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Standard Triples of Structured Matrix Polynomials

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Standard and Jordan Triples

Standard and Jordan triples $(U, \mathcal{T}, V)$ for

$$P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j, \quad A_j \in \mathbb{C}^{n\times n}, \ \det(A_m) \neq 0.$$

- Introduced and developed by Gohberg, Lancaster and Rodman.
- Play a central role in the theory of matrix polynomials.
- Extend notion of Jordan pair (X, J) for $A \in \mathbb{C}^{n\times n}$.

Aim: study standard and Jordan triples of **structured matrix polynomials**.

Structured Matrix Polynomials

$P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$, $A_j \in \mathbb{F}^{n\times n}$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, $\det(A_m) \neq 0$.

Structure	**Definition**	**Coeffs property**
Hermitian	$P(\lambda) = P^*(\lambda)$	$A_j = A_j^*$
symmetric	$P(\lambda) = P^T(\lambda)$	$A_j = A_j^T$
skew-Hermitian	$P(\lambda) = -P^*(\lambda)$	$A_j = -A_j^*$
$\star$-even	$P(\lambda) = P^\star(-\lambda)$	$A_j = (-1)^j A_j^\star$
$\star$-odd	$P(\lambda) = -P^\star(-\lambda)$	$A_j = (-1)^{j+1} A_j^\star$
$\star$-palindromic	$P(\lambda) = \lambda^m P^\star(\frac{1}{\lambda})$	$A_j = A_{m-j}^\star$
$\star$-antipalindromic	$P(\lambda) = -\lambda^m P^\star(\frac{1}{\lambda})$	$A_j = -A_{m-j}^\star$

Here $\star = T$ (transpose) or $\star = *$ (conjugate transpose).

NLEVP Toolbox

Collection of Nonlinear Eigenvalue Problems: T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, F. T., 2011.

- Quadratic, polynomial, rational and other nonlinear eigenproblems.
- Provided in the form of a MATLAB Toolbox.
- Problems from real-life applications + specifically constructed problems.

http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html

Structured Quadratics from NLEVP

$n \times n$ quadratic $Q(\lambda) = \lambda^2 M + \lambda D + K$.

speaker box (`pep,qep,real,` `symmetric`).
$n = 107$. Finite element model of a speaker box.
$\|M\|_2 = 1$, $\|D\|_2 = 5.7 \times 10^{-2}$, $\|K\|_2 = 1.0 \times 10^7$.

wiresaw1 (`pep,qep,` `t-even` `, ..., scalable`).
Gyroscopic QEP from vibration analysis of a wiresaw.
$M = M^T$, $D = -D^T$, $K = K^T$.

railtrack (`pep,qep,` `t-palindromic` `,sparse`).
$n = 1005$. Model of vibration of rail tracks under the excitation of high speed trains. $M = K^T$, $D = D^T$.

gen_tantipal2 (`pep,qep,real,` `T-antipalindromic` `, ..., random`). T-anti-palindromic QEP with eigenvalues on the unit circle.

Inverse Polynomial Eigenvalue Problem

Construct $n \times n$ $P(\lambda)$ of degree m having a given list of elementary divisors, $(\lambda - \lambda_i)^{\alpha_{ij}}$, $i = 1\colon s$, $j = 1\colon t_j \leq n$.

- Use procedure in the proof of Thm. 1.7 in *Matrix Polynomials*, Gohberg, Lancaster, Rodman, 1982.
  - This procedure does not generate structured matrix polynomials.
- Use ($\mathcal{S}$-structured) **standard triples**.

Standard Triples for Matrix Polynomials

$(U, \mathcal{T})$ is an **(m, n)-standard pair** over $\mathbb{F}$ if $\mathcal{T} \in \mathbb{F}^{mn \times mn}$ and $U \in \mathbb{F}^{n \times mn}$ are such that $\det Q(U, \mathcal{T}) \neq 0$, where

$$Q(U, \mathcal{T}) := \begin{bmatrix} U\mathcal{T}^{m-1} \\ \vdots \\ U\mathcal{T} \\ U \end{bmatrix}.$$

$(U, \mathcal{T}, V)$ is an **(m, n)-standard triple** over $\mathbb{F}$ if $(U, \mathcal{T})$ is a standard pair over $\mathbb{F}$ and $V = Q(U, \mathcal{T})^{-1}(e_1 \otimes N) \in \mathbb{F}^{mn \times n}$ for some nonsingular $N \in \mathbb{F}^{n \times n}$.

A **Jordan triple** (X, J, Y) is a standard triple for which $\mathcal{T} = J$ is in Jordan form.

Generating $P(\lambda)$ from Jordan Triple

An mn-standard triple (X, J, Y) uniquely generates an $n \times n$ matrix polynomial $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ via

$$A_m = (XJ^{m-1}Y)^{-1},$$

$$A_{m-j} = -A_m \sum_{i=m-j+1}^{m} XJ^{i+j-1}YA_i, \quad j = 1, \dots m.$$

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When $P(\lambda)$ is structured, (X, J, Y) has extra properties.

E.g., for symmetric structures, (X, J, Y) is self-adjoint, i.e., $Y = SX^T$, $S = S^T$, $JS = (JS)^T$.

Assumptions

- $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$, $A_j \in \mathbb{F}^{n\times n}$, $\det A_m \neq 0$

- Consider structures in $\mathcal{S} \in \mathbb{S}$, where

$$\mathbb{S} = \{\text{Hermitian, symm., } *\text{-even, } *\text{-odd, } T\text{-even, } T\text{-odd,} \\ *\text{-palindromic, } *\text{-antipalind., } T\text{-palind., } T\text{-antipalind.}\}.$$

- If $\mathcal{S} \in \{T\text{-palindromic, } T\text{-antipalindromic}\}$ and $m = 2k$ then either $-1 \notin \Lambda(P)$ or $1 \notin \Lambda(P)$.

$\mathcal{S}$-structured Standard Triple

Definition (Al-Ammari, T. 11)

A standard triple $(U, \mathcal{T}, V)$ is **$\mathcal{S}$-structured** if there exists a nonsingular S s.t.

$$US = V^\star u_\mathcal{S}(\mathcal{T}), \quad S^{-1}\mathcal{T}S = t_\mathcal{S}(\mathcal{T}), \quad S^{-1}V = v_\mathcal{S}(\mathcal{T})U^\star.$$

Structure $\mathcal{S}$	$u_\mathcal{S}(\mathcal{T})$	$t_\mathcal{S}(\mathcal{T})$	$v_\mathcal{S}(\mathcal{T})$
Hermitian/symmetric	I	$\mathcal{T}^\star$	I
$\star$-even	$-I$	$-\mathcal{T}^\star$	I
$\star$-odd	I	$-\mathcal{T}^\star$	I
$\star$-palind., $m = 2k+1$	$-\mathcal{T}^{\star(k-1)}$	$\mathcal{T}^{-\star}$	$\mathcal{T}^{\star k}$
$\star$-palind., $m = 2k$	$-\mathcal{T}^{\star(k-1)}(I + \alpha\mathcal{T}^\star)^{-1}$	$\mathcal{T}^{-\star}$	$(I + \alpha\mathcal{T}^\star)\mathcal{T}^{\star(k-1)}$

$\alpha \in \mathbb{F}$ s.t. $\alpha^\star\alpha = 1$ and $-\alpha \notin (\mathcal{T})$

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Theorem (Al-Ammari, T., 11)

$P(\lambda)$ has structure $\mathcal{S}$ if and only if $P(\lambda)$ admits an $\mathcal{S}$-structured standard triple, in which case every standard triple for $P(\lambda)$ is $\mathcal{S}$-structured.

S is the **S-matrix** of the $\mathcal{S}$-structured triple $(U, \mathcal{T}, V)$.

Example: $*$-even Structure

- (X, J, Y) with matrix S is $\mathcal{S}$-structured iff
  $Y = SX^*, \quad S = -S^*, \quad JS = (JS)^*.$
- J is in Lie algebra of scalar product defined by S^{-1}.
- Eigenvalues of J occur in pairs $(\lambda, -\bar{\lambda})$.
- If $\lambda = i\beta$, $\mu \in \mathbb{R}$ then $\lambda = -\bar{\lambda}$ so no pairing for purely imaginary eigenvalues but have **sign characteristic**.

$$iJ = \bigoplus_{j=1}^{r} J_{\ell_j}(-\beta_j) \oplus \bigoplus_{j=1}^{s} \big(J_{m_j}(-\bar{\mu}_j) \oplus J_{m_j}(-\mu_j)\big),$$

$$iS = \bigoplus_{j=1}^{r} \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^{s} F_{2m_j}, \quad (\varepsilon_j = \pm 1), \quad F_k := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}_{k\times k}.$$

Construction of $*$-even Jordan Triples

$m = 2$, (X, J, SX^*), $S = -S^*$, $JS = (JS)^*$.

- Build J, S from given list of elementary divisors and assign sign characteristic s.t. $\sum_{j=1}^{r}(1-(-1)^{\ell_j})\varepsilon_j = 0$.

- Construct $X \in \mathbb{C}^{n\times 2n}$ s.t.

$$\det \begin{bmatrix} XJ \\ X \end{bmatrix} \neq 0, \quad \det(XJSX^T) \neq 0, \quad XSX^T = 0.$$

- Can show that $X = [X_1 \ X_1\Theta]W^T$ satisfies $XSX^T = 0$,
  - $X_1 \in \mathbb{C}^{n\times n}$ nonsingular,
  - $\Theta \in \mathbb{C}^{n\times n}$ unitary,
  - W orthogonal s.t. $W^T SW = -i\left[\begin{smallmatrix} I_n & 0 \\ 0 & -I_n \end{smallmatrix}\right]$.

Example of $*$-even Quadratic

$$J = \left([0] \oplus [2i] \oplus \begin{bmatrix} i & -i \\ & i \end{bmatrix} \oplus [-1+i] \oplus [1+i] \right),$$

$$S = -i\left((-1)[1] \oplus (+1)[1] \oplus (-1)\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \oplus \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \right),$$

$$X = [X_1 \ X_1\Theta]W^T,$$

with $W^T S W = -i\left[\begin{smallmatrix} I_n & 0 \\ 0 & -I_n \end{smallmatrix}\right]$ and $X_1 = \Theta = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$. Then,

$$A_2 = (XJSX^*)^{-1} = 0.5 \begin{bmatrix} 1 & 1 & -i \\ 1 & -3 & 3i \\ i & -3i & -1 \end{bmatrix}, \quad (A_2 = A_2^*)$$

$$A_1 = -A_2 X J^2 S X^* A_2 = \begin{bmatrix} -i & 0 & -1 \\ 0 & i & 4 \\ 1 & -4 & 3i \end{bmatrix}, \quad (A_1 = -A_1^*)$$

$$A_0 = -A_2(XJ^2SX^*A_1 + XJ^3SX^*A_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2i \\ 0 & 2i & 4 \end{bmatrix}, \quad (A_0 = A_0^*).$$

$*$-palindromic Structure

If $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ is **$*$-even** then

$$\mathcal{M}(Q)(\lambda) = (\lambda+1)^2 Q\left(\frac{\lambda-1}{\lambda+1}\right) \quad \text{(**Möbius transform**)}$$
$$= \lambda^2 Q(1) + \lambda(-2A_2 + 2A_0) + Q(1)^*$$

is **$*$-palindromic**.

- Use Möbius transform $\lambda_j \mapsto \rho_j := \frac{\lambda_j - 1}{\lambda_j + 1}$ to map list of elementary divisor for $\mathcal{M}(Q)(\lambda)$ to one for $Q(\lambda)$.
- Solve the $*$-even inverse quadratic eigenproblem, i.e., compute A_2, A_1, A_0.
- Compute $\mathcal{M}(Q)(\lambda)$.

Summary

- Introduced the notion of $\mathcal{S}$**-structured standard triples**.
- Used $\mathcal{S}$-structured Jordan triples to solve **structured inverse quadratic eigenvalue problems**.
- Need to better understand:
  - Signature constraint for structured polynomials.
  - Sign characteristic at ∞.
  - Effect of Möbius transforms on standard triples.

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