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Triangularization of Quadratic Matrix Polynomials

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Reduction to Triangular Forms

Let $A, B \in \mathbb{C}^{n\times n}$ and let T_A, T_B denote triangular matrices.

- Single matrices: there exists $U \in \mathbb{C}^{n\times n}$ such that

$$U^*AU = T_A, \quad U^*U = I, \quad \text{(Schur decomp.)}.$$

- Pair of matrices: there exist unitary $U, V \in \mathbb{C}^{n\times n}$ s.t.

$$U^*AV = T_A, \quad U^*BV = T_B, \quad \text{(generalized Schur decomp.)}.$$

These reductions have many applications (eigenvalue computation, matrix function computation, . . .).

Can we extend these reductions to matrix triples?

Extension to Matrix Triples

Let $Q(\lambda) = \lambda^2 M + \lambda D + K$ be **regular** ($\det Q(\lambda) \not\equiv 0$).

Suppose there exist U, V nonsingular s.t.
$UQ(\lambda)V = \lambda^2 T_M + \lambda T_D + T_K = T(\lambda)$ is triangular.

- Roots $\lambda_j^{(1)}, \lambda_j^{(2)}$ of $\lambda^2 (T_M)_{jj} + \lambda (T_D)_{jj} + (T_K)_{jj}$ are e'vals of $Q(\lambda)$.
- $Q(\lambda_1^{(k)})v_1 = U^{-1} T(\lambda_1^{(k)}) e_1 = 0$, $k = 1, 2$.
  So $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$ must have the same e'vec — a strong condition!

"We can forget about simultaneous triangularization ..."
Charlie Van Loan, CP3, Monday

Use $U(\lambda)$, $V(\lambda)$ unimodular instead!

Equivalences

Definition

Two matrix polynomials $Q(\lambda)$, $T(\lambda)$ are **equivalent** if there are unimodular $U(\lambda)$, $V(\lambda)$ s.t. $T(\lambda) = U(\lambda)Q(\lambda)V(\lambda)$.

$Q(\lambda)$ and $T(\lambda)$ have the same finite elementary divisors.

Need equivalence transformations preserving

- the degree (want $Q(\lambda)$, $T(\lambda)$ to be quadratic),
- the elementary divisors at infinity (i.e., the elementary divisors at 0 of $\mathrm{rev}\, Q(\lambda) = \lambda^2 Q(1/\lambda) = \lambda^2 K + \lambda D + M$).

Elementary Divisors at Infinity

$$L(\lambda) = \lambda \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] + \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

has one linear elementary divisors at 0 and two linear elementary at ∞. Now

$$\underbrace{\begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{unimodular}} L(\lambda) = \lambda \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] + \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

has one linear elementary divisor at 0 and one elementary divisor at ∞ with partial multiplicity 2.

Equivalences can modify the partial multiplicities at ∞.

Triangularization by Strong Equivalences

Theorem (Zaballa & T, 2012)

Any regular $Q(\lambda) = \lambda^2 M + \lambda D + K$ *is* ***strongly equivalent*** *over* $\mathbb{C}[\lambda]$ *to triangular* $T(\lambda) = \lambda^2 T_M + \lambda T_D + T_K$*, i.e., there are unimodular* $U(\lambda)$, $V(\lambda)$ *s.t.* $T(\lambda) = U(\lambda)Q(\lambda)V(\lambda)$ *has same finite and infinite elementary divisors as* $Q(\lambda)$*.*

- Byproduct of solution to quadratic realizability problem, D.S. Mackey (MS68).
- Proved by Gohberg, Lancaster & Rodman (1982) for monic polynomials of arbitrary degree.
- Extended to regular/singular polynomials of arbitrary degree by Taslaman, Nakatsukasa, Zaballa & T. (2012).

How to numerically compute $T(\lambda)$?

Structure Preserving Transformation (SPT)

$$Q(\lambda) = \lambda^2 M + \lambda D + K,$$

$$C_Q(\lambda) = \lambda \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} + \begin{bmatrix} 0 & K \\ -I & D \end{bmatrix}.$$

$Q(\lambda)$ —— **(strong) equivalence**, $U(\lambda)Q(\lambda)V(\lambda)$ ——→ $T(\lambda)$

$Q(\lambda)$ ↓ **lineari- zation** ↓ $C_Q(\lambda)$

$C_T(\lambda)$ ↑ **quadrati- zation** ↑ $T(\lambda)$

$C_Q(\lambda)$ —— **SPT**, $S_L C_Q(\lambda) S_R$ ——→ $C_T(\lambda)$

A Simple MATLAB Code

Let $Q(\lambda) = \lambda^2 I + \lambda D + K$ be $n \times n$.

```
A = [zeros(n) -K; eye(n) -D];
[U,T] = schur(A,'complex');
X =(U(:,1:2:2*n-1)+U(:,2:2:2*n))/sqrt(2);
S = [X A*X];
At = S\A*S;
```

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```

- $A_T = \begin{bmatrix} 0 & -T_K \\ I & -T_D \end{bmatrix}$ is in companion form.
- $T(\lambda) = \lambda^2 I + \lambda T_D + T_K$: **upper triang.**, $\Lambda(Q) = \Lambda(T)$.
- S is an SPT. **Cols of $X \in \mathbb{C}^{2n \times n}$ are orthonormal**.

Schur's Theorem for Complex Matrices

Matrix version: if $A \in \mathbb{C}^{n \times n}$ then there exists a unitary U such that $U^*AU = T$ is a triangular matrix.

Subspaces version:

Theorem

Let $A \in \mathbb{C}^{n \times n}$. There are subspaces $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of $\mathbb{C}^n$ satisfying

(i) $\mathbb{C}^n = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n$,

(ii) *for $k = 1\colon n$, $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k$, is A-invariant,*

(iii) *for $k = 1\colon n$, $\mathcal{V}_k = \langle u_k \rangle$, where $u_1, \ldots u_n$ form an orthonormal system of vectors of $\mathbb{C}^n$.*

Schur-like Theorem for $Q(\lambda)$

Let $Q(\lambda) = \lambda^2 M + \lambda D + K$, $\det M \neq 0$.

Theorem (Zaballa, T., 2012)

Let $\lambda I - A \in \mathbb{C}[\lambda]^{2n\times 2n}$ be a linearization of $n \times n$ $Q(\lambda)$. There are subspaces $\mathcal{V}_1, \dots, \mathcal{V}_n$ of $\mathbb{C}^{2n}$ satisfying

(i) $\mathbb{C}^{2n} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n$,

(ii) *for $k = 1\colon n$, $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k$ is A-invariant,*

(iii) *for $k = 1\colon n$, $\dim \mathcal{V}_k = 2$ and $\mathcal{V}_k = \langle x_k, Ax_k \rangle$, where $x_1, \dots x_n$ form an orthogonal system of vectors of $\mathbb{C}^{2n}$.*

- $\mathcal{V}_k = \langle x_k, Ax_k \rangle$ is a **Krylov subspace** of dimension 2.
- The x_j are **generating vectors**.
- If $X = [x_1 \dots x_n]$ then $S = [X \;\; AX]$ is nonsingular.

Triangularizing SPT for $Q(\lambda) = \lambda^2 M + \lambda D + K$

Let $S = [X\ AX]$, where X contains **generating vectors** for linearization $\lambda I - A$ of $Q(\lambda)$.

$$S^{-1}AS =: B \quad \Longleftrightarrow \quad A[X\ AX] = [X\ AX]\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

- $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and hence B has **companion form**.
- $\langle x_1, Ax_1, \cdots x_k, Ax_k \rangle$ A-invariant $\Rightarrow$ B_{12} and B_{22} are **upper triangular**.

$S = [X\ AX]$ is a **triangularizing SPT**.

Schur-like Theorem: Matrix Form

Let $Q(\lambda) = \lambda^2 M + \lambda D + K \in \mathbb{C}[\lambda]^{n\times n}$ with $\det(M) \neq 0$.

Theorem (Zaballa, T., 2012)

For any linearization $\lambda I - A$ of $Q(\lambda)$, there exists $U \in \mathbb{C}^{2n\times n}$ with ***orthonormal columns*** *s.t. $[U\ AU]$ is nonsingular and*

$$[U\ AU]^{-1} A [U\ AU] = \begin{bmatrix} 0 & -T_0 \\ I_n & -T_1 \end{bmatrix},$$

where $I_n\lambda^2 + T_1\lambda + T_0$ is triangular and equivalent to $Q(\lambda)$.

- The columns of U are generating vectors for A.
- Extends to **arbitrary degree matrix polynomials**.

Generating Vectors from Schur Vectors

Theorem (Nakatsukasa, Taslaman, Zaballa & T, 2012)

Let $A \in \mathbb{C}^{2n\times 2n}$ have Schur decomposition

$$U^*AU = \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ & \ddots & \vdots \\ & & T_{nn} \end{bmatrix}, \quad T_{ij} \in \mathbb{C}^{2\times 2}.$$

*If there is $v_j = \begin{bmatrix} \nu_{j1} \\ \nu_{j2} \end{bmatrix}$ s.t. $T_{jj}v_j \neq \alpha_j v_j$, $j = 1\colon n$ then cols of $X = U\mathrm{diag}(v_1, \dots, v_n) = [\nu_{11}u_1 + \nu_{12}u_2, \dots \nu_{n1}u_{2n-1} + \nu_{n2}u_{2n}]$ are **generating vectors**.*

Hence

$$[X\ AX]^{-1}A[X\ AX] = \begin{bmatrix} 0 & -T_K \\ I & -T_D \end{bmatrix},$$

where T_D, T_K are **upper triangular**.

Schur form with Nonderogatory Blocks

If $T_{jj} \neq \alpha I_2$, i.e., T_{jj} is **nonderogatory** then there exists $0 \neq v_j \in \mathbb{C}^2$ s.t. v_j is not an e'vec of T_{jj}.

Theorem (Nakatsukasa, Taslaman, Zaballa, T., 2012)

Any Schur form T of a linearization $A \in \mathbb{C}^{2n\times 2n}$ of $Q(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ is unitarily similar to a Schur form $\widetilde{T}$ with nonderogatory 2×2 diagonal blocks.

- Proof is constructive and nontrivial.
- Relies on the property that $\lambda \in \Lambda(A) = \Lambda(Q)$ has geometric multiplicity less or equal to n.

Triangular Matrix Coefficients

$Q(\lambda)$ is equivalent to triangular $T(\lambda) = \lambda^2 T_M + \lambda T_D + T_K$.

Have explicit expressions for T_M, T_D, T_K in terms of either

- the n orthonormal generating vectors for A, or
- the Schur form T of A,
  (or Jordan form of A, see Taslaman MS 35)

where $\lambda I - A$ is a linearization of $Q(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ with nonsingular leading coeff.

Triangular Coefficients from Schur Form

Let $\lambda I - A$ be a linearization of $Q(\lambda) = \lambda^2 M + \lambda D + K$ with

Schur form $T = \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ & \ddots & \vdots \\ & & T_{nn} \end{bmatrix}$, $T_{jj} \in \mathbb{C}^{2\times 2}$ **nonderogatory**.

Take $v_i, w_i \in \mathbb{C}^2$ s.t. $\|v_i\|_2 = \|w_i\|_2 = 1$, $T_{ii} v_i \neq \alpha v_i$, $w_i^* v_i = 0$,

$$V = \text{diag}(v_1, \dots, v_n) \in \mathbb{C}^{2n\times n}, \quad W = \text{diag}(w_1, \dots, w_n) \in \mathbb{C}^{2n\times n}.$$

$Q(\lambda)$ is equivalent to $T(\lambda) = \lambda^2 T_M + \lambda T_D + T_K$, where

$$T_M = W^* T V,$$
$$T_D = -W^* T^2 V,$$
$$T_K = -T_M((V^* T V) T_D - (V^* T^2 V))$$

are **triangular**.

Triangularizing SPT S

Let $S = [X \; AX]$, where X contains generating vectors for $A = \begin{bmatrix} 0 & -KM^{-1} \\ I & -DM^{-1} \end{bmatrix}$ so that $S^{-1}AS = \begin{bmatrix} 0 & -T_K \\ I & -T_D \end{bmatrix} =: A_T$.

- $S^{-1} = [\, Y \quad A_T Y \,]$, $Y \in \mathbb{C}^{2n\times n}$ [Garvey et al., 2011].
- Can construct Y from generating vectors X and M, D, K.

No need to factorize (invert) S.

Linear Systems

Let $S = [X\ AX] = [Y\ A_T Y]^{-1}$ and $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$.

For every $\lambda \notin \Lambda(Q)$,

$$Q(\lambda)^{-1} = (\widetilde{X}_1 + \lambda X_2) T(\lambda)^{-1} (Y_1 + \lambda Y_2) - X_2 Y_2,$$

where $\widetilde{X}_1 = X_1 - DX_2 + X_2 T_D$.

Linear Systems

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Applications

- Solving for x, $Q(\omega)x = b$ for many values of $|\omega| \in [\omega_l, \omega_h]$, $\omega_l \ll \omega_h$.
- Evaluation of transfer function: $G(s) = c^T Q(s)^{-1} b$.

Triangularization over $\mathbb{R}$

Theorem (Zaballa & T, 2012)

$Q(\lambda) \in \mathbb{R}[\lambda]^{n\times n}$ is triangularizable over $\mathbb{R}[\lambda]$ if and only if

$$p \le n - n_c,$$

where

- *$2n_c$ is the number of nonreal e'vals and*
- *p is the largest geometric multiplicity of the real e'vals and e'vals at infinity.*

$Q(\lambda) \in \mathbb{R}[\lambda]^{3\times 3}$ with elementary divisors in $\mathbb{R}$: $(\lambda - 1)$, $(\lambda - 1)$, $(\lambda^2 + 1)^2$ is **not triangularizable over $\mathbb{R}[\lambda]$** since $p = 2 > n - n_c = 3 - 2 = 1$.

Quasi-triangular Quadratics

[Zaballa & T, 2012]

Any quadratic $Q(\lambda) \in \mathbb{R}[\lambda]^{n\times n}$ is strongly equivalent to a quadratic of the form

$$T(\lambda) = \begin{array}{c} \\ 2r \\ 2n-2r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-2r \end{array} \\ \begin{bmatrix} T_1(\lambda) & T_3(\lambda) \\ 0 & T_2(\lambda) \end{bmatrix} \end{array},$$

where $r = \max\{0, p + n_c - n\}$ and

- $T_1(\lambda)$ **is quasi-triangular** with r 2×2 diag. blocks with elementary divisors $(\lambda - \lambda_1)$, $(\lambda - \lambda_1)$, $(\lambda^2 + d_i\lambda + k_i)$. Here, $\lambda_1 \in \mathbb{R}$ has largest geometric multiplicity p.
- $T_2(\lambda)$ **is triangular**.

($Q(\lambda)$ has $2n_c$ nonreal e'vals.)

Summary and Concluding Remarks

- Any regular quadratic is strongly equivalent to a triangular quadratic.
- There is a Schur-like theorem for quadratic matrix polynomials.
- Triangularizing SPTs are defined by n orthonormal (generating) vectors.
- Can competitive algorithms be designed that compute sets of n orthonormal generating vectors?
- Subspace iteration, Krylov subspace methods are worth exploring.

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