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A Review of Nonlinear Eigenvalue Problems

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Nonlinear Eigenvalue Problems (NEPs)

Let $F\colon \Omega \to \mathbb{C}^{m\times n}$ be analytic on open set $\Omega \subseteq \mathbb{C}$.

The **nonlinear eigenvalue problem**: Find scalars λ and nonzero $x, y \in \mathbb{C}^n$ satisfying $F(\lambda)x = 0$ and $y^* F(\lambda) = 0$.

- λ is an e'val, x, y are corresponding right and left e'vecs.
- When $m = n$, e'vals are solutions of $\det(F(\lambda)) = 0$.

In practice, elements of F most often **polynomial, rational or exponential functions of λ**.

NLEVP Toolbox

Collection of Nonlinear Eigenvalue Problems : T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, F. T., 2010.

- Quadratic, polynomial, rational and other nonlinear eigenproblems.
- Provided in the form of a MATLAB Toolbox.
- Problems from real-life applications + specifically constructed problems.

`http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html`

(See C. Schröder's talk in MS392.)

Sample of Quadratic Problems

$n \times m$ quadratic $Q(\lambda) = \lambda^2 M + \lambda D + K$.

Speaker box (`pep,qep,real,symmetric`).
$n = m = 107$. Finite element model of a speaker box.
$\|M\|_2 = 1$, $\|C\|_2 = 5.7 \times 10^{-2}$, $\|K\|_2 = 1.0 \times 10^7$.

Railtrack (`pep,qep,t-palindromic,sparse`).
$n = m = 1005$. Model of vibration of rail tracks under the excitation of high speed trains. $M = K^T$, $D = D^T$.

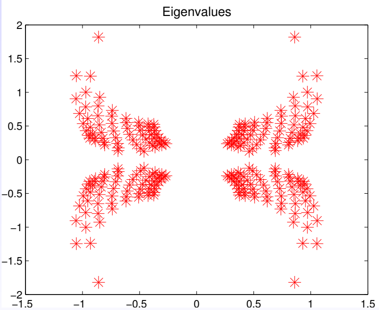
Surveillance (`pep,qep,real,nonsquare, nonregular`). $n = 21$, $m = 16$. From calibration of surveillance camera using human body as calibration target.

Sample of Higher Degree Problems

Plasma drift(`pep`). Cubic polynomial from modeling of drift instabilities in the plasma edge inside a Tokamak reactor.

Orr-Sommerfeld(`pep,parameter-dependent`). Quartic arising in spatial stability analysis of Orr-Sommerfeld eq.

Butterfly (`pep,real, T-even, scalable`) quartic matrix polynomial with T-even structure.



Sample of Rational/Nonlinear Problems

Loaded string (`rep,real,symmetric,scalable`) rational eigenvalue problem describing eigenvibration of a loaded string.

$$R(\lambda)x = \left(A - \lambda B + \frac{\lambda}{\lambda - \sigma} C\right) x = 0.$$

Gun (`nep,sparse`) nonlinear eigenvalue problem modeling a radio-frequency gun cavity.

$$F(\lambda)x = \left[K - \lambda M + i(\lambda - \sigma_1^2)^{\frac{1}{2}} W_1 + i(\lambda - \sigma_2^2)^{\frac{1}{2}} W_2\right] x = 0.$$

- C, W_1 and W_2 have low rank.

Comments

- NEPs underpin many areas of computational science and engineering.
- Can be very difficult to solve:
  - nonlinear,
  - large problem size,
  - poor conditioning,
  - lack of good numerical methods.

Outline

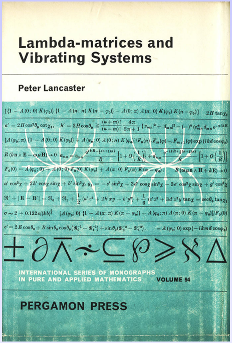
Review of recent progress in

- Polynomial eigenvalue problems (PEPs)
  - Historical aspects
  - General background
  - Unstructured PEPs
  - Structured PEPs
- Rational and nonlinear eigenvalue problems
- Structure preserving transformations

Historical Aspects

- In the 1930s, **Frazer, Duncan & Collar** were developing matrix methods for analyzing **flutter** in aircraft.
- Worked in Aerodynamics Division of NPL.
- Developed matrix structural analysis.
- Wrote **Elementary Matrices & Some Applications to Dynamics and Differential Equations, 1938**.
- **Olga Taussky**, in Frazer's group at NPL, 1940s. 6×6 quadratic eigenvalue problems from flutter in supersonic aircraft.

Historical Aspects (cont.)



- **Peter Lancaster**, English Electric Co., 1950s solved quadratic eigenvalue problems of dimension 2 to 20.

Books

- Lancaster, **Lambda-Matrices and Vibrating Systems**, 1966 (Pergamon), 2002 (Dover).
- Gohberg, Lancaster, Rodman, **Matrix Polynomials**, 1982 (Academic Press), 2009 (SIAM).
- Gohberg, Lancaster, Rodman, **Indefinite Linear Algebra and Applications**, 2005 (Birkhäuser).
- Gohberg, Lancaster, Rodman, **Invariant Subspaces of Matrices with Applications**, 1986 (Wiley), 2006 (SIAM).

Polynomial Eigenvalue Problems (PEP)

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i, \quad A_i \in \mathbb{C}^{n\times n}, \quad A_k \neq 0.$$

- Assume $P(\lambda)$ is **regular**, i.e., $\det(P(\lambda)) \not\equiv 0$.
- $P(\lambda)$ has kn **e'vals**. Finites e'vals are roots of $\det(P(\lambda)) = 0$.
- Zero e'vals when A_0 is singular and infinite e'vals when A_k is singular.

Example 1

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 0 & 8 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & -6 & 0 \\ 2 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Regular: $\det Q = -6\lambda^5 + 11\lambda^4 - 12\lambda^3 + 12\lambda^2 - 6\lambda + 1 \not\equiv 0.$

Six eigenpairs (λ_k, x_k), $k = 1: 6$, given by

k	1	2	3	4	5	6
λ_k	$1/3$	$1/2$	1	i	$-i$	∞
x_k	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Jordan pairs

- Jordan canonical decomposition of $A \in \mathbb{C}^{n \times n}$, $A = XJX^{-1}$ extends to matrix polynomials. (X, J) is a **Jordan pair** for $\lambda I - A$.

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- (X, J) is a **Jordan pair** for $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, $A_i \in \mathbb{C}^{n\times n}$, $\det A_k \neq 0$ if
  - $X \in \mathbb{C}^{n\times kn}$ and $J \in \mathbb{C}^{kn\times kn}$ Jordan matrix,
  - $Q(X, J) := \begin{bmatrix} XJ^{k-1} \\ \vdots \\ XJ \\ X \end{bmatrix}$ is nonsingular,
  - $A_k XJ^k + \cdots + A_1 XJ + A_0 X = 0$.

X contains right e'vecs and generalized e'vecs of $P(\lambda)$.

Jordan Triples

Let (X, J) be a Jordan pair for $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$.

- Define $Y = (Q(X, J))^{-1}(e_1 \otimes A_k^{-1}) \in \mathbb{C}^{kn \times n}$.
- Y contains left e'vecs and generalized e'vecs of $P(\lambda)$.
- (X, J, Y) is a **Jordan triple** for $P(\lambda)$.

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Can express coeffs A_j in terms of spectral data (X, J, Y).

For example, if $k = 2$,

$$\begin{aligned} A_2 &= (XJY)^{-1}, \\ A_1 &= -A_2 XJ^2 YA_2, \\ A_0 &= -A_2(XJ^2 YA_1 + XJ^3 YA_2). \end{aligned}$$

Condensed Forms

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i, \quad A_i \in \mathbb{C}^{n\times n}, \quad A_k \neq 0.$$

- **Generalized Schur decomposition**: there exist U, V unitary s.t. $U(\lambda A_1 + A_0)V = \lambda T + S$ is upper triangular.
  - $\lambda_j = -s_{jj}/t_{jj}$, $j = 1: n$.
  - Can be computed by the QZ algorithm.
  - Useful for purging and locking of e'vals, implicit restart, ...
- **No analog of generalized Schur decomposition** when $k > 1$.
- Eigensolvers often **absent from numerical libraries.**

Eigensolvers for PEPs (cont.)

$$P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0, \quad P(\lambda)x = 0.$$

Dfn: $\mathcal{A} - \lambda\mathcal{B}$ is a linearization of $P(\lambda)$ if there exist unimodular $E(\lambda), F(\lambda)$ s.t. $E(\lambda)(\mathcal{A} - \lambda\mathcal{B})F(\lambda) = \mathrm{diag}(P(\lambda), I)$.

- PEPs commonly solved by **linearization**.
  - Convert P into linear pencil $\mathcal{A} - \lambda\mathcal{B}$, e.g., $(k = 2)$,

$$\left(\begin{bmatrix} A_0 & 0 \\ 0 & I \end{bmatrix} - \lambda \begin{bmatrix} -A_1 & -A_2 \\ I & 0 \end{bmatrix}\right) \begin{bmatrix} x \\ \lambda x \end{bmatrix} =: (\mathcal{A} - \lambda\mathcal{B})z = 0.$$

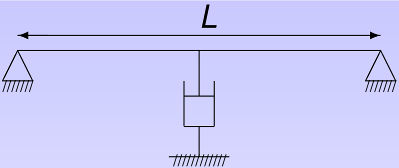
  - Solve generalized eigenvalue problem (GEP).
  - Recover e'vecs of $P(\lambda)$ from those of $\mathcal{A} - \lambda\mathcal{B}$.

Numerical Issues

- **Infinitely many linearizations** to choose from.
- Solving PEPs with a backward stable alg. (e.g., QZ alg.) applied to a linearization can be **backward unstable** for the PEP.
- Linearizations can have **widely varying eigenvalue condition numbers**.

Numerical solution of PEPs requires special attention.

Example 2: Beam Problem



Transverse displacement $u(x, t)$ governed by

$$\rho A \frac{\partial^2 u}{\partial t^2} + c(x)\frac{\partial u}{\partial t} + EI\frac{\partial^4 u}{\partial x^4} = 0.$$

Boundary conditions: $u(0,t) = u''(0,t) = u(L,t) = u''(L,t) = 0.$

$u(x,t) = e^{\lambda t} v(x,\lambda)$ yields

eigenvalue problem for the free vibrations :

$$\lambda^2 \rho A v(x,\lambda) + \lambda c(x) v(x,\lambda) + EI\frac{\partial^4}{\partial x^4} v(x,\lambda) = 0.$$

Discretized Beam Problem

Finite element method leads to

$$Q(\lambda) = \lambda^2 M + \lambda D + K$$

with **symmetric** $M, D, K \in \mathbb{R}^{n\times n}$. $M > 0$, $K > 0$, $D \geq 0$.

Roots of $x^* Q(\lambda) x = 0$, $x \in \mathbb{C}^n$,

$$\lambda = \frac{-(x^* D x) \pm \sqrt{(x^* D x)^2 - 4(x^* M x)(x^* K x)}}{2(x^* M x)}.$$

- $M > 0$, $K > 0$, $D \geq 0$ $\Rightarrow$ all e'vals have $\mathrm{Re}(\lambda) \leq 0$.
- D is rank 1. Can show n pure imaginary e'vals.

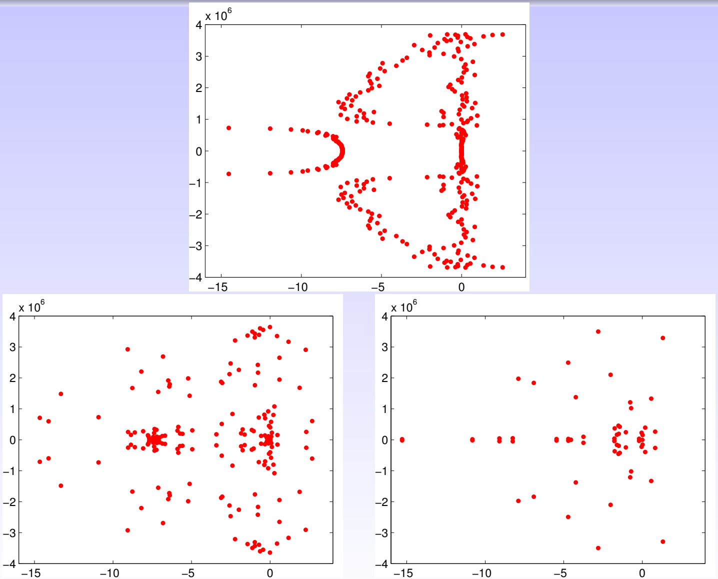
Eigenvalues of $Q(\lambda) = \lambda^2 M + \lambda D + K$

$$C_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} D & K \\ -I & 0 \end{bmatrix},$$

$$L_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} + \begin{bmatrix} D & K \\ K & 0 \end{bmatrix}, \quad L_2(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}.$$

```
coeffs = nlevp('damped_beam',100);
K = coeffs{1}; D = coeffs{2}; M = coeffs{3};
I = eye(2*nele); O = zeros(2*nele);
eval = eig([D K; -I O],-[M O; O I]; % C1
%eval = eig([D K; K O],-[M O; O -K]; % L1
%eval = eig([-M O; O K],-[O M; M D]; % L2
plot(eval,'.r');
```

Computed Spectra of C_1, L_1 and L_2



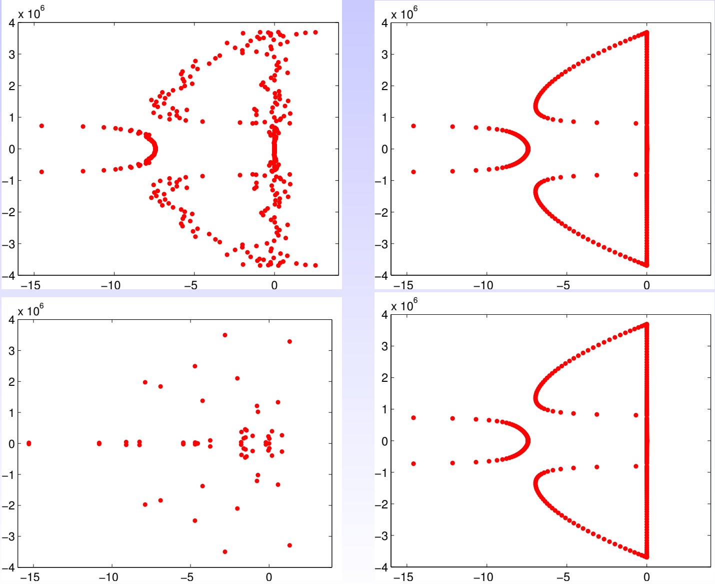
Sensitivity and Stability of Linearizations

- Developed theory concerning the sensitivity and stability of linearizations [Higham, Mackey, T. 06, Higham, Li, T. 07, Grammont, Higham, T., 11].
- Importance of **scaling** QEPs/PEPs before computing e'vals via linearization.
  - Eigenvalue parameter scaling:

$$\lambda = \gamma\mu, \quad \widetilde{P}(\mu) := \delta P(\gamma\mu).$$

  - Does not affect sparsity of matrix coeffs.
  - γ, δ chosen to improve growth factors $\kappa_L(\lambda)/\kappa_P(\lambda)$ (conditioning), $\eta_P(z_i, \lambda)/\eta_L(z, \lambda)$ (backward error).

Spectrum of C_1, L_2 before/after Scaling



quadeig

- Eigensolver for dense (small to medium size quadratics)—**quadeig**.
- Incorporate:
  - Appropriate choice of linearization.
  - Deflation of 0 and ∞ eigenvalues.
  - Eigenvalue parameter scaling.
  - Exploitation of block structure.
  - Careful recovery of the eigenvectors.
- MATLAB and Fortran implementations [Hammarling, Munro, T. 2011].

Eigensolver for Large Sparse QEPs

Two types of projection methods:

- those applied directly to the quadratic (e.g., residual iteration method, Jacobi Davidson method).
- those applied to linearized problem $(\mathcal{A} - \lambda\mathcal{B})z = 0$ (e.g., Arnoldi/Two-sided Lanczos methods).

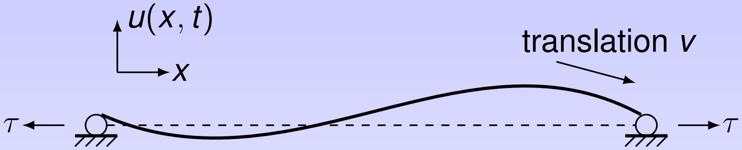
Some recent contributions:

- Second Order Arnoldi (SOAR) method [Bai & Su, 05]. Projection applied directly to quadratic.
- Quadratic Arnoldi method [Meerbergen, 08].

Structured Matrix Polynomials

- In application, $P(\lambda)$ is often **structured**:
  - Symmetries in the matrix coefficients, e.g., (cplx) symm., palindromic, alternating structures, ...
  - Definiteness properties, e.g., hyperbolic/ overdamped systems, gyroscopic systems, ...
- Structured $P(\lambda)$ often have **spectral symmetries**.
- New **structure preserving linearizations** derived along with **algorithms preserving spectral properties** in finite precision arithmetic.

Example 3: Traveling Band



The band's transverse displacement described by

$$\left[\frac{\partial^2}{\partial t^2} + 2v\frac{\partial^2}{\partial x \partial t} + (\kappa v^2 - \tau)\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right] u(x,t) = 0, \quad x \in [0,1],$$

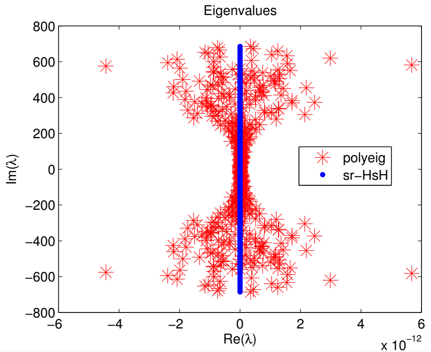
where

- τ is the tension,
- $\kappa \in [0, 1]$ depends on the pulley mounting system.

Computed Spectra

Galerkin method yields $M\ddot{q}(t) + G\dot{q}(t) + Kq(t) = 0$,
$M = M^T > 0$, $G = -G^T$, $K = K^T > 0$ (**gyroscopic system**).

All e'vals are purely imaginary and the system is (weakly) stable.



`nlevp('wiresaw1',250,.1)`

MATLAB `polyeig`: not structure preserving.

`sr-HsH`: Square-reduced alg. for Hamiltonian/skew-Hamiltonian pencils.

Typical Structures

$\mathrm{rev}\,(P(\lambda)) := \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^{k-i} A_i$.
$\star$ = transpose T or conjugate transpose $*$.

Structure	Property of $P(\lambda)$	Eigenvalue pairing
Hermitian/symm.	$P(\lambda) = P^\star(\lambda)$	$(\lambda, \bar{\lambda})$
$\star$-even/odd	$P(\lambda) = \pm P^\star(-\lambda)$	$(\lambda, -\lambda^\star)$
$\star$-(anti)palindromic	$P(\lambda) = \pm \mathrm{rev}\, P^\star(\lambda)$	$(\lambda, 1/\lambda^\star)$

Example: $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ is ***T*-even** if

$$P(\lambda) = P^T(-\lambda) \quad \Leftrightarrow \quad A_2 = A_2^T,\ A_1 = -A_1^T,\ A_0 = A_0^T.$$

Moreover, if $A_2, A_0 > 0$, $P(\lambda)$ is **gyroscopic**.

Solving Structured PEPs

- Use a **structured linearization** $\lambda\mathcal{A} + \mathcal{B}$.
- $\star$-congruence transformations preserve typical structures,

$$\lambda\mathcal{A} + \mathcal{B} \mapsto Z^\star(\lambda\mathcal{A} + \mathcal{B})Z, \quad \det(Z) \neq 0.$$

- Look for **condensed forms** under $\star$-congruences (in particular look for unitary Z) that reveal the e'vals.
- **Derive structure preserving algorithms** to compute the condensed forms.

Vector Space $\mathbb{L}_1(P)$, $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$

Mackey, Mackey, Mehl & Mehrmann (2006) introduce

$$\mathbb{L}_1(P) = \left\{ L(\lambda) : L(\lambda) \begin{bmatrix} \lambda^{k-1} I \\ \vdots \\ \lambda I \\ I \end{bmatrix} = \begin{bmatrix} v_1 P(\lambda) \\ v_2 P(\lambda) \\ \vdots \\ v_k P(\lambda) \end{bmatrix}, \; v \in \mathbb{C}^k \right\}.$$

- Almost all $L(\lambda) = \lambda X + Y$ in $\mathbb{L}_1$ are **linearizations**.
- E'vecs of P easily recovered from e'vecs of $L \in \mathbb{L}_1$.
  [M^4, 2006], [Higham, Li, T., 2007]
- $\mathbb{L}_1(P)$ is a **rich source of structured linearizations**.

Structured Linearizations

$$\mathbb{L}_1(P) = \Big\{ L(\lambda) : L(\lambda)(\Lambda \otimes I) = v \otimes P(\lambda),\ v \in \mathbb{C}^k \Big\}.$$

- For some v satisfying
  - $v \in \mathbb{R}^k$ for Hermitian/symm. structures,
  - $v = \Sigma v$ for T-even, T-odd structures,
  - $v = Rv$ for T-(anti)palindromic structures,

  there is a unique structured $L(\lambda) \in \mathbb{L}_1(P)$, where

  $$\Sigma = \mathrm{diag}((-1)^{k-1}, \dots, (-1)^0),\ R = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}.$$

- $L(\lambda)$ is a linearization of P if no roots of $\mathrm{p}(x; v) = v_k t^k + \cdots v_1 t + v_0 t$ is an e'val of $P(\lambda)$.

- Procedures to construct structured $L(\lambda) \in \mathbb{L}_1(P)$.

Example 3: Symmetric Linearizations

For symmetric quadratics $Q(\lambda) = \lambda^2 M + \lambda D + K$,

$$L(\lambda) = v_1 L_1(\lambda) + v_2 L_2(\lambda), \quad v_1, v_2 \in \mathbb{R}$$

where

$$L_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} + \begin{bmatrix} D & K \\ K & 0 \end{bmatrix}, \; L_2(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix},$$

is a symmetric linearization in $\mathbb{L}_1(Q)$ if $-v_2/v_1 \notin \Lambda(Q)$.

- Construction extends to arbitrary degree k.
- For hyperbolic (overdamped) $P(\lambda)$, there is $v \in \mathbb{R}^k$ s.t. $L(\lambda) = \lambda X + Y$ with vector v is definite ($X > 0$).

Eigensolvers for Structured Pencils

- Symm./Hermitian pencils: Jacobi-like algorithm (computes antitriang. form of Hermitian pencil) [Mehl, 04].
- T-even/T-palindromic pencils (antitriang. Schur form).
  - PEPACK [Poppe, Schröder, Thies, 09] based on generalized Laub trick.
  - Jacobi type method for T-palindromic pencils [Mackey, Mackey, Mehl, Mehrmann, 09].
- New Krylov methods, doubling algorithms, ... [Chu, Huang, Lin, Mehrmann, Simoncini, Schröder, Watkins, ...]

Non-polynomial Nonlinear Eigenproblems

$F(\lambda)x = 0$ with analytic $F\colon \Omega \to \mathbb{C}^{n\times n}$, $\Omega \subseteq \mathbb{C}$.

Some applications:

- PDE models with λ-dependent boundary conditions or material conditions.
- Use of special basis functions in the discretization.
- Characteristic function of delay differential equations.

See [Mehrmann & Voss, Nonlinear eigenvalue problems: A challenge for modern eigenvalue methods, GAMM-Mitteilungen, 04] for an overview of applications and numerical methods.

Rational Eigenvalue Problems (REPs)

- An **emerging class of nonlinear eigenproblems**.
- Arise in a variety of physical applications, e.g.,
  - acoustic emissions of high speed trains,
  - calculations of quantum dots, ...
- Matrix function takes the form

$$R(\lambda) = P(\lambda) + \sum_{i=1}^{m} \frac{s_i(\lambda)}{q_i(\lambda)} E_i,$$

  - $P(\lambda)$ is a matrix polynomial,
  - s_i and q_i are scalar polynomials,
  - E_i are constant matrices.

In practice, m is small and rank(E_i) is low.

Numerical Methods

$$R(\lambda) = P(\lambda) + \sum_{i=1}^{m} \frac{s_i(\lambda)}{q_i(\lambda)} E_i.$$

- Brute-force approach: turn the REP into a PEP.
  - Employed by practitioners.
  - Not practical when $q_i(\lambda)$ has several poles.
- Treat the REP as a general nonlinear eigenproblem.
  - Limits the exploitation of the underlying structure and properties of the REP.
- Linearizations for rational problems [Bai & Su, 11].

Linearizations for Rational Problems

- Rewrite $R(\lambda) = P(\lambda) + \sum_{i=1}^{m} \frac{s_i(\lambda)}{q_i(\lambda)} E_i$ as

$$R(\lambda) = P(\lambda) + U(C - \lambda D)^{-1} V^*, \qquad (*)$$

where U, V are $n \times r$. In practice $r \ll n \sum_{i=1}^{m} deg(q_i)$.

- Convert $(*)$ into linear pencil of dim. $nk + r$.
(See Y. Su's talk, MS475 at 3:30pm)

Example: if $P(\lambda) = \lambda A + B$, $R(\lambda)x = 0$ becomes a **linear eigenproblem**

$$\left(\lambda \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} B & U \\ V^* & -C \end{bmatrix}\right) \begin{bmatrix} x \\ (C - \lambda D)^{-1} V^* x \end{bmatrix} = 0.$$

Methods for Nonlinear Eigenproblems

- Typically interested in only a few e'vals closest to a target point, a line or small region of complex plane.
- Numerical methods include
  - Newton methods, inverse iteration,. . . [Ruhe, 73].
  - Nonlinear Rayleigh-Ritz iterative procedure, e.g., nonlinear Arnoldi [Voss, 04], NRRIT, [Liao, Bai, Lee, Ko, 06], rational Krylov [Jarlebring, Voss, 05], Jacobi-Davidson [Voss, 04].

  (see MS 342, part I and part II)
- All methods require a good choice of initial eigenpairs.

Summary and Concluding Remarks

- Better understanding of linearization process.
- New (structure preserving) linearizations derived along with algorithms preserving spectral properties in finite precision arithmetic. Appropriate software is needed.
- More research needed for singular PEPs.
- More research needed on the development of methods for PEPs that work on $P(\lambda)$ and not its linearization.
- Need to improve current nonlinear eigensolvers, analyze their stability/convergence, generate implementations that can be used by non-experts.

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