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Structured Matrix Polynomials and their Sign Characteristic

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ILAS Conference, Providence, June 3-7, 2013.

Structured Matrices Associated with $\langle \cdot, \cdot \rangle_{\mathrm{M}}$

Consider scalar product on $\mathbb{C}^n$, $\langle x, y \rangle_{\mathrm{M}} = x^* M y$, $\det(M) \neq 0$.

Adjoint $A^{\star_M}$ of $A \in \mathbb{C}^{n \times n}$ wrt $\langle \cdot, \cdot \rangle_{\mathrm{M}}$: $A^{\star_M} = M^{-1} A^* M$.

Three important classes of matrices associated with $\langle \cdot, \cdot \rangle_{\mathrm{M}}$:

- **Jordan algebra**: $\mathbb{J}_M = \{A \in \mathbb{C}^{n \times n} : A^{\star_M} = A\}$,
- **Lie algebra**: $\mathbb{L}_M = \{A \in \mathbb{C}^{n \times n} : A^{\star_M} = -A\}$,
- **Automorphism group**: $\mathbb{G}_M = \{A \in \mathbb{C}^{n \times n} : A^{\star_M} = A^{-1}\}$.

E'vals of $A \in \mathbb{J}_M$, $\mathbb{L}_M$ or $\mathbb{G}_M$ occur in pairs $(\lambda, f(\lambda))$, where

$$f(\lambda) = \begin{cases} \overline{\lambda} & \text{for } A \in \mathbb{J}_M \\ -\overline{\lambda} & \text{for } A \in \mathbb{L}_M, \\ 1/\overline{\lambda} & A \in \mathbb{G}_M. \end{cases}$$

Sign Characteristic

When $\langle \cdot, \cdot \rangle_M$ is **orthosymmetric**, i.e., $M^* = \beta M$, $|\beta| = 1$, e'vals for which $\lambda = f(\lambda)$, i.e.,

- real e'vals when $A \in \mathbb{J}_M$,
- purely imaginary e'vals if $A \in \mathbb{L}_M$,
- unit modulus e'vals when $A \in \mathbb{G}_M$,

have a sign $+1$ or -1 attached to them forming the **sign characteristic of** (A, M). (See Leiba Rodman's talk)

If λ is simple with e'vec x then its sign characteristic $\varepsilon(\lambda)$ is

$$\varepsilon(\lambda) = \text{sign}(\bar{\beta} x^* M x).$$

M-selfadjoint A, $M = M^*$: Canonical Form

There exists nonsingular X s.t. $J = X^{-1}AX$, $S = X^*MX$, where

$$J = \bigoplus_{j=1}^{r} \underbrace{J_{\ell_j}(\lambda_j)}_{\text{real}} \oplus \bigoplus_{j=1}^{s} \underbrace{(J_{m_j}(\bar{\mu}_j) \oplus J_{m_j}(\mu_j))}_{\text{nonreal}},$$

$$S = \bigoplus_{j=1}^{r} \varepsilon_j S_{\ell_j} \oplus \bigoplus_{j=1}^{s} S_{2m_j}.$$

S_j: sip matrices with sizes consistent with that of $J_j(\lambda_j)$.

$\varepsilon = \{\varepsilon_1, \cdots, \varepsilon_r\}$: ordered set of signs ± 1 forming the **sign characteristic** of the pair (A, M).

Definite Pencils

Let $L(\lambda) = K - \lambda M$ be Hermitian with $\det(M) \neq 0$.

Then $(M^{-1}K)^{\star_M} = M^{-1}(M^{-1}K)^* M = M^{-1}K$ so $M^{-1}K \in \mathbb{J}_M$.

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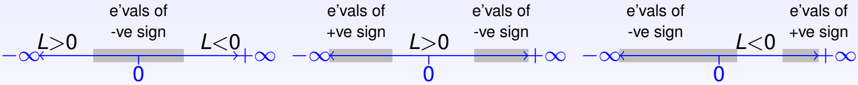
$L(\lambda)$ is **definite** if it satisfies any one of (a), (b), (c).

(a) $(x^*Kx, x^*Mx) \neq 0$ for all nonzero $x \in \mathbb{C}^n$.

(b) $K - \mu M > 0$ for some $\mu \in \mathbb{R} \cup \{\infty\}$.

(c) The e'vals lie in $\mathbb{R}$ and the e'vals with $+$ve sign characteristic are separated from the e'vals of $-$ve sign characteristic.



K indefinite, $M > 0$ | $K > 0$, M indefinite | K, M both indefinite

How to Detect Definiteness?

Let $K - \lambda M$ be Hermitian. Is it definite?

- Can use
  - J-orthogonal Jacobi algorithm [Veselic, 93],
  - Level set algorithm [Higham,Van Dooren & T, 02],
  - improved arc algorithm [Higham, Guo & T, 09],
- For a quick check (if all e'vals are real):

```
[X,D] = eig(K,M);
[e,ind] = sort(diag(D)); X = X(:,ind);
for j = 1:length(K)
  x = X(:,j); s(j) = -sign(x'*M*x);
end
```

If M is 5×5 then

`s = 1 1 1 -1 -1` implies $K - \lambda M$ is definite.

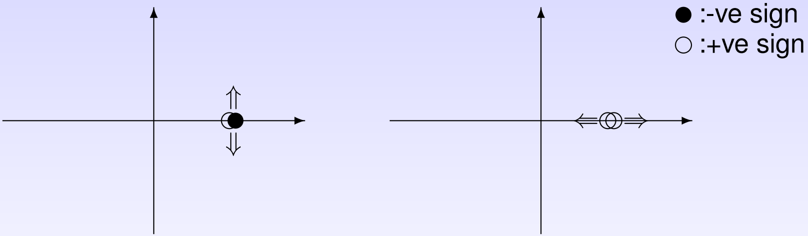
`s = 1 -1 1 -1 -1` implies $K - \lambda M$ is not definite.

Structured Eigenvalue Perturbation Theory

Let $A \in \mathbb{J}_M$ with $M = M^*$, $\det(M) \neq 0 \;\Rightarrow MA = (MA)^*$.

Suppose A is perturbed into $\widetilde{A} = A + E \in \mathbb{J}_{\widetilde{M}}$ with $\widetilde{M}$ Hermitian, i.e., E **preserves the structure**.

- If two real e'vals collide they can become nonreal if they have opposite sign characteristic.



- Extend to other structured matrices, e.g., Hamiltonians and purely imaginary e'vals.

Passivity of Linear Systems

To check the passivity of a control system

$$\begin{aligned} \dot{x} &= Ax + Bu, \;\; x(0) = x_0, \\ y &= Cx + Du \end{aligned} \qquad (*)$$

we can check whether the **Hamiltonian** matrix

$$H = \begin{bmatrix} A - BX^{-1} & -BX^{-1}B^* \\ C^*XC & -(A - BX^{-1}C)^* \end{bmatrix}, \qquad X = D + D^*$$

has **no purely imaginary eigenvalue**.

H is Hamiltonian if $JH = (JH)^*$, i.e., $H \in \mathbb{L}_J$, $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

If $(*)$ not passive, perturb A, B, C, D to move e'vals from imaginary axes. **Sign characteristic** plays an important role [Merhmann & Xu, 08].

Structured Matrix Polynomials

$P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$, $A_j \in \mathbb{C}^{n\times n}$, $\det(A_m) \neq 0$.

Structure	**Definition**	**Coeffs property**
Hermitian	$P(\lambda) = P^*(\lambda)$	$A_j = A_j^*$
skew-Hermitian	$P(\lambda) = -P^*(\lambda)$	$A_j = -A_j^*$
$*$-even	$P(\lambda) = P^*(-\lambda)$	$A_j = (-1)^j A_j^*$
$*$-odd	$P(\lambda) = -P^*(-\lambda)$	$A_j = (-1)^{j+1} A_j^*$
$*$-palindromic	$P(\lambda) = \lambda^m P^*(\frac{1}{\lambda})$	$A_j = A_{m-j}^*$
$*$-antipalindromic	$P(\lambda) = -\lambda^m P^*(\frac{1}{\lambda})$	$A_j = -A_{m-j}^*$

Structured Quadratics From NLEVP

$n \times n$ quadratic $Q(\lambda) = \lambda^2 M + \lambda D + K$.

speaker box (`pep, qep, real, symmetric`).
$n = 107$. Finite element model of a speaker box.
$\|M\|_2 = 1$, $\|D\|_2 = 5.7 \times 10^{-2}$, $\|K\|_2 = 1.0 \times 10^7$.

wiresaw1 (`pep, qep, t-even, ..., scalable`).
Gyroscopic QEP from vibration analysis of a wiresaw.
$M = M^T$, $D = -D^T$, $K = K^T$.

railtrack (`pep, qep, t-palindromic, sparse`).
$n = 1005$. Model of vibration of rail tracks under the excitation of high speed trains. $M = K^T$, $D = D^T$.

gen_tantipal2 (`pep, qep, real, t-antipalindromic, ..., random`). T-anti-palindromic QEP with eigenvalues on the unit circle.

Standard and Jordan Triples

Standard and Jordan triples $(U, \mathcal{T}, V)$ for

$$P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j, \quad A_j \in \mathbb{C}^{n \times n}, \;\; \det(A_m) \neq 0.$$

- Introduced and developed by **Gohberg**, **Lancaster** and **Rodman** (early 80's).
- Play a central role in the theory of matrix polynomials.
- Extend notion of Jordan pair (X, J) for $A \in \mathbb{C}^{n \times n}$.

Standard Triples

If $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ then $(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1})$ with

$$\mathcal{C} = -\begin{bmatrix} A_m^{-1}A_{m-1} & A_m^{-1}A_{m-2} & \dots & A_m^{-1}A_0 \\ -I_n & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}$$

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$(U, \mathcal{T}, V)$ is a standard triple for $P(\lambda)$ if it is **similar** to the primitive standard triple, i.e., $\exists$ nonsingular Q s.t.

$$U = (e_m^T \otimes I_n)Q, \quad \mathcal{T} = Q^{-1}\mathcal{C}Q, \quad V = Q^{-1}(e_1 \otimes A_m^{-1}).$$

Jordan triple: standard triple for which $\mathcal{T}$ is in Jordan form.

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Example: (I, A, I) and (X, J, X^{-1}) are standard triples for $\lambda I - A$, where $A = XJX^{-1} \in \mathbb{C}^{n\times n}$.

Structured Standard Triples

The "T-matrix" of any standard triple (U, T, V) for a structured $P(\lambda)$ is structured:

- $T \in \mathbb{J}_S$ for (skew)-Hermitian $P(\lambda)$,
- $T \in \mathbb{L}_S$ for $*$-even/$*$-odd $P(\lambda)$,
- $T \in \mathbb{G}_S$ for $*$-(anti)palindromic $P(\lambda)$.

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Issues:

- Is $\langle \cdot, \cdot \rangle_S$ orthosymmetric (i.e., $S^* = \beta S$, $|\beta| = 1$)?
- Is S uniquely defined? Answer: **No!**

Eigenvalue Functions

[Gohberg, Lancaster, Rodman '05 & 09]

Let

- $P(\lambda)$ be Hermitian and $n \times n$,
- $\xi_1\big(P(\lambda)\big), \ldots, \xi_n\big(P(\lambda)\big)$ be real e'vals of $P(\lambda), \lambda \in \mathbb{R}$,
- $\lambda_1, \ldots, \lambda_r$ be the r distinct real e'vals of $P(\lambda)$,

and write $\xi_k\big(P(\lambda)\big) = (\lambda - \lambda_j)^{\ell_{jk}} \nu_{jk}(\lambda)$, $\nu_{jk}(\lambda_j) \neq 0$.

Then,

- $\ell_{jk} \neq 0$ is a partial multiplicity of λ_j,
- $\text{sign}\big(\nu_{jk}(\lambda_j)\big)$ is the **sign characteristic** attached to ℓ_{jk}

If λ_j is simple then $\text{sign}\big(\nu_{jk}(\lambda_j)\big) = \text{sign}\left(\frac{\partial \xi_i\big(P(\lambda)\big)}{\partial \lambda}\right)\Big|_{\lambda=\lambda_j}$.

Eigenvalue Functions (Cont.)

For Hermitian A, E and $Ax = \xi_j(A)x$ with $\xi_j(A)$ simple,

$$\xi_j(A+E) = \xi_j(A) + \frac{x^*Ex}{x^*x} + O(\|E\|^2).$$

Also, $P(\lambda_0 + \Delta\lambda) = P(\lambda_0) + P'(\lambda_0)\Delta\lambda + O(\Delta\lambda^2)$.

Since $P(\lambda_0)$ is Hermitian for $\lambda_0 \in \mathbb{R}$, we have

$$\xi_i(P(\lambda_0 + \Delta\lambda)) = \xi_i(P(\lambda_0)) + \frac{x^*P'(\lambda_0)x}{x^*x}\Delta\lambda + O(\Delta\lambda^2).$$

Hence $\text{sign}\left(\frac{\partial\xi_i\left(P(\lambda)\right)}{\partial\lambda}\right)\Big|_{\lambda=\lambda_0} = \text{sign}\left(x^*P'(\lambda_0)x\right)$.

λ_0 is of positive type if $x^*P'(\lambda_0)x > 0$ and negative type if $x^*P'(\lambda_0)x < 0$.

Sign Characteristic of Hermitian $P(\lambda)$

If $\lambda_0 \in \mathbb{R}$ is a simple e'val of $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ with e'vec x then

- $z = [\lambda_0^{m-1} \ \ldots \ \lambda_0 \ 1]^T \otimes x$ is a right e'vec of companion form $\mathcal{C}$ of P with e'val λ_0,
- $x^* P'(\lambda_0) x = z^* S z$, where

$$S = \begin{bmatrix} 0 & \cdots & 0 & A_m \\ \vdots & & \iddots & \vdots \\ \vdots & \iddots & \iddots & A_2 \\ A_m & \cdots & A_2 & A_1 \end{bmatrix}.$$

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- $\mathcal{C} \in \mathbb{J}_S$, i.e., $\mathcal{C}$ is S-selfadjoint.

The sign characteristic of Hermitian $P(\lambda)$ is that of the pair $(\mathcal{C}, S)$. [GLR 05]

Other Structured Matrix Polynomials

For a given structured $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$, associate an Hermitian matrix function:

- If $P(\lambda)$ is skew-Herm then $iP(\lambda)$ is Hermitian for $\lambda \in \mathbb{R}$.
- If $P(\lambda)$ is $*$-even then $P(i\lambda)$ is Hermitian for $\lambda \in \mathbb{R}$.
- If $P(\lambda)$ is $*$-odd then $iP(i\lambda)$ is Hermitian for $\lambda \in \mathbb{R}$.
- If $P(\lambda)$ is $*$-palindromic then $e^{-\frac{i\theta m}{2}} P(e^{i\theta})$ is Hermitian for $\theta \in \mathbb{R}$.

Other Structured Matrix Polynomials (cont.)

Let λ_0 be a simple e'val of P with e'vec x and let $z = [\lambda_0^{m-1} \ \ldots \ \lambda_0 \ 1]^T \otimes x$.

- For **skew-Herm.** P, define sign characteristic of $\lambda_0 \in \mathbb{R}$ as $\varepsilon(\lambda_0) = \mathrm{sign}(\frac{\partial \xi_j(iP(\lambda))}{\partial \lambda})\big|_{\lambda=\lambda_0}$. Can show that

$$\varepsilon(\lambda_0) = \mathrm{sign}(ix^* P'(\lambda_0)x) = \mathrm{sign}(z^* S z)$$

for some Hermitian S s.t. $\mathcal{C} \in \mathbb{J}_S$.

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- For $*$**-palindromic** P, define sign characteristic of $\lambda_0 = e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ as $\varepsilon(\lambda_0) = \text{sign}(\frac{\partial \xi_j(H(e^{i\theta}))}{\partial \theta})\big|_{\theta=\theta_0}$, where $H(\lambda) = \lambda^{-m/2} P(\lambda)$. Can show that

$$\varepsilon(\lambda_0) = \text{sign}(x^* \frac{\partial H(\theta)}{\partial \theta} x) = \text{sign}(z^* S z)$$

for some Hermitian S s.t. $\mathcal{C} \in \mathbb{G}_S$.

Solvability of a palindromic matrix equation

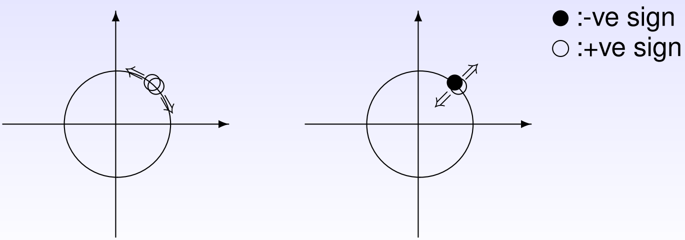
[Brüll, Poloni, Sbrana & Schröder '13]

Given A and $B > 0$ find $X > 0$ s.t. $\rho(X^{-1}A^T) < 1$ and

$$X + A^T X^{-1} A = B.$$

There exists a solution if $Q(\lambda) = \lambda^2 A + \lambda B + A^T$ has no e'val on the unit circle and $\frac{1}{\lambda}Q(\lambda) > 0$ for one such λ.

If $Q(\lambda)$ has unit modulus e'vals, perturb A, B such that new problem is solvable.



Signature constraint for Hermitian $P(\lambda)$

Let $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ with $\det(A_m) \neq 0$ be Hermitian.

If $P(\lambda)$ has r real elementary divisors $(\lambda - \lambda_j)^{\ell_j}$, $j = 1: r$ with corresponding sign characteristic ε_j, $j = 1: r$ then

$$\sum_{j=1}^{r} \frac{1}{2}(1 - (-1)^{\ell_j})\varepsilon_j = \begin{cases} 0 & \text{if } m \text{ is even} \\ \operatorname{sig}(A_m) & \text{if } m \text{ is odd,} \end{cases} \qquad (1)$$

where $\operatorname{sig}(A_m)$ is the signature of A_m.

- (1) is the **signature constraint** for $P(\lambda)$.
- Signature constraint important when solving the **inverse Hermitian polynomial e'val problem**.

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Do we have a signature constraint for other structures?

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