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Hermitian Matrix Polynomials with Real Eigenvalues

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Hermitian Eigenvalue Problem

Consider

$$Ax = \lambda x, \qquad A = A^* \in \mathbb{C}^{n \times n}$$

- Many desirable properties. In particular,
  - real eigenvalues,
  - diagonalizable by congruences,
  - well-conditioned eigenvalues.
- Variety of special algorithms.

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  - well-conditioned eigenvalues.
- Variety of special algorithms.

What are the closest analogues of this class of problems for Hermitian matrix polynomials?

Hermitian Eigenproblem

We consider

- Generalized eigenvalue problem: $L(\lambda)x = 0$,

$$L(\lambda) = \lambda A - B, \quad A = A^*, \; B = B^*.$$

- Polynomial eigenvalue Problem: $P(\lambda)x = 0$,

$$P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j, \qquad A_j = A_j^*, \quad j = 0\colon m.$$

  - mn e'vals, all finite when A_m nonsingular,
  - ∞ and 0 e'vals when A_m and A_0 singular, resp.,
  - $\Lambda(P)$ is symmetric with respect to the real axis.

Hermitian Pencils/Polynomials

The following are known to have **real e'vals**:

- definite pencils,
- definitizable pencils,
- overdamped quadratics,
- gyroscopically stabilized quadratics
- hyperbolic matrix polynomials,
- quasihyperbolic matrix polynomials,
- definite matrix polynomials.

Another common feature: their **e'vals are all of definite type**.

Eigenvalue Types

A finite real e'val λ_0 of $P(\lambda)$ Hermitian is of

- **positive type** if $x^* P'(\lambda_0) x > 0$ for all $0 \neq x \in \ker P(\lambda_0)$,
- **negative type** if $x^* P'(\lambda_0) x < 0$ for all $0 \neq x \in \ker P(\lambda_0)$.
- **definite type** if it is either of positive or negative type.

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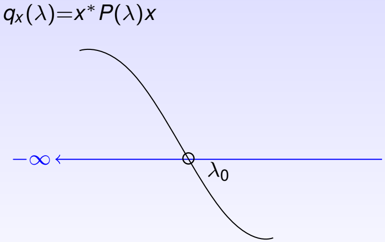
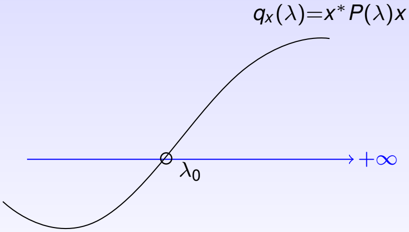


negative type



positive type

Examples

- Simple e'vals are always of definite type.

- The pencil $L(\lambda) = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -\alpha & 0 \\ 0 & \alpha \end{bmatrix}$ has a semisimple e'val $\lambda_0 = \alpha$ with e'vecs $e_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $e_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, and

$$e_1^* L'(\alpha) e_1 = 1, \quad e_2^* L'(\alpha) e_2 = -1.$$

$\Rightarrow \lambda_0 = \alpha$ is of **mixed type**.

Eigenvalue Type at ∞

$$\mathrm{rev}P(\lambda) := \lambda^m P(1/\lambda) = \lambda^m A_0 + \lambda^{m-1} A_1 + \cdots + \lambda A_{m-1} + A_m.$$

$\lambda = \infty$ is an e'val of $P(\lambda)$ iff 0 is an e'val of $\mathrm{rev}P(\lambda)$.

Can show that

$$x^* P'(\lambda_0) x = -\lambda_0^{m-2} x^* (\mathrm{rev}P)'(1/\lambda_0) x, \quad \lambda_0 \neq 0.$$

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$\lambda_0 = \infty$ as an e'val of P is of

- positive type if $x^* A_{m-1} x < 0$ for all $0 \neq x \in \ker A_m$,
- negative type if $x^* A_{m-1} x > 0$ for all $0 \neq x \in \ker A_m$.

Systems of Differential Equations

The solutions to

$$\sum_{j=0}^{m} i^j A_j \frac{d^j u(t)}{dt^j} = 0, \quad t \in \mathbb{R},$$

are bounded on $[0, \infty)$ iff $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$, $\det(A_m) \neq 0$ has real semisimple e'vals.

Solutions remain bounded under small perturbations of A_j iff the e'vals of P are real and of definite type [Gohberg, Lancaster, Rodman 82].

Quasidefinite Matrix Polynomials

Definition (Al-Ammari, T., 10)

A Hermitian matrix polynomial P is **quasidefinite** if

- $\Lambda(P) \subset \mathbb{R} \cup \{\infty\}$, and
- all e'vals are of definite type.

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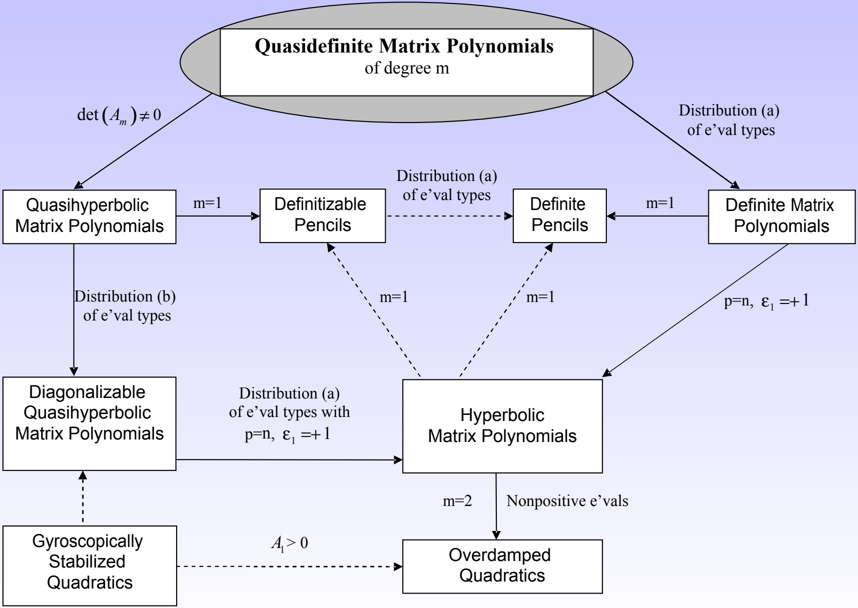
Quasidefinite polynomials include

- definite pencils,
- definitizable pencils,
- overdamped quadratics,
- gyroscopically stabilized quadratics,
- hyperbolic matrix polynomials,
- quasihyperbolic matrix polynomials,
- definite matrix polynomials.

Classification of Quasidefinite Polynomials



Definite Pencils $L(\lambda) = \lambda A - B$

$L(\lambda)$ is **definite** if it satisfies any one of (P1), (P2), (P3).

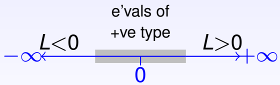
Theorem

The following are equivalent:

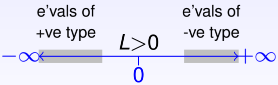
(P1) $\Lambda(L) \subset \mathbb{R} \cup \{\infty\}$ *with all e'vals of definite type and e'vals of +ve type are separated from the e'vals of -ve type.*

(P2) *The matrix* $L(\mu)$ *is definite for some* $\mu \in \mathbb{R} \cup \{\infty\}$.

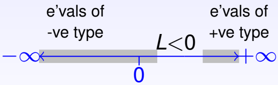
(P3) *For every nonzero vector* x, *the scalar equation* $x^*L(\lambda)x = 0$ *has exactly one zero in* $\mathbb{R} \cup \{\infty\}$.



$A > 0$, B indefinite



A indefinite, $B < 0$

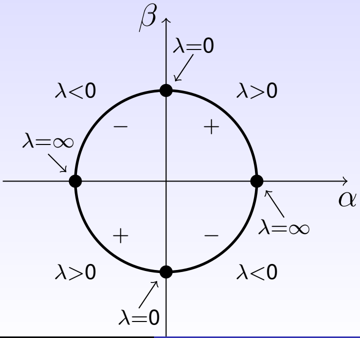


A, B both indefinite

Homogeneous Form

$$P(\alpha,\beta)=\sum_{j=0}^{m}\alpha^j\beta^{m-j}A_j, \qquad L(\alpha,\beta)=\alpha A-\beta B.$$

- E'val λ identified with any pair $(\alpha,\beta)\neq(0,0)$ s.t. $\lambda=\alpha/\beta$.
- $\lambda=0$ represented by $(0,\beta)$, $\lambda=\infty$ represented by $(\alpha,0)$.
- With $\alpha^2+\beta^2=1$, have direct correspondence between $\lambda\in\mathbb{R}\cup\{\infty\}$ and (α,β) on unit circle:



Homogeneous Rotation

$\widetilde{P}(\widetilde{\alpha}, \widetilde{\beta})$ is obtained from $P(\alpha, \beta)$ by **homogenous rotation** if

$$G \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} =: \begin{bmatrix} \widetilde{\alpha} \\ \widetilde{\beta} \end{bmatrix}, \quad c, s \in \mathbb{R}, \quad c^2 + s^2 = 1$$

and

$$P(\alpha, \beta) = \sum_{j=0}^{m} \alpha^j \beta^{m-j} A_j = \sum_{j=0}^{m} \widetilde{\alpha}^j \widetilde{\beta}^{m-j} \widetilde{A}_j := \widetilde{P}(\widetilde{\alpha}, \widetilde{\beta}).$$

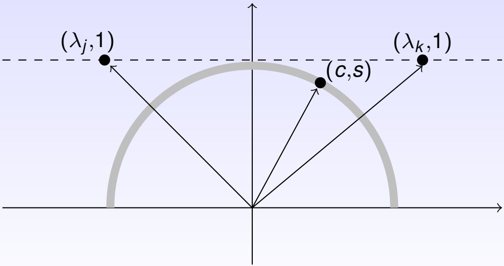
- E'vecs remain the same but e'vals are rotated.
- $\widetilde{A}_m = P(c, s)$.
- Use rotation G to transform P with $\det(A_m) = 0$ or A_m indefinite to $\widetilde{P}$ with $\widetilde{A}_m$ nonsingular or $\widetilde{A}_m > 0$.

Homogeneous Rotation and Types

Let λ_j be an e'val of P rotated to $\widetilde{\lambda}_j$. E'val types are related by:

$$x^* \widetilde{P}'_{\widetilde{\lambda}}(\widetilde{\lambda}_j)x = (c - \lambda_j s)^{m-2} \cdot x^* P'_{\lambda}(\lambda_j)x \quad \text{if } \lambda_j, \widetilde{\lambda}_j \text{ are finite.}$$

$c - \lambda_j s = \det \begin{bmatrix} c & \lambda_j \\ s & 1 \end{bmatrix} > 0$ if $\lambda_j = (\lambda_j, 1)$ that lies counterclockwise from (c, s).



Hyperbolic Polynomials $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$

$P(\lambda)$ $n \times n$ and Hermitian is **hyperbolic** if it satisfies any one of (P1) [Al-Ammari, T. 10], (P2), (P3) [Markus 88]. .

Theorem

The following are equivalent:

(P1) *All e'vals are real, finite, of definite type, and s.t.*

$$\underbrace{\lambda_{mn} \le \cdots \le \lambda_{(m-1)n+1}}_{(-1)^{m-1} \text{ type}} < \cdots < \underbrace{\lambda_{2n} \le \cdots \le \lambda_{n+1}}_{\text{negative type}} < \underbrace{\lambda_n \le \cdots \le \lambda_1}_{\text{positive type}}.$$

(P2) *There exist* $\mu_j \in \mathbb{R} \cup \{\infty\}$ *s.t.* $\infty = \mu_0 > \mu_1 > \cdots > \mu_{m-1}$,

$$(-1)^j P(\mu_j) > 0, \quad j = 0\colon m-1.$$

(P3) $A_m > 0$ *and for every nonzero* $x \in \mathbb{C}^n$*, the scalar equation* $x^* P(\lambda) x = 0$ *has* m *distinct real and finite zeros.*

Acoustic Fluid-structure Interaction Problem

Consider generalized eigenproblem

$$\omega \begin{bmatrix} M_s & 0 \\ M_{fs} & M_f \end{bmatrix} + \begin{bmatrix} K_s & -M_{fs}^* \\ 0 & K_f \end{bmatrix},$$

where $0 < M_s, K_s \in \mathbb{C}^{n\times n}$ and $0 < M_f, K_f \in \mathbb{C}^{m\times m}$.
Multiplying 1st block row by $-\omega$ yields Hermitian quadratic

$$Q(\omega) = \omega^2 \begin{bmatrix} -M_s & 0 \\ 0 & 0 \end{bmatrix} + \omega \begin{bmatrix} -K_s & M_{fs}^* \\ M_{fs} & M_f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_f \end{bmatrix}.$$

- Q is not hyperbolic,
- $\mathrm{rev}\,Q(\omega) = \omega^2 Q(1/\omega)$ is not hyperbolic.
- However, Q is a definite polynomial.

Definitizable Pencils

Definition: An $n \times n$ Hermitian pencil $L(\lambda) = \lambda A - B$ is is **definitizable** if is satisfies any one of (P1), (P2), (P3).

Theorem

The following are equivalent:

(P1) *L has real, finite e'vals of definite type.*

(P2) $\det(A) \neq 0$ *and there exists a real polynomial q s.t. $Aq(A^{-1}B) > 0$.*

(P3) $\det(A) \neq 0$ *and the scalar equation $x^* L(\lambda) x = 0$ has one zero in $\mathbb{R}$ for all e'vecs $x \in \mathbb{C}^n$ of L.*

Proofs in [Lancaster, Ye, 93], except (P3).

Saddle Point Problems

Want to solve large linear systems $\mathcal{A}x = b$ with

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix},$$

where $A = A^T \in \mathbb{R}^{n\times n}$, $A > 0$ and $C = C^T \in \mathbb{R}^{m\times m}$, $C \geq 0$.

$\mathcal{A}$ is **indefinite**: it has n positive e'vals and rank$(C + BA^{-1}B^T)$ negative e'vals.

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If $\lambda\mathcal{J} - \mathcal{A}$ is definitizable with $\mathcal{J} = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix}$ then there exists a well-define CG method for solving linear systems with $\mathcal{J}\mathcal{A}$ (see [Liesen & Parlett 08]).

($\mathcal{J}\mathcal{A}$ is $\mathcal{J}q(\mathcal{J}\mathcal{A})$ symmetric for some q s.t. $\mathcal{J}q(\mathcal{J}\mathcal{A}) > 0$.)

Hermitian Linearizations

Let $\mathbb{L}_1(P) = \{\, L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\ v \in \mathbb{C}^m \,\}$,

where $\Lambda = [\lambda^{m-1}, \lambda^{m-2}, \dots, 1]^T \in \mathbb{C}^m$.

$$\mathbb{H}(P) := \{\, L(\lambda) = \lambda A - B \in \mathbb{L}_1(P) :\ A^* = A,\ B^* = B \,\}$$
$$= \left\{ \sum_{j=1}^{m} v_j(\lambda B_j - B_{j-1}),\ v \in \mathbb{R}^m \right\},$$

where B_j is a direct sum of block Hankel matrices.

Almost all pencils in $\mathbb{H}(P)$ are Hermitian linearizations of P.

Do they preserve additional properties?

Linearizations and E'val Types

For an eigenpair (λ_0, x) of P and $L(\lambda) \in \mathbb{H}(P)$ with vector v, we have

$$z^* L'(\lambda_0) z = \Lambda_0^T v \cdot x^* P'(\lambda_0) x,$$

where (z, λ_0) is an eigenpair of L, $\Lambda_0 = [\lambda_0^{m-1}, \lambda_0^{m-2}, \dots, 1]^T$.

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Theorem

- *P is quasihyperbolic iff any $L \in \mathbb{H}(P)$ is definitizable [Al-Ammari, T., 10].*
- *P is definite iff P has a definite linearization $L \in \mathbb{H}(P)$. [Higham, Mackey, T. 09].*
- *P is hyperbolic iff P has a definite linearization $\lambda A - B \in \mathbb{H}(P)$ with A definite.*

Diagonalizable Pencils

Hermitian pencils are diagonalizable by congruence iff e'vals belong to $\mathbb{R} \cup \{\infty\}$ and are semisimple (see [Lancaster, Rodman 05]).

- Definite pencils are diagonalizable.
- Definitizable pencils are diagonalizable.

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What can we say about (quasi)hyperbolic and definite matrix polynomials?

Strictly Isospectral Polynomials

P is **isospectral** to $\widehat{P}$ if $\Lambda(P) = \Lambda(\widehat{P})$ with same partial multiplicities.

P and $\widehat{P}$ are **strictly isospectral** if they are isospectral and share the **same sign characteristic**.

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Let P, $\widehat{P}$ be quasihyperbolic and strictly isospectral and let $L \in \mathbb{H}(P)$, $\widehat{L} \in \mathbb{H}(\widehat{P})$ with vector v.

There exist nonsingular $X, \widehat{X}$ s.t.

$$XL(\lambda)X^* = \lambda \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} - \begin{bmatrix} J_+ & 0 \\ 0 & -J_- \end{bmatrix} = \widehat{X}\widehat{L}(\lambda)\widehat{X}^*.$$

$\widehat{X}^{-1}X$ defines a **structure preserving congruence**.

Diagonalizable by SPC

Definition: $P(\lambda)$, Hermitian and of degree m is **diagonalizable by structure preserving congruence** (SPC) if it is strictly isospectral to a real diagonal matrix polynomial of degree m.

- Quasidefinite quadratics are always strictly isospectral to quasidefinite diagonal quadratics.
- Definite matrix polynomial are always strictly isospectral to definite diagonal matrix polynomials.

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Theorem (Al-Ammari, T. 10)

An $n \times n$ quasihyperbolic matrix polynomial of degree m is diagonalizable by SPC iff there is a grouping of its e'vals and their types into n subsets of m distinct e'vals, which when ordered have alternating types.

- Quasidefinite quadratics are always strictly isospectral to quasidefinite diagonal quadratics.
- Definite matrix polynomial are always strictly isospectral to definite diagonal matrix polynomials.

Concluding Remarks

- Gave a unified treatment of the many subclasses of Hermitian matrix polynomials with real eigenvalues.
- Identified classes of Hermitian matrix polynomials that are diagonalizable by SPC.
- Results useful in the solution of the inverse problem.
- Investigate analogous results for palindromic and odd/even matrix polynomials.

For paper see:

M. Al-Ammari and F. Tisseur. *Hermitian Matrix Polynomials with Real Eigenvalues of Definite Type. Part I: Classification*, MIMS EPrint 2010.9, The University of Manchester, 2010.