Deflating Quadratic Matrix Polynomials with Structure Preserving Transformations

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Consider

\[ Q(\lambda) = \lambda^2 M + \lambda C + K, \]

with \( M, C, K \in \mathbb{R}^{n \times n} \) and \( M \) nonsingular.

QEP: Find scalars \( \lambda \) and nonzero \( x, y \in \mathbb{C}^n \) satisfying \( Q(\lambda)x = 0 \) and \( y^* Q(\lambda) = 0 \).

- \( \lambda \) is an e’val, \( x, y \) are corresponding right and left e’vecs.
- \( Q(\lambda) \) has \( 2n \) eigenvalues, solutions of \( \det(Q(\lambda)) = 0 \).
- No simple canonical form analogous to Schur form

\[ W^*(A - \lambda B)Z = (S - \lambda T), \]

\( W, Z \) unitary, \( S, T \) upper triangular.
Standard way of treating QEPs both theoretically and numerically.

Convert $Q(\lambda) = \lambda^2 M + \lambda C + K$ into a linear pencil such as

$$L(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} D & K \\ -I & 0 \end{bmatrix}.$$  

$L(\lambda)$ is a **linearization** of $Q$ if

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} Q(\lambda) & 0 \\ 0 & I \end{bmatrix}$$

for some $E(\lambda)$ and $F(\lambda)$ with constant, nonzero determinants.
Recent Work

- Better understanding of linearization process and effects of scaling on
  - conditioning of eigenvalues,
  - backward error of computed eigenpairs.

  [Adhikari, Alam, Betcke, Higham, Kressner, Li, Mackey, Mehl, Mehrmann, T., ...]

- New structure preserving linearizations derived along with algorithms preserving spectral properties in finite precision arithmetic.

  [Antoniou, Higham, Lin, Mackey, Mackey, Mehl, Mehrmann, T., Vologiannidis, ...]
Let \((\lambda_j, x_j), j = 1, 2\) be two given eigenpairs of \(Q(\lambda)\).

Want to transform \(n \times n Q(\lambda)\) into

\[
\tilde{Q}(\lambda) = \begin{bmatrix} Q_d(\lambda) & 0 \\ 0 & q(\lambda) \end{bmatrix}
\]

such that

\[
\Lambda(Q) = \Lambda(\tilde{Q}), \text{ (same spectrum)}
\]

\[
q(\lambda_j) = 0, j = 1, 2.
\]

Deflation procedure decoupling \(Q(\lambda)\) into two quadratics.
Suppose there exist nonsingular $S_L, S_R$ s.t.

$$S_L Q(\lambda) S_R = \tilde{Q}(\lambda) = \begin{bmatrix} Q_d(\lambda) & 0 \\ 0 & q(\lambda) \end{bmatrix}.$$ 

The roots $\lambda_1, \lambda_2$ of $q(\lambda)$ are e’vals of $Q$ and $\tilde{Q}$

$$Q(\lambda_j)x_j = 0, \quad \tilde{Q}(\lambda_j)e_n = 0, \quad j = 1, 2$$

with e’vecs related by

$$S_R^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e_n \\ e_n \end{bmatrix}.$$ 

Decoupling possible only if e’vecs $x_1$ and $x_2$ are parallel.
Work with a linearization $L(\lambda)$ of $Q(\lambda)$ rather than directly with $Q$.

Use similarity transformations $S_L, S_R$ that

- preserve block structure of $L(\lambda) \Rightarrow S_L L(\lambda) S_R$ is still a linearization of a quadratic $Q_1(\lambda)$,
- e’vecs of $Q_1(\lambda)$ associated with $\lambda_1, \lambda_2$ are parallel,
- e’vecs of $Q(\lambda)$ are easily recovered from those of $Q_1(\lambda)$. 
Structure Preserving Transformation (SPT)

\[ L(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \]

is a linearization of \( Q(\lambda) \).

**Definition**

\( W_L, W_R \) nonsingular define an **SPT** for \( Q \) with nonsingular \( M \) if

\[ W_L^T L(\lambda) W_R = \tilde{L}(\lambda) = \lambda \begin{bmatrix} 0 & \tilde{M} \\ \tilde{M} & \tilde{C} \end{bmatrix} + \begin{bmatrix} -\tilde{M} & 0 \\ 0 & \tilde{K} \end{bmatrix}. \]

- \( W_L, W_R \) preserve the block structure of \( L(\lambda) \).
- \( \tilde{L}(\lambda) \) is a linearization of \( \tilde{Q}(\lambda) = \lambda^2 \tilde{M} + \lambda \tilde{C} + \tilde{K} \).
Let \( T = I_{2n} + \begin{bmatrix} ab^T & ad^T \\ af^T & ah^T \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \) for \( a, b, d, f, h \in \mathbb{R}^n \).

- Rank-2 modification of \( I_{2n} \).

- For almost all \( a \in \mathbb{R}^n \), any solution \( V = [b \ d \ f \ h] \) to \( VA = B \) defines an SPT \( T \), \( A \in \mathbb{R}^{4 \times 3} \), \( B \in \mathbb{R}^{n \times 3} \) depend on \( a, M, C \) and \( K \).

- If \( (M, C, K) \xleftarrow{T} (\tilde{M}, \tilde{C}, \tilde{K}) \) then \( \tilde{M}, \tilde{C}, \tilde{K} \) are low rank modifications of \( M, C, K \).
(\(\lambda_1, x_1\)), (\(\lambda_2, x_2\)) to be deflated with \(\lambda_1 \neq \lambda_2\) and \(x_1 \neq \alpha x_2\).

**Aim:** construct \( T = I_{2n} + \begin{bmatrix} ab^T & ad^T \\ af^T & ah^T \end{bmatrix} \) and nonzero \( z \in \mathbb{R}^n \) s.t.

- \( Q(\lambda) \xrightarrow{T} \tilde{Q}(\lambda) \) with \( \tilde{Q}(\lambda_j)z = 0, \ j = 1, 2 \).

  Yields \( a, z \) and \( z^T \begin{bmatrix} b & d & f & h \end{bmatrix} = z^TV \).

- \( T \) is an SPT \( \iff VA = B \).
Existence of SPT $T$

**Theorem**

*Eigenpairs* $(\lambda_1, x_1), (\lambda_2, x_2)$ with $\lambda_1 \neq \lambda_2$ either real or complex conjugate can be mapped to $(\lambda_1, z), (\lambda_2, z)$ by elementary SPTs if

- $x_j^T Q'(\lambda_j) x_j \neq 0, j = 1, 2,$
- *real eigenpairs have opposite type:*

$$\text{sign}(x_1^T Q'(\lambda_1) x_1) = -\text{sign}(x_2^T Q'(\lambda_2) x_2).$$

Can generate a family of SPTs mapping $(\lambda_j, x_j)$ to $(\lambda_j, z)$, $j = 1, 2$. 

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Structured Deflation
Lemma

If \((\lambda_j^2 M + \lambda_j C + K)z = 0, \ j = 1, 2\) with \(\lambda_1 \neq \lambda_2\) then

\[(M, C, K)z = (mp, cp, kp), \ p \in \mathbb{R}^n, \ p^T z = 1,\]

\[c = -m(\lambda_1 + \lambda_2), \ k = m\lambda_1\lambda_2.\]
Lemma

If \((\lambda_j^2 M + \lambda_j C + K)z = 0, j = 1, 2\) with \(\lambda_1 \neq \lambda_2\) then

\((M, C, K)z = (mp, cp, kp), p \in \mathbb{R}^n, p^Tz = 1,\)

\[c = -m(\lambda_1 + \lambda_2), \quad k = m\lambda_1\lambda_2.\]

Let nonsingular \(G\) be such that

\[Ge_n = z, \quad G^Tp = e_n.\]

Then \(G^TMGe_n = G^TMz = mG^Tp = me_n\) and Lemma \(\Rightarrow\)

\[G^T(M, C, K)G = \left(\begin{bmatrix} \tilde{M} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{C} & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} \tilde{K} & 0 \\ 0 & k \end{bmatrix}\right).\]
Example 1

\[ Q(\lambda) = \lambda^2 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \]

Given \( \lambda_{1,2} = -0.34 \pm 1.84i \) and associated e’vecs, our deflation procedure yields

\[ \lambda^2 \begin{bmatrix} 5.6 & 2.0e-16 \\ 2.0e-16 & -1.4e-1 \end{bmatrix} + \lambda \begin{bmatrix} -1.6 & -9.4e-16 \\ -9.4e-16 & -9.3e-2 \end{bmatrix} + \begin{bmatrix} 1.6 & -9.8e-17 \\ -9.8e-17 & -4.8e-1 \end{bmatrix}, \]

with \( \kappa_2(T) = 7.9 \) and \( \kappa_2(G) \approx 1. \)

Decoupling accomplished to within the working precision.
Example 2: Damped Beam Problem

$M, C, K$ generated by \texttt{nlevp('damped\_beam',nele)}.

$Q(\lambda) = \lambda^2 M + \lambda C + K$ and undamped $Q_u(\lambda) = \lambda^2 M + K$ have $n$ e’vals in common that we deflate by

- our decoupling procedure: $Q \xrightarrow{S} \left[ \begin{array}{cc} Q_1(\lambda) & 0 \\ 0 & Q_2(\lambda) \end{array} \right]$,
- using special property of $Q(\lambda)$ to orthogonally block diag’lize it and then diag’lize one block with Cholesky-QR (transformation $W$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\kappa_2(S)$</th>
<th>$\kappa_2(W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>4.47e1</td>
<td>3.79e1</td>
</tr>
<tr>
<td>32</td>
<td>9.57e1</td>
<td>7.84e1</td>
</tr>
<tr>
<td>64</td>
<td>1.95e2</td>
<td>1.57e2</td>
</tr>
</tbody>
</table>

$k_2(E) = \|E\|_2\|E^{-1}\|_2$

$k_2(S)$ not much larger that $k_2(W)$. 
Concluding Remarks

- Deflation procedure extends to nonsymmetric quadratics.
- First attempt at defining an SPT with a well-defined action.
- Deflation procedure finds application in
  - second-order model reduction,
  - model updating with no spill-over.

For papers and Eprints,
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