Hermitian Matrix Polynomials with Real Eigenvalues

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Joint work with Maha Al-Ammari

Consider

\[ Ax = \lambda x, \quad A = A^* \in \mathbb{C}^{n \times n} \]

Many desirable properties. In particular,

- real eigenvalues,
- diagonalizable by congruences,
- well-conditioned eigenvalues.

Variety of special algorithms.
Hermitian Eigenvalue Problem

Consider

\[ Ax = \lambda x, \quad A = A^* \in \mathbb{C}^{n\times n} \]

- Many desirable properties. In particular,
  - real eigenvalues,
  - diagonalizable by congruences,
  - well-conditioned eigenvalues.

- Variety of special algorithms.

What are the closest analogues of this class of problems for Hermitian matrix polynomials?
We consider

- **Generalized eigenvalue problem**: \( \lambda A - B = 0 \),
  \[
  L(\lambda) = \lambda A - B, \quad A = A^*, \quad B = B^*.
  \]

- **Polynomial eigenvalue Problem**: \( P(\lambda) x = 0 \),
  \[
  P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j, \quad A_j = A_j^*, \quad j = 0 : m.
  \]

- \( mn \) e’vals, all finite when \( A_m \) nonsingular,
- \( \infty \) and 0 e’vals when \( A_m \) and \( A_0 \) singular, resp.,
- \( \Lambda(P) \) is symmetric with respect to the real axis.
The following are known to have real e’vals:

- definite pencils,
- definitizable pencils,
- overdamped quadratics,
- gyroscopically stabilized quadratics
- hyperbolic matrix polynomials,
- quasi-hyperbolic matrix polynomials,
- definite matrix polynomials.

Another common feature: their e’vals are all of definite type.
A finite real e’val $\lambda_0$ of $P(\lambda)$ Hermitian is of

- **positive type** if $x^* P'(\lambda_0) x > 0$ for all $0 \neq x \in \ker P(\lambda_0)$,
- **negative type** if $x^* P'(\lambda_0) x < 0$ for all $0 \neq x \in \ker P(\lambda_0)$.
- **definite type** if it is either of positive or negative type.
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- **definite type** if it is either of positive or negative type.

$q_x(\lambda) = x^* P(\lambda)x$
Examples

- Simple e’vals are always of definite type.

- The pencil $L(\lambda) = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -\alpha & 0 \\ 0 & \alpha \end{bmatrix}$ has a semisimple e’val $\lambda_0 = \alpha$ with e’vecs $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and

  $$e_1^* L'(\alpha)e_1 = 1, \quad e_2^* L'(\alpha)e_2 = -1.$$  

  $\Rightarrow \lambda_0 = \alpha$ is of mixed type.
\[ \text{rev} P(\lambda) := \lambda^m P(1/\lambda) = \lambda^m A_0 + \lambda^{m-1} A_1 + \cdots + \lambda A_{m-1} + A_m. \]

\( \lambda = \infty \) is an e’val of \( P(\lambda) \) iff 0 is an e’val of \( \text{rev} P(\lambda) \).

Can show that

\[ x^* P'(\lambda_0) x = -\lambda_0^{m-2} x^* (\text{rev} P)'(1/\lambda_0) x, \quad \lambda_0 \neq 0. \]
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\( \lambda_0 = \infty \) as an e’val of \( P \) is of

- positive type if \( x^* A_{m-1} x < 0 \) for all \( 0 \neq x \in \ker A_m \),
- negative type if \( x^* A_{m-1} x > 0 \) for all \( 0 \neq x \in \ker A_m \).
The solutions to

\[ \sum_{j=0}^{m} \lambda^j A_j \frac{d^j u(t)}{dt^j} = 0, \quad t \in \mathbb{R}, \]

are bounded on \([0, \infty)\) iff \(P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j, \det(A_m) \neq 0\) has real semisimple e’vals.

Solutions remain bounded under small perturbations of \(A_j\) iff the e’vals of \(P\) are real and of definite type [Gohberg, Lancaster, Rodman 82].
A Hermitian matrix polynomial \( P \) is **quasidefinite** if

- \( \Lambda(P) \subset \mathbb{R} \cup \{\infty\} \), and
- all eigenvalues are of definite type.
Definition (Al-Ammari, T., 10)

A Hermitian matrix polynomial $P$ is **quasidefinite** if:
- $\Lambda(P) \subset \mathbb{R} \cup \{\infty\}$, and
- all e’vals are of definite type.

Quasidefinite polynomials include:
- definite pencils,
- definitizable pencils,
- overdamped quadratics,
- gyroscopically stabilized quadratics,
- hyperbolic matrix polynomials,
- quasihyperbolic matrix polynomials,
- definite matrix polynomials.
Classification of Quasidefinite Polynomials

Quasidefinite Matrix Polynomials
of degree m

Quasihyperbolic Matrix Polynomials

Definitizable Pencils

Diagonalizable Quasihyperbolic Matrix Polynomials

Gyroscopically Stabilized Quadratics

Det \( A_m \) \( \neq 0 \)

Definite Matrix Polynomials

Definite Pencils

Hyperbolic Matrix Polynomials

Distribution (a) of e’val types

Distribution (b) of e’val types

Distribution (a) of e’val types with
p=n, \( \varepsilon_1 = +1 \)

m=1

m=1

m=2

Nonpositive e’vals

m=1

A_i > 0

Overdamped Quadratics

Det \( A_m \) \( \neq 0 \)

Distribution (a) of e’val types
Definite Pencils $L(\lambda) = \lambda A - B$

$L(\lambda)$ is **definite** if it satisfies any one of (P1), (P2), (P3).

**Theorem**

The following are equivalent:

(P1) $\Lambda(L) \subset \mathbb{R} \cup \{\infty\}$ with all e’vals of definite type and e’vals of +ve type are separated from the e’vals of -ve type.

(P2) The matrix $L(\mu)$ is definite for some $\mu \in \mathbb{R} \cup \{\infty\}$.

(P3) For every nonzero vector $x$, the scalar equation $x^* L(\lambda) x = 0$ has exactly one zero in $\mathbb{R} \cup \{\infty\}$.

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$A > 0$, $B$ indefinite

$A$ indefinite, $B < 0$

$A$, $B$ both indefinite
Homogeneous Form

\[ P(\alpha, \beta) = \sum_{j=0}^{m} \alpha^j \beta^{m-j} A_j, \quad L(\alpha, \beta) = \alpha A - \beta B. \]

- \( \text{Eval} \ \lambda \) identified with any pair \((\alpha, \beta) \neq (0, 0)\) s.t. \( \lambda = \alpha/\beta \).
- \( \lambda = 0 \) represented by \((0, \beta)\), \( \lambda = \infty \) represented by \((\alpha, 0)\).
- With \( \alpha^2 + \beta^2 = 1 \), have direct correspondence between \( \lambda \in \mathbb{R} \cup \{\infty\} \) and \((\alpha, \beta)\) on unit circle:

\[\begin{align*}
\lambda > 0 & \quad \lambda < 0 \\
\lambda = 0 & \quad \lambda = \infty
\end{align*}\]
Homogeneous Rotation

$\tilde{P}(\tilde{\alpha}, \tilde{\beta})$ is obtained from $P(\alpha, \beta)$ by homogenous rotation if

$$G \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} =: \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}, \quad c, s \in \mathbb{R}, \quad c^2 + s^2 = 1$$

and

$$P(\alpha, \beta) = \sum_{j=0}^{m} \alpha^j \beta^{m-j} A_j = \sum_{j=0}^{m} \tilde{\alpha}^j \tilde{\beta}^{m-j} \tilde{A}_j := \tilde{P}(\tilde{\alpha}, \tilde{\beta}).$$

- E’vecs remain the same but e’vals are rotated.
- $\tilde{A}_m = P(c, s)$.
- Use rotation $G$ to transform $P$ with $\det(A_m) = 0$ or $A_m$ indefinite to $\tilde{P}$ with $\tilde{A}_m$ nonsingular or $\tilde{A}_m > 0$. 
Let $\lambda_j$ be an e’val of $P$ rotated to $\tilde{\lambda}_j$. E’val types are related by:

$$x^* \tilde{P}'_{\tilde{\lambda}}(\tilde{\lambda}_j)x = (c - \lambda_j s)^{m-2} \cdot x^* P'_{\lambda}(\lambda_j)x$$

if $\lambda_j$, $\tilde{\lambda}_j$ are finite.

$$c - \lambda_j s = \det \begin{bmatrix} c & \lambda_j \\ s & 1 \end{bmatrix} > 0$$

if $\lambda_j = (\lambda_j, 1)$ that lies counterclockwise from $(c, s)$. 

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Hermitian Matrix Polynomials
Hyperbolic Polynomials $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$

$P(\lambda)$ $n \times n$ and Hermitian is **hyperbolic** if it satisfies any one of (P1) [Al-Ammari, T. 10], (P2), (P3) [Markus 88].

**Theorem**

The following are equivalent:

(P1) **All e’vals are real, finite, of definite type, and s.t.**

\[
\lambda_{mn} \leq \cdots \leq \lambda_{(m-1)n+1} < \cdots < \lambda_{2n} \leq \cdots \leq \lambda_{n+1} < \lambda_n \leq \cdots \leq \lambda_1.
\]

($(-1)^{m-1}$ type) **negative type** **positive type**

(P2) **There exist $\mu_j \in \mathbb{R} \cup \{\infty\}$ s.t.** $\infty = \mu_0 > \mu_1 > \cdots > \mu_{m-1}$,

\[
(-1)^j P(\mu_j) > 0, \quad j = 0 : m - 1.
\]

(P3) $A_m > 0$ and for every nonzero $x \in \mathbb{C}^n$, the scalar equation $x^* P(\lambda)x = 0$ has $m$ distinct real and finite zeros.
Consider generalized eigenproblem

\[
\omega \begin{bmatrix}
M_s & 0 \\
M_{fs} & M_f
\end{bmatrix} + \begin{bmatrix}
K_s & -M_{fs}^* \\
0 & K_f
\end{bmatrix},
\]

where \( 0 < M_s, K_s \in \mathbb{C}^{n \times n} \) and \( 0 < M_f, K_f \in \mathbb{C}^{m \times m} \).

Multiplying 1st block row by \(-\omega\) yields Hermitian quadratic

\[
Q(\omega) = \omega^2 \begin{bmatrix}
-M_s & 0 \\
0 & 0
\end{bmatrix} + \omega \begin{bmatrix}
-K_s & M_{fs}^* \\
M_{fs} & M_f
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & K_f
\end{bmatrix}.
\]

\[
\begin{align*}
\triangleright & \quad Q \text{ is not hyperbolic,} \\
\triangleright & \quad \text{rev} Q(\omega) = \omega^2 Q(1/\omega) \text{ is not hyperbolic.} \\
\triangleright & \quad \text{However, } Q \text{ is a definite polynomial.}
\end{align*}
\]
Definitizable Pencils

Definition: An \( n \times n \) Hermitian pencil \( L(\lambda) = \lambda A - B \) is is definitizable if it satisfies any one of (P1), (P2), (P3).

Theorem

The following are equivalent:

(P1) \( L \) has real, finite e’vals of definite type.

(P2) \( \det(A) \neq 0 \) and there exists a real polynomial \( q \) s.t. \( Aq(A^{-1}B) > 0 \).

(P3) \( \det(A) \neq 0 \) and the scalar equation \( x^*L(\lambda)x = 0 \) has one zero in \( \mathbb{R} \) for all e’vecs \( x \in \mathbb{C}^n \) of \( L \).

Proofs in [Lancaster, Ye, 93], except (P3).
Want to solve large linear systems $\mathcal{A}x = b$ with

$$
\mathcal{A} = \begin{bmatrix}
  A & B^T \\
  B & \ -C
\end{bmatrix},
$$

where $A = A^T \in \mathbb{R}^{n \times n}$, $A > 0$ and $C = C^T \in \mathbb{R}^{m \times m}$, $C \geq 0$.

$\mathcal{A}$ is **indefinite**: it has $n$ positive e’vals and $\text{rank}(C + BA^{-1}B^T)$ negative e’vals.
Want to solve large linear systems \( Ax = b \) with

\[
A = \begin{bmatrix}
    A & B^T \\
    B & -C
\end{bmatrix},
\]

where \( A = A^T \in \mathbb{R}^{n \times n}, A > 0 \) and \( C = C^T \in \mathbb{R}^{m \times m}, C \geq 0 \).

\( A \) is **indefinite**: it has \( n \) positive e’vals and \( \text{rank}(C + BA^{-1}B^T) \) negative e’vals.

If \( \lambda J - A \) is definitizable with \( J = \begin{bmatrix}
    I_n & 0 \\
    0 & -I_m
\end{bmatrix} \) then there exists a well-define CG method for solving linear systems with \( JA \) (see [Liesen & Parlett 08]).

(\( JA \) is \( Jq(JA) \) symmetric for some \( q \) s.t. \( Jq(JA) > 0 \).)
Let \( \mathbb{L}_1(P) = \{ L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda), \ v \in \mathbb{C}^m \} \),

where \( \Lambda = [\lambda^{m-1}, \lambda^{m-2}, \ldots, 1]^{T} \in \mathbb{C}^{m} \).

\[
\mathbb{H}(P) := \left\{ L(\lambda) = \lambda A - B \in \mathbb{L}_1(P) : A^{*} = A, \quad B^{*} = B \right\} 
= \left\{ \sum_{j=1}^{m} v_j (\lambda B_j - B_{j-1}), \quad v \in \mathbb{R}^m \right\},
\]

where \( B_j \) is a direct sum of block Hankel matrices.

Almost all pencils in \( \mathbb{H}(P) \) are Hermitian linearizations of \( P \).

Do they preserve additional properties?
For an eigenpair \((\lambda_0, x)\) of \(P\) and \(L(\lambda) \in \mathbb{H}(P)\) with vector \(v\), we have

\[
z^* L'(\lambda_0) z = \Lambda_0^T v \cdot x^* P'(\lambda_0) x,
\]

where \((z, \lambda_0)\) is an eigenpair of \(L\), \(\Lambda_0 = [\lambda_0^{m-1}, \lambda_0^{m-2}, \ldots, 1]^T\).
Linearizations and E’val Types

For an eigenpair \((\lambda_0, x)\) of \(P\) and \(L(\lambda) \in \mathbb{H}(P)\) with vector \(v\), we have

\[
z^* L'(\lambda_0) z = \Lambda_0^T v \cdot x^* P'(\lambda_0) x,
\]

where \((z, \lambda_0)\) is an eigenpair of \(L\), \(\Lambda_0 = [\lambda_0^{-1}, \lambda_0^{-2}, \ldots, 1]^T\).

Theorem

- \(P\) is quasihyperbolic iff any \(L \in \mathbb{H}(P)\) is definitizable [Al-Ammari, T., 10].
- \(P\) is definite iff \(P\) has a definite linearization \(L \in \mathbb{H}(P)\). [Higham, Mackey, T. 09].
- \(P\) is hyperbolic iff \(P\) has a definite linearization \(\lambda A - B \in \mathbb{H}(P)\) with \(A\) definite.
Hermitian pencils are diagonalizable by congruence iff e’vals belong to $\mathbb{R} \cup \{\infty\}$ and are semisimple (see [Lancaster, Rodman 05]).

- Definite pencils are diagonalizable.
- Definitizable pencils are diagonalizable.
Hermitian pencils are diagonalizable by congruence iff e’vals belong to $\mathbb{R} \cup \{\infty\}$ and are semisimple (see [Lancaster, Rodman 05]).

- Definite pencils are diagonalizable.
- Definitizable pencils are diagonalizable.

What can we say about (quasi)hyperbolic and definite matrix polynomials?
Strictly Isospectral Polynomials

$P$ is **isospectral** to $\hat{P}$ if $\Lambda(P) = \Lambda(\hat{P})$ with same partial multiplicities.

$P$ and $\hat{P}$ are **strictly isospectral** if they are isospectral and share the **same sign characteristic**.
Strictly Isospectral Polynomials

$P$ is isospectral to $\hat{P}$ if $\Lambda(P) = \Lambda(\hat{P})$ with same partial multiplicities.

$P$ and $\hat{P}$ are strictly isospectral if they are isospectral and share the same sign characteristic.

Let $P$, $\hat{P}$ be quasihyperbolic and strictly isospectral and let $L \in \mathbb{H}(P)$, $\hat{L} \in \mathbb{H}(\hat{P})$ with vector $v$.

There exist nonsingular $X$, $\hat{X}$ s.t.

$$XL(\lambda)X^* = \lambda \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} - \begin{bmatrix} J_+ & 0 \\ 0 & -J_- \end{bmatrix} = \hat{X}\hat{L}(\lambda)\hat{X}^*.$$  

$\hat{X}^{-1}X$ defines a structure preserving congruence.
Diagonalizable by SPC

Definition: \( P(\lambda) \), Hermitian and of degree \( m \) is diagonalizable by structure preserving congruence (SPC) if it is strictly isospectral to a real diagonal matrix polynomial of degree \( m \).

- Quasidefinite quadratics are always strictly isospectral to quasidefinite diagonal quadratics.
- Definite matrix polynomial are always strictly isospectral to definite diagonal matrix polynomials.
Definition: $P(\lambda)$, Hermitian and of degree $m$ is **diagonalizable by structure preserving congruence** (SPC) if it is strictly isospectral to a real diagonal matrix polynomial of degree $m$.

**Theorem (Al-Ammari, T. 10)**

An $n \times n$ quasihyperbolic matrix polynomial of degree $m$ is diagonalizable by SPC iff there is a grouping of its e’vals and their types into $n$ subsets of $m$ distinct e’vals, which when ordered have alternating types.

- Quasidefinite quadratics are always strictly isospectral to quasidefinite diagonal quadratics.
- Definite matrix polynomial are always strictly isospectral to definite diagonal matrix polynomials.
Concluding Remarks

- Gave a unified treatment of the many subclasses of Hermitian matrix polynomials with real eigenvalues.
- Identified classes of Hermitian matrix polynomials that are diagonalizable by SPC.
- Results useful in the solution of the inverse problem.
- Investigate analogous results for palindromic and odd/even matrix polynomials.

For paper see: