Recent Advances in the Numerical Solution of Quadratic Eigenvalue Problems

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Let $F: \Omega \rightarrow \mathbb{C}^{n \times n}$ be analytic on open set $\Omega \subseteq \mathbb{C}$.

The **nonlinear eigenvalue problem**: Find scalars $\lambda$ and nonzero $x, y \in \mathbb{C}^n$ satisfying $F(\lambda)x = 0$ and $y^*F(\lambda) = 0$.

- $\lambda$ is an e’val, $x, y$ are corresponding right and left e’vecs.
- E’vals are solutions of $\det(F(\lambda)) = 0$.

In practice, elements of $F$ most often **polynomial, rational or exponential functions of $\lambda$**.

Can be very difficult to solve (poor conditioning, algebraic structure to be preserved).
NLEVP Toolbox

Collection of Nonlinear Eigenvalue Problems:
T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, F. T.

- Quadratic, polynomial, rational and other nonlinear eigenproblems.
- Provided in the form of a MATLAB Toolbox.
- Problems from real-life applications + specifically constructed problems.

http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html

- New release to come. Further contributions welcome.
**Sample of Problems**

**Bicycle** *(pep,qep,real,parameter-dependent)*

2 × 2 quadratic poly. arising in study of bicycle self-stability.

**Railtrack** *(pep,qep,t-palindromic,sparse)*

T-palindromic quadratic of size 1005. Stems from a model of vibration of rail tracks under the excitation of high speed trains.

**Butterfly** *(pep,real, T-even,scalable)*

quartic matrix polynomial with T-even structure.

**Loaded string** *(rep,real,symmetric,scalable)*

rational eigenvalue problem describing eigenvibration of a loaded string.

**Gun** *(nep,sparse) nonlinear* eigenvalue problem modeling a radio-frequency gun cavity.

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Quadratic Eigenvalue Problem (QEP)

Concentrate on

\[ Q(\lambda) = \lambda^2 M + \lambda D + K, \]

with \( M, D, K \in \mathbb{C}^{n \times n} \).

- Appears in many practical applications.

- Recent work on quadratization: convert degree \( \ell \) matrix polynomials to degree 2 (Al-Ammari & T, 2009; Huang, Lin & Su, 2008)

  - Mainly for structured matrix polynomials.
  
  - Use new efficient algorithms for quadratics.
  
  - Theoretical reasons.
Review of recent progress.

Structure Preserving Transformations.
Standard way of treating QEPs both theoretically and numerically.

Convert $Q(\lambda) = \lambda^2 M + \lambda D + K$ into a linear pencil such as

$$C_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} D & K \\ -I & 0 \end{bmatrix}.$$  

(first companion form)

$C_1(\lambda)$ is a linearization of $Q$, i.e., there exist $E(\lambda)$ and $F(\lambda)$ with constant, nonzero determinants s.t.

$$\begin{bmatrix} Q(\lambda) & 0 \\ 0 & I \end{bmatrix} = E(\lambda)C_1(\lambda)F(\lambda).$$
Work on Linearizations

New (structure preserving) linearizations derived along with algorithms preserving spectral properties in finite precision arithmetic.

[Antoniou, Higham, Lin, Mackey, Mackey, Mehl, Mehrmann, T., Vologiannidis, ...]

Mackey, Mackey, Mehl & Mehrmann (2006) introduce

\[ L_1(Q) = \{ L(\lambda) : L(\lambda) \begin{bmatrix} \lambda I \\ I \end{bmatrix} = \begin{bmatrix} v_1 Q(\lambda) \\ v_2 Q(\lambda) \end{bmatrix}, \quad v \in \mathbb{C}^2 \}. \]

- Almost all pencils in \( L_1 \) are linearizations.
- E’vecs of \( Q \) easily recovered from e’vecs of \( L \in L_1 \) ([M^4, 2006], [Higham, Li, T., 2007]).
- \( L_1(Q) \) is a rich source of interesting linearizations.
Higham, Mackey, Mackey and T. (2006) define

$$\mathcal{B}(Q) := \{ \lambda X + Y \in \mathbb{L}_1(Q) : X^B = X, \ Y^B = Y \}$$

$$= \{ v_1 L_1(\lambda) + v_2 L_2(\lambda), \ v \in \mathbb{C}^2 \},$$

where

$$L_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} + \begin{bmatrix} D & K \\ K & 0 \end{bmatrix}, \quad L_2(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}. $$
Higham, Mackey, Mackey and T. (2006) define

$$\mathbb{B}(Q) := \{ \lambda X + Y \in \mathbb{L}_1(Q) : X^B = X, \ Y^B = Y \}$$

$$= \{ v_1 L_1(\lambda) + v_2 L_2(\lambda), \ v \in \mathbb{C}^2 \},$$

where

$$L_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} + \begin{bmatrix} D & K \\ K & 0 \end{bmatrix}, \quad L_2(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}.$$

- $L \in \mathbb{B}(Q)$ with vector $v$ is a linearization of $Q$ iff e’val of $Q$ is not a root of $p(x; v) = v_1 x + v_2$.

- [Higham, Mackey, T., 2009] Identified **definite pencils** in $\mathbb{B}(Q)$ for **hyperbolic** (overdamped) $Q$. 

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Quadratic eigenproblem

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Better understanding of linearization process and effects of scaling on

- conditioning of eigenvalues,
- backward error of computed eigenpairs.

[Adhikari, Alam, Betcke, Higham, Kressner, Li, Mackey, Mehl, Mehrmann, T., ...]
Linearization Process

Better understanding of linearization process and effects of scaling on

- conditioning of eigenvalues,
- backward error of computed eigenpairs.

[Adhikari, Alam, Betcke, Higham, Kressner, Li, Mackey, Mehl, Mehrmann, T., ...]

Illustration: free vibrations of aluminium beam.

- $M > 0$, $K > 0$, $D \geq 0 \Rightarrow$ all ei’vals have $\text{Re}(\lambda) \leq 0$.
- $D$ is rank 1. Can show $n$ pure imaginary e’vals.
Eigenvalues of \( Q(\lambda) = \lambda^2 M + \lambda D + K \)

\[
C_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} D & K \\ -I & 0 \end{bmatrix}.
\]

\[
L_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} + \begin{bmatrix} D & K \\ K & 0 \end{bmatrix}, \quad L_2(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}.
\]

coeffs = nlevp('damped_beam',100);
K = coeffs{1}; D = coeffs{2}; M = coeffs{3};
I = eye(2*nele); O = zeros(2*nele);
eval = eig([D K; -I 0],-[M O; O I]); % C_1
%eval = eig([D K; -I 0],-[M O; O I]); % L_1
%eval = eig([D K; -I 0],-[M O; O I]); % L_2
plot(eval,'.r');
Computed Spectrum of $C_1$, $L_1$ and $L_2$
- Show practical value of condition numbers and backward errors for understanding the quality of computed results.
- Importance of **scaling** QEPs before computing e’vals via linearization.
- Results not confined to the beam problem but apply to any QEP.
- [Grammont, Higham, T., 2009] General **framework** for analyzing the linearization process (see Grammont’s talk, MS12, Monday).
Spectrum of $C_1, L_2$ before/after Scaling
A Black Box (Dense Case)

[Hammarling, Higham, Munro, T.]

**step 1**: Apply Fan, Lin and Van Dooren scaling.

**step 2**: Construct “a” companion linearization.

**step 3**: Deflate zeros and $\infty$ e’vals.

**step 4**: Solve generalized eigenproblem with QZ.

Optional:

**step 5**: Recover right/left e’vecs of $Q$ from those of companion form.

**step 6**: Compute e’vals condition number.
Basic Facts

- Triples \((M, D, K)\) cannot, in general, be simultaneously diagonalized.
  (See Lancaster, MS35, Wednesday)

- No analog of generalized Schur form for matrix triples.

- Cannot simultaneously \textbf{tridiagonalize} \((M, D, K)\).
For $n \times n$ symmetric $Q(\lambda) = \lambda^2 M + \lambda D + K$ with all e’vals distinct, there exists $U \in \mathbb{R}^{2n \times 2n}$ s.t.

$$U^T \left( \lambda \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \right) U = \lambda \begin{bmatrix} 0 & \Lambda_M \\ \Lambda_M & \Lambda_D \end{bmatrix} + \begin{bmatrix} -\Lambda_M & 0 \\ 0 & \Lambda_K \end{bmatrix},$$

where $\Lambda_M$, $\Lambda_D$ and $\Lambda_K$ are diagonal matrices. (Garvey, Friswell, Prells, 2002)

**New diagonal quadratic matrix polynomial**

$$Q_\Lambda(\lambda) = \lambda^2 \Lambda_M + \lambda \Lambda_C + \Lambda_K.$$

- $Q(\lambda)$ and $Q_\Lambda(\lambda)$ have the same e’vals.
- $U$ is a structure preserving transformation.
Structure Preserving Transformation (SPT)

Recall: \( L(\lambda) \in \mathbb{B}(Q) \iff L(\lambda) = v_1 L_1(\lambda) + v_2 L_2(\lambda), \)

\[
L_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} + \begin{bmatrix} D & K \\ K & 0 \end{bmatrix}, \quad L_2(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}.
\]

Assume \( M \) nonsingular.

**Definition**

\( S_L, S_R \) nonsingular define an SPT for \( Q(\lambda) \) if

\[
S_L^T L_2(\lambda) S_R = \lambda \begin{bmatrix} 0 & \tilde{M} \\ \tilde{M} & \tilde{D} \end{bmatrix} + \begin{bmatrix} -\tilde{M} & 0 \\ 0 & \tilde{K} \end{bmatrix} =: \tilde{L}_2(\lambda).
\]

\( \tilde{L}(\lambda) \) is a linearization of \( \tilde{Q}(\lambda) = \lambda^2 \tilde{M} + \lambda \tilde{D} + \tilde{K}. \)
Some Properties

Let \((S_L, S_R)\) be an SPT mapping \(Q(\lambda)\) to \(\tilde{Q}(\lambda)\).

- **SPTs preserve block structure of pencils in \(\mathbb{B}(Q)\).**
  \[
  S_L^T \mathbb{B}(Q) S_R = \mathbb{B}(\tilde{Q}).
  \]
  Moreover, \(L(\lambda) \in \mathbb{B}(Q)\) with vector \(v\)
  \(\Leftrightarrow \tilde{L}(\lambda) = S_L^T L(\lambda) S_R \in \mathbb{B}(\tilde{Q})\) with vector \(v\).

- **Well-defined relations between e’vecs of \(Q\) and \(\tilde{Q}\).**
  Let \((\lambda, x, y)\) be an eigentriple of \(Q(\lambda)\). Then
  \[
  \begin{bmatrix}
  \lambda x \\
  x
  \end{bmatrix} = S_R \begin{bmatrix}
  \lambda \tilde{x} \\
  \tilde{x}
  \end{bmatrix},
  \begin{bmatrix}
  \bar{\lambda} y \\
  y
  \end{bmatrix} = S_L \begin{bmatrix}
  \bar{\lambda} \bar{y} \\
  \bar{y}
  \end{bmatrix},
  \]
  where \(\tilde{x}, \tilde{y}\) such that \(\tilde{Q}(\lambda)\tilde{x} = 0, \tilde{y}^* \tilde{Q}(\lambda) = 0\).
\[ S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad S_{ij}, T_{ij} \in \mathbb{R}^{n \times n}. \]

\((S, T)\) defines a SPT iff

\[
S_{11}^T M T_{21} + S_{21}^T M T_{11} + S_{21}^T C T_{21} = 0, \\
-S_{11}^T M T_{12} + S_{21}^T K T_{22} = 0, \\
-S_{12}^T M T_{12} + S_{22}^T K T_{22} = 0, \\
S_{11}^T M T_{22} + S_{21}^T M T_{12} + S_{21}^T D T_{22} = S_{11}^T M T_{11} - S_{21}^T K T_{21}, \\
S_{22}^T M T_{11} + S_{12}^T M T_{21} + S_{22}^T D T_{21} = S_{11}^T M T_{11} - S_{21}^T K T_{21}.
\]

5n^2 constraints.
Elementary SPTs (symmetric case)

Let \( T = I_{2n} + \begin{bmatrix} ab^T & ad^T \\ af^T & ah^T \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \ a, b, d, f, h \in \mathbb{R}^n. \)

- Rank-2 modification of \( I_{2n}. \)

- Let \( V = \begin{bmatrix} b & d & f & h \end{bmatrix} \in \mathbb{R}^{n \times 4}. \)
  For almost all \( a \in \mathbb{R}^n, \) any solution \( V \) to \( VA = B \) defines an SPT \( T. \)
  \( A \in \mathbb{R}^{4 \times 3}, \ B \in \mathbb{R}^{n \times 3} \) depend on \( a, M, D \) and \( K. \)

- If \( (M, D, K) \xrightarrow{T} (\tilde{M}, \tilde{D}, \tilde{K}) \) then \( \tilde{M}, \tilde{D}, \tilde{K} \) are **low rank modifications** of \( M, D, K. \)
Let \((\lambda_j, x_j), j = 1, 2\) be two given eigenpairs of \(Q(\lambda)\).

Want to transform \(n \times n\) \(Q(\lambda)\) into

\[
\tilde{Q}(\lambda) = \begin{bmatrix}
Q_d(\lambda) & 0 \\
0 & q(\lambda)
\end{bmatrix}
\]

such that

\[
\Lambda(Q) = \Lambda(\tilde{Q}), \text{ (same spectrum)}
\]
\[
q(\lambda_j) = 0, j = 1, 2.
\]

Deflation procedure decoupling \(Q(\lambda)\) into two quadratics.
Suppose there exist nonsingular $S_L$, $S_R$ s.t.

$$S_L Q(\lambda) S_R = \begin{bmatrix} Q_d(\lambda) & 0 \\ 0 & q(\lambda) \end{bmatrix} = \tilde{Q}(\lambda).$$

The roots $\lambda_1, \lambda_2$ of $q(\lambda)$ are e’vals of $Q$ and $\tilde{Q}$

$$Q(\lambda_j)x_j = 0, \quad \tilde{Q}(\lambda_j)e_n = 0, \quad j = 1, 2$$

with e’vecs related by $S_R^{-1} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} e_n & e_n \end{bmatrix}$.

Decoupling possible only if e’vecs $x_1$ and $x_2$ are parallel.
Suppose there exist nonsingular $S_L, S_R$ s.t.

$$S_L Q(\lambda) S_R = \begin{bmatrix} Q_d(\lambda) & 0 \\ 0 & q(\lambda) \end{bmatrix} = \tilde{Q}(\lambda).$$

The roots $\lambda_1, \lambda_2$ of $q(\lambda)$ are e’vals of $Q$ and $\tilde{Q}$

$$Q(\lambda_j)x_j = 0, \quad \tilde{Q}(\lambda_j)e_n = 0, \quad j = 1, 2$$

with e’vecs related by $S_R^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e_n \\ e_n \end{bmatrix}$.

Decoupling possible only if e’vecs $x_1$ and $x_2$ are parallel.

Is there an elementary SPT mapping $Q$ to $\tilde{Q}$ s.t. e’vecs of $\tilde{Q}$ with e’vals $\lambda_1, \lambda_2$ are parallel?
(\lambda_1, x_1), (\lambda_2, x_2) to be deflated with \lambda_1 \neq \lambda_2 and x_1 \neq \alpha x_2.

**Aim:** construct \( T = I_{2n} + \begin{bmatrix} ab^T & ad^T \\ af^T & ah^T \end{bmatrix} \) and nonzero \( z \in \mathbb{R}^n \) s.t.

- \( Q(\lambda) \overset{T}{\rightarrow} \tilde{Q}(\lambda) \) with \( \tilde{Q}(\lambda_j)z = 0, j = 1, 2. \)

- Yields \( a, z \) and \( z^T \begin{bmatrix} b & d & f & h \end{bmatrix} = z^T V. \)

- \( T \) is an SPT \( \iff \) \( VA = B. \)
Theorem

Eigenpairs \((\lambda_1, x_1), (\lambda_2, x_2)\) with \(\lambda_1 \neq \lambda_2\) either real or complex conjugate can be mapped to \((\lambda_1, z), (\lambda_2, z)\) by elementary SPTs if

- \(x_j^T Q'(\lambda_j) x_j \neq 0, j = 1, 2,\)
- real eigenpairs have opposite type:

\[
\text{sign}(x_1^T Q'(\lambda_1) x_1) = -\text{sign}(x_2^T Q'(\lambda_2) x_2).
\]

Can generate a family of SPTs mapping \((\lambda_j, x_j)\) to \((\lambda_j, z), j = 1, 2.\)
Lemma

If \((\lambda_j^2 M + \lambda_j C + K)z = 0, j = 1, 2\) with \(\lambda_1 \neq \lambda_2\) then

\[
(M, C, K)z = (mp, cp, kp), \quad p \in \mathbb{R}^n, \quad p^Tz = 1,
\]

\[
c = -m(\lambda_1 + \lambda_2), \quad k = m\lambda_1\lambda_2.
\]
Deflation of \((\lambda_1, z), (\lambda_2, z)\)

**Lemma**

If \((\lambda_j^2 M + \lambda_j C + K)z = 0, j = 1, 2\) with \(\lambda_1 \neq \lambda_2\) then

\[
(M, C, K)z = (mp, cp, kp), \ p \in \mathbb{R}^n, \ p^Tz = 1,
\]

\[
c = -m(\lambda_1 + \lambda_2), \ k = m\lambda_1\lambda_2.
\]

Let nonsingular \(G\) be such that

\[
Ge_n = z, \ G^T p = e_n.
\]

Then \(G^T MGe_n = G^T Mz = mG^T p = me_n\) and Lemma \(\Rightarrow\)

\[
G^T(M, C, K)G = \left(\begin{bmatrix} \tilde{M} & 0 \\ 0 & m \end{bmatrix}, \begin{bmatrix} \tilde{C} & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} \tilde{K} & 0 \\ 0 & k \end{bmatrix}\right).
\]
Example 1

\[ Q(\lambda) = \lambda^2 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \]

Given \( \lambda_{1,2} = -0.34 \pm 1.84i \) and associated e’vecs, our deflation procedure yields

\[ \lambda^2 \begin{bmatrix} 5.6 & 2.0e-16 \\ 2.0e-16 & -1.4e-1 \end{bmatrix} + \lambda \begin{bmatrix} -1.6 & -9.4e-16 \\ -9.4e-16 & -9.3e-2 \end{bmatrix} + \begin{bmatrix} 1.6 & -9.8e-17 \\ -9.8e-17 & -4.8e-1 \end{bmatrix}, \]

with \( \kappa_2(T) = 7.9 \) and \( \kappa_2(G) \approx 1. \)

Decoupling accomplished to within the working precision.
Example 2: Damped Beam Problem

\( M, C, K \) generated by \texttt{nlevp('damped\_beam',nele)}.

\[ Q(\lambda) = \lambda^2 M + \lambda C + K \] and undamped \( Q_u(\lambda) = \lambda^2 M + K \) have \( n \) e’vals in common that we deflate by

- our decoupling procedure: \( Q \xrightarrow{S} \begin{bmatrix} Q_1(\lambda) & 0 \\ 0 & Q_2(\lambda) \end{bmatrix} \),
- using special property of \( Q(\lambda) \) to orthogonally block diag’lize it and then diag’lize one block with Cholesky-QR (transformation \( W \)).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \kappa_2(S) )</th>
<th>( \kappa_2(W) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>4.47e1</td>
<td>3.79e1</td>
</tr>
<tr>
<td>32</td>
<td>9.57e1</td>
<td>7.84e1</td>
</tr>
<tr>
<td>64</td>
<td>1.95e2</td>
<td>1.57e2</td>
</tr>
</tbody>
</table>

\[ \kappa_2(E) = \| E \|_2 \| E^{-1} \|_2 \]

\( \kappa_2(S) \) not much larger that \( \kappa_2(W) \).
Deflation procedure extends to nonsymmetric quadratics.

First attempt at defining an SPT with a well-defined action.

Deflation procedure finds application in
- second-order model reduction,
- model updating with no spill-over.

For papers and Eprints,
http://www.ma.man.ac.uk/~ftisseur/