(C, J)$ generated by \texttt{mdsmax}.

$\kappa_2(Q) = 3.02, \kappa_{\text{max}}(G) = 2 \times 10^3$

$$R = \frac{\|Q^T C Q - T\|_2}{\|C\|_2 \|Q\|_2}, \quad \mathcal{E} = \max_{i=1:n} \frac{|\lambda_i - \tilde{\lambda}_i|}{|\lambda_i|}.$$  

<table>
<thead>
<tr>
<th>$\mathcal{R}_{\text{direct}}$</th>
<th>$\mathcal{R}_{\text{implicit}}$</th>
<th>$\mathcal{E}_{\text{direct}}$</th>
<th>$\mathcal{E}_{\text{implicit}}$</th>
<th>$\text{cond}(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5 \times 10^{-12}$</td>
<td>$4 \times 10^{-15}$</td>
<td>$4 \times 10^{-10}$</td>
<td>$2 \times 10^{-13}$</td>
<td>$4 \times 10^2$</td>
</tr>
</tbody>
</table>
Comparison Between Three Reductions

- **trd\_ur**: tridiag. by unified rotations
- **trd\_uh**: tridiag. by unified Householder reflectors.
- **trd\_hr**: tridiag. by mixed Householder/hyp. rot.

300 matrices from `randn`, `randsvd`, `randjorth`.

► 85% of the time, all residuals are equivalent.

► **trd\_uh** appears the least stable.

► Residual means: $R_{ur} \approx R_{hr} = O(10^{-15})$, $R_{uh} = O(10^{-14})$.

Can generate matrices s.t. $R_{ur} \gg R_{hr}$ but harder for the converse.
Eigensolvers for Tridiagonal-Diagonal Pencils

Assume symm. indef. \((A, B)\) reduced to tridiag.-diag. form \((T, J)\).

- HZ algorithm [F.T, M. Berhanu]
  Related to HR alg. \((Brebner & Grad, Bunse-Gerstner)\)
  HZ iteration driven by polynomial \(p:\)

  \[
  \hat{T} = H^{-1}TH^{-T}, \quad \hat{J} = H^{-1}JH^{-T},
  \]
  where \(p(TJ) = HR\) (HR factorization).

- Aberth’s iteration [D. Bini, L. Gemignani, F. T.]

  \[
  \lambda_j^{(k+1)} = \lambda_j^{(k)} - \frac{\zeta(\lambda_j^{(k)})}{1 - \zeta(\lambda_j^{(k)}) \sum_{k=1}^{m} (\lambda_j^{(k)} - \lambda_k^{(k)})^{-1}}, \quad j = 1: m,
  \]
  where \(\zeta(\lambda) = p(\lambda)/p'(\lambda) = \text{trace}((TJ - \lambda I)^{-1})\).
Structure Preserving Transformation

Consider \( Q(\lambda) = \lambda^2 M + \lambda C + K \) and the symmetric linearization

\[
A = \begin{bmatrix} 0 & K \\ K & C \end{bmatrix}, \quad B = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}.
\]

For QEP with nondefective \( \lambda \), \( \exists \mathcal{W} \) real nonsingular s.t.

\[
\mathcal{W}^T A \mathcal{W} = \begin{bmatrix} 0 & D_K \\ D_K & D_C \end{bmatrix}, \quad \mathcal{W}^T B \mathcal{W} = \begin{bmatrix} D_K & 0 \\ 0 & -D_M \end{bmatrix},
\]

with \( D_K \), \( D_C \) and \( D_M \) real diagonal. [Garvey et al., 2001].

\((\tilde{A}, \tilde{B}) = \mathcal{W}^T (A, B) \mathcal{W}\) is symm. linearization of \( \lambda^2 D_M + \lambda D_C + D_K \) with diagonal coeff. matrices.
Structure Preserving Algorithms

Work in progress.

- Jacobi-like algorithm. [S. Mackey & N. Mackey, F.T.]
  *Make use of Clifford algebra.*

- Reduction of blocks to tridiagonal form. [S. Garvey, N.J. Higham, F.T.]

\[
W^T A W = \begin{bmatrix} 0 & T_K \\ T_K & T_D \end{bmatrix}, \quad W^T B W = \begin{bmatrix} T_K & 0 \\ 0 & -T_M \end{bmatrix},
\]

where \( T_K, T_D \) and \( T_M \) are tridiagonal.

Eigenvalues of \( \lambda^2 T_M + \lambda T_C + T_K \) via Aberth’s iterations.
Summary

- Symmetric QEPs occur in several applications.
- Many interesting mathematical properties.
- Need for symmetry preserving numerical methods and software.
- New reductions to tridiagonal and tridiagonal-diagonal forms.
- New and improved eigensolvers for symmetric indefinite pairs \((T, J)\).
- Towards structure preserving algorithms.

Papers and reports:

http://www.ma.man.ac.uk/~ftisseur/