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Detecting a definite Hermitian pair and a hyperbolic or elliptic quadratic eigenvalue problem, and associated nearness problems

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Abstract

An important class of generalized eigenvalue problems $Ax = \lambda Bx$ is those in which A and B are Hermitian and some real linear combination of them is definite. For the quadratic eigenvalue problem (QEP) $(\lambda^2 A + \lambda B + C)x = 0$ with Hermitian A, B and C and positive definite A, particular interest focuses on problems in which $(x^*Bx)^2 - 4(x^*Ax)(x^*Cx)$ is one-signed for all non-zero x—for the positive sign these problems are called hyperbolic and for the negative sign elliptic. The important class of overdamped problems arising in mechanics is a sub-class of the hyperbolic problems. For each of these classes of generalized and quadratic eigenvalue problems we show how to check that a putative member has the required properties and we derive the distance to the nearest problem outside the class. For definite pairs (A, B) the distance is the Crawford number, and we derive bisection and level set algorithms

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both for testing its positivity and for computing it. Testing hyperbolicity of a QEP is shown to reduce to testing a related pair for definiteness. The distance to the nearest non-hyperbolic or non-elliptic $n \times n$ QEP is shown to be the solution of a global minimization problem with $n - 1$ degrees of freedom. Numerical results are given to illustrate the theory and algorithms.

AMS classification: 65F15; 15A22

Keywords: Hermitian generalized eigenvalue problem; Definite pair; Crawford number; Hermitian quadratic eigenvalue problem; Hyperbolic system; Elliptic system; Overdamped system; Gyroscopic system; Bisection algorithm; Level set algorithm; Nearness problems

1. Introduction

Eigenvalue problems $Ax = \lambda x$ with Hermitian A have many desirable properties and are amenable to a variety of special algorithms. Here we consider what can be regarded as the closest analogues of this class of problems for the generalized eigenvalue problem and the quadratic eigenvalue problem (QEP): definite generalized eigenvalue problems and hyperbolic QEPs. A property in common to all these problems is that the eigenvalues are real. We also consider elliptic QEPs, which have eigenvalues that are all non-real.

We have two aims: to determine, for a given generalized or quadratic eigenvalue problem, whether the property of interest holds, and, if it does, to compute the distance to the nearest problem without that property. The first problem is non-trivial because definiteness, hyperbolicity and ellipticity are all defined by optimization problems over the unit ball. The second problem is analogous to stability radii computations in control theory, whereby it is determined by how much a problem can be perturbed without it losing its defining property.

The definite generalized eigenvalue problem is treated in Section 2. We show that the distance from a definite problem to the nearest non-definite one is given by the Crawford number—the number whose positivity is used to define definiteness. We show how to compute the Crawford number using a bisection algorithm. Although each iteration of this algorithm is expensive, only a few iterations may be required to detect definiteness and an interval guaranteeing to contain the Crawford number is produced. We also derive a more efficient and rapidly converging level set algorithm that, in particular, tests for definiteness at the cost of solving just one quadratic eigenvalue problem.

In Section 3 we give definitions and characterizations of hyperbolic QEPs (including the subclass of overdamped QEPs) and elliptic QEPs. We show that testing for hyperbolicity can be reduced to testing for definiteness of an associated generalized eigenvalue problem. We also show that the distances to the nearest non-hyperbolic or non-elliptic QEP can both be expressed in terms of a global minimization problem over the unit ball.

Numerical examples are given in both sections to illustrate the theory and algorithms.

2. Definite pair

Two Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$ form a definite pair if

$$\gamma(A, B) := \min_{\substack{z\in\mathbb{C}^n \\ \|z\|_2=1}} \sqrt{(z^*Az)^2 + (z^*Bz)^2} > 0. \tag{2.1}$$

The quantity $\gamma(A, B)$ is known as the Crawford number. A sufficient condition for definiteness is that one of A and B is definite, but it is the definiteness of a suitable linear combination of A and B that characterizes definiteness of the pair, as shown by the following result [19], [20, Theorem 6.1.18]. We denote by $\lambda_{\min}$ and $\lambda_{\max}$ the smallest and largest eigenvalues, respectively, of a Hermitian matrix.

Theorem 2.1. *Let (A, B) be a definite Hermitian pair, and for $\theta \in \mathbb{R}$ let*

$$\begin{aligned} A_\theta &= A\cos\theta + B\sin\theta, \\ B_\theta &= -A\sin\theta + B\cos\theta. \end{aligned} \tag{2.2}$$

Then there is a $\theta \in [0, 2\pi)$ such that B_θ is positive definite and

$$\gamma(A, B) = \lambda_{\min}(B_\theta).$$

Definite pairs have the desirable properties that they are simultaneously diagonalizable and, in the associated eigenproblem, $Ax = \lambda Bx$, the eigenvalues are real. The practical importance of definite pairs stems from their appearance in physical problems, particularly vibration problems in engineering; in practice the matrix B is usually positive definite.

An alternative interpretation of γ in (2.1) is

$$\gamma(A, B) = \min\{|w| : w \in F(A + \mathrm{i}B)\}, \tag{2.3}$$

where the field of values of a general $G \in \mathbb{C}^{n\times n}$ is defined by

$$F(G) = \{z^*Gz : z \in \mathbb{C}^n, \|z\|_2 = 1\}.$$

Thus (A, B) is a definite pair if and only if $F(A + \mathrm{i}B)$ does not contain the origin, and $\gamma(A, B)$ is the distance from the origin to the nearest point in $F(A + \mathrm{i}B)$.

Two questions are of interest: "How can we test whether a given Hermitian pair (A, B) is definite?", and "How far is a definite pair from being non-definite?" We consider the second question first, since the answers to the two questions are closely related.

We define the distance from a Hermitian pair (A, B) to the nearest non-definite pair by

$$d(A, B) = \min\left\{ \|[\Delta A \quad \Delta B]\|_2 : z^*(A + \Delta A + \mathrm{i}(B + \Delta B))z = 0, \text{ some } z \neq 0 \right\}. \tag{2.4}$$

Here, and throughout, perturbations of Hermitian matrices are assumed to be Hermitian. A feasible perturbation in the definition of d is ΔA and ΔB such that $[A + \Delta A, B + \Delta B]$ has rank less than n, and the minimum of $\|[\Delta A, \Delta B]\|_2$ over all such perturbations is

$$\sigma_n\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) \geqslant d(A, B), \tag{2.5}$$

where σ_n denotes the nth largest singular value. However, this upper bound can be arbitrarily weak: for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \mu A$$

we have $\gamma(A, B) = 0$ but the upper bound in (2.5) is $\sqrt{1 + \mu^2}$.

In [3] a computable formula is obtained for the 2-norm distance from an arbitrary Hermitian pair to the nearest definite pair with a given value of γ. Making use of the ideas from [3] we can readily solve the converse problem (2.4).

Theorem 2.2. *For Hermitian $A, B \in \mathbb{C}^{n\times n}$, $d(A, B) = \gamma(A, B)$.*

Proof. We assume that (A, B) is a definite pair, otherwise the result is trivial. Suppose $z^*(A + \Delta A + \mathrm{i}(B + \Delta B))z = 0$ for some z with $\|z\|_2 = 1$. Then

$$\begin{aligned}
\gamma(A, B) &\leqslant |z^*(A + \mathrm{i}B)z| \\
&= |z^*(\Delta A + \mathrm{i}\Delta B)z| \\
&= \|[z^*\Delta A z \quad z^*\Delta B z]\|_2 \\
&= \max_\theta \left\| \begin{bmatrix} z^*\Delta A z & z^*\Delta B z \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\|_2 \\
&= \max_\theta \left\| z^*[\Delta A \quad \Delta B] \begin{bmatrix} z\cos\theta \\ z\sin\theta \end{bmatrix} \right\|_2 \\
&\leqslant \|[\Delta A \quad \Delta B]\|_2.
\end{aligned} \tag{2.6}$$

Thus $d(A, B) \geqslant \gamma(A, B)$. Suppose, first, that the nearest point to the origin in the convex set $F(A + \mathrm{i}B)$ is $\gamma(A, B)$ on the real axis. Then a perturbation to A of $-\gamma(A, B)I$ shifts $F(A + \mathrm{i}B)$ left to touch the origin. This perturbation gives equality throughout (2.6) and so is optimal. In the general case we can rotate the field of values, as in (2.7) below, to reduce to the previous case. □

To summarize, computing $d(A, B)$ is equivalent to computing $\gamma(A, B)$, whereas testing whether (A, B) is a definite pair requires only determining whether $\gamma(A, B) > 0$.

2.1. Existing methods for testing definiteness

Crawford and Moon [4,5] present a bisection-like algorithm for computing a θ such that B_θ in (2.2) is positive definite or determining that such a θ does not exist. At each step the algorithm performs a Cholesky factorization to test the definiteness of B_θ for the current estimate of θ. The algorithm takes a finite number of steps that in the worst case is $O(n)$, and therefore $O(n^4)$ flops can be required. This algorithm does not compute $\gamma(A, B)$.

Another way to test definiteness is to apply the J-orthogonal Jacobi algorithm of Veselić [24], which breaks down when applied to an indefinite pair. However, this algorithm uses hyperbolic transformations so is potentially unstable.

Theorem 2.1 shows that the problem of checking definiteness can be expressed as that of checking feasibility (for some real α and β) of the linear matrix inequality (LMI)

$$\alpha A + \beta B > 0,$$

where the inequality denotes positive definiteness of the left-hand side. Various interior point methods are available for the solution of convex optimization problems with LMI constraints [23]. However, none of them is likely to be as efficient for this problem as Algorithm 2.4.

Finally, we note that an algorithm given by Doyle [6, Appendix] and Fan [9, Algorithm 2.2] for finding the vector of minimum 2-norm in the convex hull of a set of vectors can be used to compute $\gamma(A, B)$, in view of (2.3). However, this algorithm appears to be only linearly convergent and does not bracket $\gamma(A, B)$, so it is less attractive than Algorithms 2.3 and 2.4.

2.2. Bisection and level set methods

We now examine how to evaluate $\gamma(A, B)$ for an arbitrary Hermitian pair (A, B). Write $C = A + \mathrm{i}B$ and note that

$$F(\mathrm{e}^{-\mathrm{i}\theta}C) = \mathrm{e}^{-\mathrm{i}\theta}F(C), \tag{2.7}$$

that is, the field of values of $\mathrm{e}^{-\mathrm{i}\theta}C$ is that of C rotated clockwise through θ radians about the origin. Moreover, it is easy to see that each point $z \in F(\mathrm{e}^{-\mathrm{i}\theta}C)$ satisfies

$$\tfrac{1}{2}\lambda_{\min}(\mathrm{e}^{-\mathrm{i}\theta}C + \mathrm{e}^{\mathrm{i}\theta}C^*) \leqslant \mathrm{Re}(z) \leqslant \tfrac{1}{2}\lambda_{\max}(\mathrm{e}^{-\mathrm{i}\theta}C + \mathrm{e}^{\mathrm{i}\theta}C^*).$$

This leads to the formulae (essentially [3, Theorem 2.1]), for a Hermitian pair (A, B),

$$\begin{aligned}\gamma(A, B) &= -\min\left(\min_{0\leqslant\theta\leqslant 2\pi} \lambda_{\max}(A\cos\theta + B\sin\theta), 0\right)\\ &= \max\left(\max_{0\leqslant\theta\leqslant 2\pi} \lambda_{\min}(A\cos\theta + B\sin\theta), 0\right),\end{aligned} \tag{2.8}$$

where the outer min and max account for the case where $0 \in F(C)$, so that (A, B) is not a definite pair. Computing $\gamma(A, B)$ is therefore a one-dimensional global optimization problem. Optimization techniques that are based on function minimization or on a simple grid search (as used in [3]) produce at best a local minimum. By taking advantage of the structure of the problem we will derive methods that guarantee to capture the global minimum. It was pointed out by a referee that (2.8) can also be reformulated as a convex optimization problem involving matrices of order n, but this alternative has not yet been compared with the methods proposed in this paper.

Rewriting (2.8), our aim is to compute

$$\omega(A, B) = \max\{f(z) : |z| = 1\},$$

where

$$f(z) = \lambda_{\min}(M(z)), \quad M(z) = (z^{-1}C + zC^*)/2,$$

and where $M(z)$ is Hermitian for z on the unit circle. For such z the function f is a continuous function of z and so it achieves every value between its minimum and its maximum. We can assume without loss of generality that C is non-singular, as otherwise $0 \in F(A + \mathrm{i}B)$ and (A, B) is not a definite pair.

For a given ξ we ask whether f achieves the level ξ, that is, whether ξ is the smallest eigenvalue of $M(z)$ for some z on the unit circle. For any non-zero z the following equivalence is clear:

$$\det(M(z) - \xi I) = 0 \iff \det(Q(z)) := \det(C - 2\xi z I + z^2 C^*) = 0. \quad (2.9)$$

The quadratic Q has $2n$ eigenvalues z_j. We compute the z_j, and for each one on the unit circle (if there are any) we compute $\lambda_{\min}(M(z_j))$. If $\lambda_{\min}(M(z_j)) = \xi$ for some j then we know that $f(z) = \xi$ is achieved, and otherwise it is not. This provides the basis for a bisection algorithm for maximizing f, analogous to that of Byers [2] for computing the distance to the nearest unstable matrix. On each iteration of the following algorithm $f(z) = a \leqslant \omega(A, B) \leqslant b$, for some z on the unit circle.

Algorithm 2.3. Given Hermitian A and B with $A + \mathrm{i}B$ non-singular, a bracket $[a, b]$ for $\omega(A, B)$ with $a = f(z)$ for some $|z| = 1$, and a tolerance tol, this algorithm refines the bracket to an interval of width at most tol containing $\gamma(A, B)$.

```
while b − a > tol
    ξ = (a + b)/2
    Compute the eigenvalues z_j of Q(z).
    If λ_min(M(z_j)) = ξ for some eigenvalue z_j of Q on the unit circle
        a = ξ
    else
        b = ξ
        if b ⩽ 0, return with a = b = 0, end    % γ(A, B) = 0
    end
end
```

Note that instead of bisecting using the arithmetic mean of the interval $[a, b]$, when $a > 0$ we could use the geometric mean $\sqrt{ab}$ (suitably modified when a is very small), which is preferable if $|\omega(A, B)|$ is much smaller than the initial value of b.

Algorithm 2.3 needs an initial interval $[a, b]$ that brackets $\omega(A, B)$, with $a = f(z)$ for some $|z| = 1$. From (2.5) and Theorem 2.2 we can take

$$b = \sigma_n\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) \tag{2.10}$$

and

$$a = \begin{cases} \max(a_0, 0) & \text{if } (A, B) \text{ is known to be definite,} \\ a_0 & \text{otherwise,} \end{cases} \tag{2.11}$$

where (by sampling at $z = \pm 1, \pm \mathrm{i}$),

$$a_0 := \max\left\{\lambda_{\min}(A), -\lambda_{\max}(A), \lambda_{\min}(B), -\lambda_{\max}(B)\right\}.$$

For the tests $|z_j| = 1$ and $\lambda_{\min}(M(z_j)) = \xi$ a tolerance is needed of size less than tol but greater than the error in the computed eigenvalue (which is roughly the condition number of the eigenvalue multiplied by the machine precision, if a stable solver is used).

The algorithm terminates as soon as it generates a negative b, since that guarantees $\omega(A, B) < 0$ and hence that $\gamma(A, B) = 0$, that is, the pair is not definite. If our aim is to test whether (A, B) is definite but not to compute $\gamma(A, B)$ then we can terminate Algorithm 2.3 as soon as the lower bound a is positive.

When $\omega(A, B)$ is zero or tiny many iterations may be required to verify definiteness or non-definiteness, as then $\omega(A, B)$ must be computed accurately enough to determine its sign. This limitation will affect any numerical method. In particular, if $\gamma(A, B)$ is of the order of the unit roundoff times some normwise measure of (A, B) then we cannot expect to determine definiteness in floating point arithmetic, since Theorem 2.2 implies that backward errors due to roundoff can change the definiteness of the pair.

Algorithm 2.3 guarantees to produce an interval in which $\gamma(A, B)$ lies, but each iteration is expensive as it requires the solution of a quadratic eigenvalue problem and up to $2n$ Hermitian eigenproblems.

A more efficient algorithm can be obtained by applying the level set algorithm of [18], which was derived for stability radii computations. The idea is to consider all the eigenvalues of the Hermitian matrix

$$A\cos\theta + B\sin\theta - \xi I = \tfrac{1}{2}(\mathrm{e}^{-\mathrm{i}\theta}C + \mathrm{e}^{\mathrm{i}\theta}C^*) - \xi I,$$

where the level ξ is a parameter. Since this matrix is analytic in the real variable θ, the eigenvalues are piecewise analytic functions of θ, losing analyticity only at θ corresponding to a multiple eigenvalue [13]. Moreover, the unit modulus zeros of $\det(M(z) - \xi I)$ determine the "zero crossings" of these functions, since

$$\det(A\cos\theta + B\sin\theta - \xi I) = 0 \iff \det(M(z) - \xi I) = 0, \quad z = \mathrm{e}^{\mathrm{i}\theta}.$$

By (2.9), these are also the unit modulus eigenvalues of the quadratic polynomial $Q(z) = C - 2\xi z I + z^2 C^*$. For such an eigenvalue $z_j = \mathrm{e}^{\mathrm{i}\theta_j}$ of Q we have, recalling the notation (2.2),

$$Q(z_j) = 2z_j(A_{\theta_j} - \xi I),$$

and hence any eigenvector of Q corresponding to the eigenvalue z_j is an eigenvector of $A_{\theta_j} - \xi I$ corresponding to the eigenvalue 0. It follows from standard results (see, e.g., [13]) that for a simple zero eigenvalue of $A_{\theta_j} - \xi I$ with normalized eigenvector v, the derivative of the eigenvalue is given by

$$\frac{\partial}{\partial\theta}\lambda_i(A_\theta - \xi I)|_{\theta_j} = v^* \frac{\partial}{\partial\theta}(A_\theta - \xi I)|_{\theta_j} v = v^* B_{\theta_j} v. \tag{2.12}$$

More generally, for an eigenvector space of dimension k spanned by an $n \times k$ orthonormal matrix V, the eigenvalues of the $k \times k$ matrix $V^* B_{\theta_j} V$ are the derivatives of the k eigenvalue crossings at θ_j. The sign of the derivatives determines if an eigenvalue increases or decreases at each particular zero crossing, with a zero derivative signalling a multiple eigenvalue. Now set $\xi = 0$. If all n eigenvalues of $A\cos\theta + B\sin\theta$ become positive at a particular angle θ then they all become negative at $\theta \pm \pi$, since

$$A\cos(\theta \pm \pi) + B\sin(\theta \pm \pi) = -(A\cos\theta + B\sin\theta). \tag{2.13}$$

It follows that a definite pair must have, for some θ, at least n strictly increasing crossings in $[\theta - \pi, \theta)$ and at least n strictly decreasing crossings in $[\theta, \theta + \pi)$. Since there are at most $2n$ zeros in any interval $[\theta - \pi, \theta + \pi)$, a definite pair must have *exactly* n consecutive strictly increasing zero crossings followed by n consecutive strictly decreasing zero crossings in $[\theta - \pi, \theta + \pi)$. This is clearly also a sufficient condition for a definite pair, since n consecutive strictly increasing zero crossings in an interval $[\theta - \pi, \theta)$ ensure that all eigenvalues of $A\cos\theta + B\sin\theta$ are positive. The procedure for testing definiteness is thus very simple.

Algorithm 2.4. Given a Hermitian pair (A, B) with $A + \mathrm{i}B$ non-singular this algorithm determines whether or not the pair is definite.

```
Compute the eigenvalues of Q(z) (with ξ = 0).
If there are 2n eigenvalues of unit modulus
    Compute the derivatives in (2.12) (with ξ = 0).
    If there are n consecutive strictly increasing and n consecutive
    strictly decreasing zero crossings
        The pair is definite; return
    end
end
The pair is not definite.
```

This algorithm enables us to test for definiteness by solving just one quadratic eigenvalue problem.

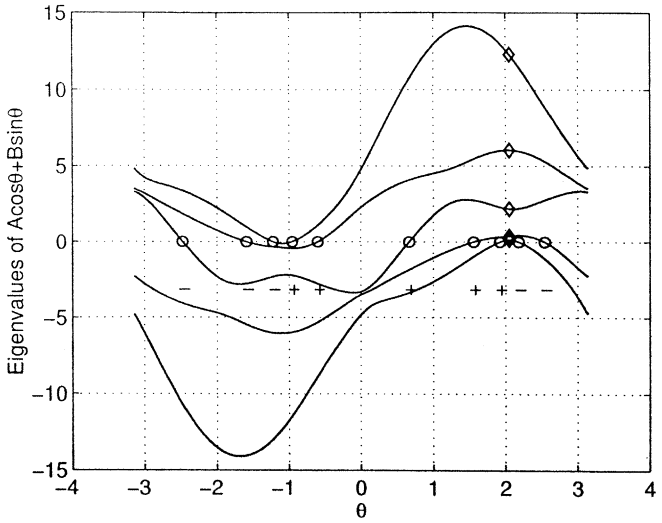

Fig. 1. Eigenvalues of a 5×5 matrix $A\cos\theta + B\sin\theta$ as a function of $\theta \in [-\pi, \pi]$.

Fig. 1 illustrates the algorithm. The matrices A and B are random, 5×5, real and symmetric. The 10 unit modulus eigenvalues of Q are marked with circles and the sign of the derivative is marked under each zero crossing. The pattern of the signs proves definiteness. The interval containing the maximum of the smallest eigenvalue is $[1.9187, 2.1851]$ and the eigenvalues of $A\cos\theta + B\sin\theta$ at the midpoint are marked with diamonds. These are all positive, which confirms the definiteness of the pair.

The same idea can be used to compute the Crawford number, $\gamma(A, B)$. Use Algorithm 2.4 to test definiteness of the pair. If definiteness is confirmed then an interval of θ is known throughout which $\lambda_{\min}(A\cos\theta + B\sin\theta) > 0$; compute the eigenvalues at its midpoint, θ_{mid}. The value $\xi = \lambda_{\min}(A\cos\theta_{\mathrm{mid}} + B\sin\theta_{\mathrm{mid}}) > 0$ is then a lower bound for the Crawford number that can be used for the next "level". If $\lambda_{\min}(A\cos\theta + B\sin\theta)$ is not multiple at its maximum value then this scheme can be shown to be quadratically convergent [11]. Moreover, variants are proposed in [11] that generically have a higher order of convergence by exploiting the computed derivatives at the zero crossings.

While the bisection algorithm is less efficient and slower to converge than the level set algorithm it does have one advantage: it produces a bracket for $\gamma(A, B)$ that shrinks to zero, whereas the level set algorithm produces only a monotonically increasing lower bound.

We give a numerical example, with $n = 10$ and

$$\begin{aligned} &A = (|i-j|) && \text{(Fiedler matrix)},\\ &B = U^{\mathrm{T}}U, \ \ U \text{ unit upper triangular with} && \\ &\qquad u_{ij} = -1, \ \ j > i && \text{(Moler matrix)}. \end{aligned} \tag{2.14}$$

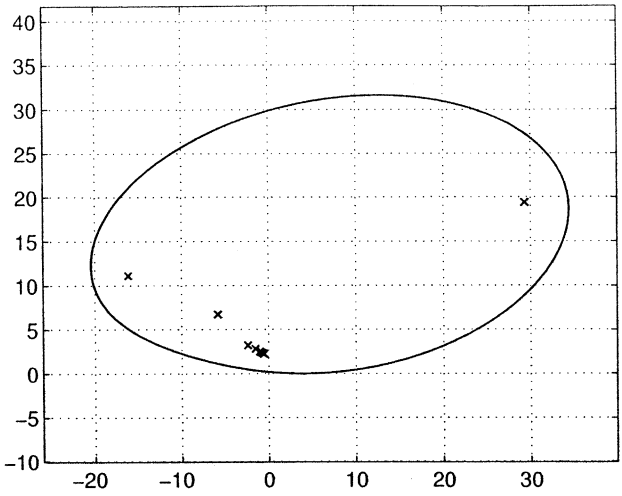

Fig. 2. $F(A + \mathrm{i}B)$, with A the Fiedler matrix and B the Moler matrix.

The pair (A, B) is obviously definite, and applying Algorithm 2.3 with $\mathrm{tol} = 10^{-4}$ we obtain after 15 iterations an interval that enables us to conclude that $d(A, B) = \gamma(A, B) = 0.1868$ to 4 significant figures. The initial interval from (2.10) and (2.11) is $[a, b] = [8.583 \times 10^{-6}, 2.314]$. Since a_0 in (2.11) is positive, the algorithm determines immediately that $\gamma(A, B) > 0$. Fig. 2 plots $F(A + \mathrm{i}B)$, with the eigenvalues of $A + \mathrm{i}B$ marked as crosses. Since B is positive definite, the field of values lies in the upper half-plane. The smallest eigenvalue of B is 8.6×10^{-6}, so while a perturbation of order 10^{-6} causes B to lose definiteness, a perturbation of order 10^{-1} to A and B is necessary to cause the pair to lose definiteness.

When we apply Algorithm 2.4 to this example we again detect that the matrix pair is positive definite; see Fig. 3. The smallest eigenvalue of $A\cos\theta + B\sin\theta$ at the midpoint of the relevant interval is 0.1867, which is a lower bound for the Crawford number.

3. Hyperbolic and elliptic systems

Now we turn our attention to the quadratic eigenvalue problem (QEP) [22]

$$Q(\lambda)x = (\lambda^2 A + \lambda B + C)x = 0, \quad A, B, C \in \mathbb{C}^{n\times n}. \tag{3.1}$$

An important class of QEPs is defined as follows [15].

Definition 3.1. The QEP (3.1) is hyperbolic if A is Hermitian positive definite, B and C are Hermitian, and

$$(x^*Bx)^2 > 4(x^*Ax)(x^*Cx) \quad \text{for all non-zero } x \in \mathbb{C}^n. \tag{3.2}$$

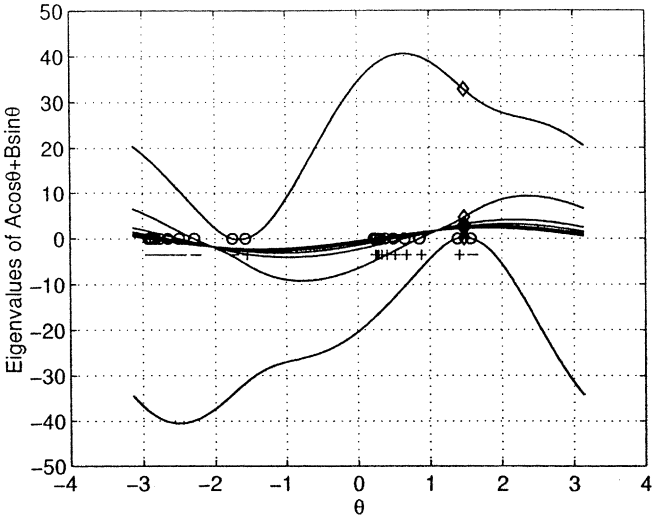

Fig. 3. Eigenvalues of $A\cos\theta + B\sin\theta$ with A the Fiedler matrix and B the Moler matrix.

For any eigenpair (x, λ), on premultiplication of (3.1) by x^* we obtain the scalar quadratic equation $\lambda^2 x^*Ax + \lambda x^*Bx + x^*Cx = 0$, with solutions

$$\lambda = \frac{-x^*Bx \pm \sqrt{(x^*Bx)^2 - 4(x^*Ax)(x^*Cx)}}{2x^*Ax}, \tag{3.3}$$

at least one of which is an eigenvalue. It follows that for a hyperbolic QEP all $2n$ eigenvalues are real. A hyperbolic QEP has further properties [7,8], [14, Section 7.6]: there is a gap between the n largest eigenvalues (the primary eigenvalues) and the n smallest eigenvalues (the secondary eigenvalues), and there are n linearly independent eigenvectors associated with the primary eigenvalues and likewise for the secondary eigenvalues (in other words, all the eigenvalues are semi-simple).

We note that certain types of gyroscopic systems are equivalent to hyperbolic systems; see the example in Section 3.3.2.

A more specialized class of QEPs is those that are overdamped.

Definition 3.2. The QEP (3.1) is overdamped if it is hyperbolic with B Hermitian positive definite and C Hermitian positive semidefinite.

It is immediate from (3.3) that for overdamped problems the eigenvalues are nonpositive. These problems are essentially shifted hyperbolic problems. To see why, consider the identity

$$\begin{aligned} Q(\lambda+\theta) &= \lambda^2 A + \lambda(B + 2\theta A) + C + \theta B + \theta^2 A \\ &=: \lambda^2\widetilde{A} + \lambda\widetilde{B} + \widetilde{C} = \widetilde{Q}(\lambda). \end{aligned}$$

It is easy to see that (3.2) holds for Q if and only if it holds for $\widetilde{Q}$. Moreover, if Q is hyperbolic then $\widetilde{Q}$ is overdamped for a large enough shift θ, since A is positive definite, and conversely any overdamped QEP can be shifted to make it non-overdamped but still hyperbolic.

Another important class of QEPs is those for which (3.2) holds with the inequality reversed.

Definition 3.3. The QEP (3.1) is elliptic if A is Hermitian positive definite, B and C are Hermitian, and

$$(x^*Bx)^2 < 4(x^*Ax)(x^*Cx) \quad \text{for all non-zero } x \in \mathbb{C}^n. \tag{3.4}$$

Elliptic QEPs clearly have non-real eigenvalues, and, necessarily, C is positive definite.

3.1. Testing for hyperbolicity and ellipticity

The first question of interest is how to test whether a given QEP is hyperbolic or elliptic. We need the following results from [15,16], which are generalizations of properties that are obvious for $n = 1$.

Theorem 3.4. *A QEP with A, B and C Hermitian and A positive definite is hyperbolic if and only if $Q(\mu)$ is negative definite for some $\mu \in \mathbb{R}$.*

Theorem 3.5. *Ellipticity of a QEP with A, B and C Hermitian and A positive definite is equivalent to either of the conditions*:
1. *$Q(\mu)$ is positive definite for all $\mu \in \mathbb{R}$,*
2. *$(x^*Bx)^2 < 4(x^*Ax)(x^*Cx)$ for all eigenvectors x of the QEP.*

Theorem 3.5 shows that to test for ellipticity is straightforward: compute the eigenvectors and check whether the discriminant condition is satisfied for all of them.

Testing for hyperbolicity is more complicated. Theorem 3.4 shows that it would suffice to solve the one-dimensional global optimization problem $\min_\mu \lambda_{\max}(Q(\mu))$. We show that testing for hyperbolicity can be reduced to testing for definiteness of a Hermitian pencil of twice the dimension, which can be done using Algorithm 2.3 or Algorithm 2.4. The following result is a slight generalization of one of Veselić [24, Theorem A5] and is essentially contained in [1].

Theorem 3.6. *A QEP with A, B and C Hermitian and A positive definite is hyperbolic if and only if the pair (A_1, B_1) is definite, where*

$$A_1 = \begin{bmatrix} -C & 0 \\ 0 & A \end{bmatrix}, \quad B_1 = -\begin{bmatrix} B & A \\ A & 0 \end{bmatrix}.$$

Proof. Recall from Theorem 2.1 that (A_1, B_1) is definite if and only if $\alpha A_1 + \beta B_1$ is positive definite for some α and β. Clearly, $\alpha = 0$ can be ruled out, since B_1 is indefinite. For $\alpha \neq 0$ we have

$$\begin{aligned}\alpha A_1 + \beta B_1 &= \begin{bmatrix} -\alpha C - \beta B & -\beta A \\ -\beta A & \alpha A \end{bmatrix} \\ &= \begin{bmatrix} I & -\frac{\beta}{\alpha} I \\ 0 & I \end{bmatrix} \begin{bmatrix} -\alpha C - \beta B - \frac{\beta^2}{\alpha} A & 0 \\ 0 & \alpha A \end{bmatrix} \begin{bmatrix} I & 0 \\ -\frac{\beta}{\alpha} I & I \end{bmatrix}.\end{aligned}$$

So $\alpha A_1 + \beta B_1$ is congruent to

$$\alpha \operatorname{diag}\big(-(\mu^2 A + \mu B + C), A\big),$$

where $\mu = \beta/\alpha$. The result now follows from Theorem 3.4, since A is positive definite. □

Also of interest are sufficient conditions for hyperbolicity and ellipticity that may be verifiable using knowledge of the QEP. The following result provides some conditions based on extremal eigenvalues of the coefficient matrices (condition (3.6) is given in [1, Theorem 3]). Here, $\sigma_{\min}$ and $\sigma_{\max}$ denote the smallest and largest singular values.

Theorem 3.7. *A QEP with A Hermitian positive definite and B and C Hermitian is hyperbolic if either of the following inequalities holds*:

$$\sigma_{\min}(B)^2 > 4\lambda_{\max}(A)\lambda_{\max}(C), \tag{3.5}$$

$$\min_i |\lambda_i(A^{-1}B)|^2 > 4\lambda_{\max}(A^{-1}C). \tag{3.6}$$

A QEP with A, B and C Hermitian and A and C positive definite is elliptic if either of the following inequalities holds:

$$\sigma_{\max}(B)^2 < 4\lambda_{\min}(A)\lambda_{\min}(C), \tag{3.7}$$

$$\max_i |\lambda_i(A^{-1}B)^2| < 4\lambda_{\min}(A^{-1}C). \tag{3.8}$$

Proof. Condition (3.5) is immediate from the variational characterization of eigenvalues of Hermitian matrices. Since A is positive definite it has a unique Hermitian positive definite square root $A^{1/2}$, and by setting $x = A^{-1/2}y$ condition (3.2) is transformed to

$$(y^* A^{-1/2} B A^{-1/2} y)^2 > 4(y^* y)(y^* A^{-1/2} C A^{-1/2} y) \quad \text{for all } y \neq 0.$$

Again applying the variational characterization and noting that $A^{-1/2}BA^{-1/2}$ is similar to $A^{-1}B$, and similarly for C, leads to (3.6). Conditions (3.7) and (3.8) are obtained in the same way. □

For both pairs of inequalities in Theorem 3.7 either member can be stronger than the other, depending on the particular QEP. Indeed, if $B = C = I$ then (3.5) requires, like (3.2), that $\lambda_{\max}(A) < 1/4$, whereas (3.6) imposes the stronger requirement that $\kappa_2(A)\lambda_{\max}(A) < 1/4$, where $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$. On the other hand, for $B = A$ and $C = I$, (3.6) requires, like (3.2), that $\lambda_{\min}(A) > 4$, whereas (3.5) insists that $\lambda_{\min}(A) > 4\kappa_2(A)$.

3.2. Distance to nearest non-hyperbolic or non-elliptic QEP

We assume throughout this section that A, B and C are Hermitian with A positive definite. For a QEP that is hyperbolic or elliptic we are interested in by how much the coefficient matrices must be perturbed for this property to be lost. Clearly, both properties are lost when A is perturbed to lose definiteness, which can be achieved by adding a perturbation of 2-norm equal to $\lambda_{\min}(A)$. The more interesting way in which hyperbolicity or ellipticity is lost is when conditions (3.2) or (3.4) fail. We will treat both properties together by making use of the 2×2 Hermitian matrix

$$W(x, A, B, C) = \begin{bmatrix} 2x^*Ax & x^*Bx \\ x^*Bx & 2x^*Cx \end{bmatrix}.$$

Note that

$$\det(W(x, A, B, C)) \begin{cases} < 0 & \text{for all } x \neq 0 \text{ if the QEP is hyperbolic,} \\ > 0 & \text{for all } x \neq 0 \text{ if the QEP is elliptic.} \end{cases}$$

For both properties we are interested in perturbing the coefficient matrices to achieve $\det(W(x, A, B, C)) = 0$. Therefore both the distance from a hyperbolic problem to the nearest non-hyperbolic problem and the distance from an elliptic problem to the nearest non-elliptic problem can be defined by

$$\begin{aligned} &d(A, B, C) \\ &\quad = \min\big\{ f(\Delta A, \Delta B, \Delta C) : \det(W(x, A + \Delta A, B + \Delta B, C + \Delta C)) = 0 \\ &\qquad\qquad\qquad\qquad \text{for some } x \neq 0 \big\}. \end{aligned} \tag{3.9}$$

Here, f is some non-negative function of the perturbation matrices ΔA, ΔB and ΔC. Also of interest as a natural measure of the degree of hyperbolicity or ellipticity is the quantity

$$\phi(A, B, C) = \min_{\|x\|_2=1} \sqrt{|(x^*Bx)^2 - 4(x^*Ax)(x^*Cx)|}. \tag{3.10}$$

We wish to obtain a simpler formulation for $d(A, B, C)$ and to compare this distance with $\phi(A, B, C)$.

There are many possible choices of f, but it is convenient to work with

$$f(A, B, C) = \left\| \begin{bmatrix} 2A & B \\ B & 2C \end{bmatrix} \right\|_{\mathrm{F}}.$$

We can rewrite (3.9) as

$$d(A, B, C) = \min_{\|x\|_2=1} \min \{ f(\Delta A, \Delta B, \Delta C) : \det(W(x, A + \Delta A, B + \Delta B, C + \Delta C)) = 0 \}.$$

Denote by $\Delta W_{\mathrm{opt}} = (\delta_{ij})$ the matrix ΔW of smallest Frobenius-norm for which $W(x, A, B, C) + \Delta W$ is singular, for a given unit 2-norm x. Since

$$W(x, A + \Delta A, B + \Delta B, C + \Delta C) = W(x, A, B, C) + W(x, \Delta A, \Delta B, \Delta C),$$

for any feasible ΔA, ΔB and ΔC in (3.9) we have

$$\begin{aligned} \|\Delta W_{\mathrm{opt}}\|_{\mathrm{F}}^2 &\leqslant \|W(x, \Delta A, \Delta B, \Delta C)\|_{\mathrm{F}}^2 \\ &\leqslant 4\|\Delta A\|_{\mathrm{F}}^2 + 2\|\Delta B\|_{\mathrm{F}}^2 + 4\|\Delta C\|_{\mathrm{F}}^2 \\ &= f(\Delta A, \Delta B, \Delta C)^2, \end{aligned}$$

and we have equality throughout for

$$\Delta A = (\delta_{11}/2)xx^*, \quad \Delta B = \delta_{12}xx^*, \quad \Delta C = (\delta_{22}/2)xx^*,$$

for which $W(x, \Delta A, \Delta B, \Delta C) = \Delta W_{\mathrm{opt}}$. We conclude that the latter perturbations are optimal for the given x.

Now $\Delta W_{\mathrm{opt}} = -\lambda_{\mathrm{opt}} v_{\mathrm{opt}} v_{\mathrm{opt}}^*$, where λ_{opt} is the eigenvalue of $W(x, A, B, C)$ of smallest modulus and v_{opt} is a corresponding unit eigenvector, and so we have the (non-convex) global minimization problem

$$d(A, B, C) = \min_{\|x\|_2=1} g(x), \tag{3.11}$$

where

$$g(x) = |\lambda_{\mathrm{opt}}(W(x, A, B, C))|.$$

Note that, from (3.10),

$$\begin{aligned} \phi(A, B, C) &= \min_{\|x\|_2=1} \sqrt{|\det(W(x, A, B, C))|} \\ &= \min_{\|x\|_2=1} \sqrt{|\lambda_{\min}(W(x, A, B, C))\lambda_{\max}(W(x, A, B, C))|} \\ &\geqslant d(A, B, C), \end{aligned}$$

so ϕ overestimates the distance to the nearest non-hyperbolic or non-elliptic system. The eigenvalues of $W(x, A, B, C)$ are $x^*(A + C)x \pm \sqrt{(x^*(A - C)x)^2 + (x^*Bx)^2}$. For a hyperbolic problem, if C is positive semidefinite then we take the minus sign to obtain

$$\lambda_{\mathrm{opt}} = x^*(A + C)x - \sqrt{(x^*(A - C)x)^2 + (x^*Bx)^2}. \tag{3.12}$$

Since W is indefinite with $\mathrm{trace}(W) > 0$, we have $\lambda_{\mathrm{opt}} < 0 < \lambda_{\max}$ with $|\lambda_{\mathrm{opt}}| < \lambda_{\max}$; thus g is differentiable for all non-zero x, since W has distinct eigenvalues, and

so local minima of g can be found by standard optimization techniques. For elliptic problems (3.12) also holds; here the eigenvalues of W are positive but repeated eigenvalues are possible, so g may not be differentiable at the global minimum.

An interesting special case is $n = 1$. Consider, for example, the elliptic quadratic $q(x) = ax^2 + bx + c = x^2 + (2-\epsilon)x + 1$, with $0 < \epsilon \ll 1$. The discriminant $b^2 - 4ac = \epsilon(\epsilon - 4)$, so $\phi(a, b, c) \approx 2\epsilon^{1/2}$. To obtain $d(a, b, c)$ we note first that

$$W(1, a, b, c) = \begin{bmatrix} 2 & 2-\epsilon \\ 2-\epsilon & 2 \end{bmatrix}, \quad \lambda_{\mathrm{opt}} = \epsilon, \quad v_{\mathrm{opt}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore $d(a, b, c) = \epsilon \ll \phi(a, b, c)$ and the nearest non-elliptic quadratic is $(1 - \epsilon/4)x^2 - (2 - \epsilon/2)x + 1 - \epsilon/4$, which has zero discriminant.

3.3. Numerical examples

We give three numerical examples to illustrate the results, each with a different type of QEP: the first is real and overdamped, the second is complex and hyperbolic but not overdamped, and the third is real and elliptic.

The question arises of whether for real data we can restrict x to be real in the minimizations (3.10) and (3.11), since the all the definitions in this section involve complex x. For $n > 2$ the answer is yes, and indeed hyperbolicity is sometimes defined in terms of real x when Q is real (see [7,8,14], for example). The exclusion of $n = 2$ relates to a subtlety in the definition (2.1) of definite pair; see [17,19] and [20, p. 290] for a discussion of this issue.

In computing $d(A, B, C)$ and $\phi(A, B, C)$ we used direct search (function `mdsmax` from [12]), taking the convergence tolerance of order the unit roundoff to obtain the best possible accuracy and trying different starting values in order to be confident that the global minima were obtained.

3.3.1. Damped mass–spring system

Our first example is from a damped mass–spring system; see [21] for the details. With four masses and a particular choice of the masses and the spring and damper constants the matrices are

$$A = I_4, \quad B = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 12 & -4 & 0 \\ 0 & -4 & 12 & -4 \\ 0 & 0 & -4 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

All three matrices are positive definite and the quadratic eigenvalue problem (3.1) is overdamped. The sufficient conditions (3.5) and (3.6) for hyperbolicity are identical in this case and are not satisfied. On applying Algorithm 2.3 to the pair (A_1, B_1) in Theorem 3.6, the pair is diagnosed definite, and hence the QEP hyperbolic, after just one iteration.

Minimization of $g(x)$ yielded $d(A, B, C) = 2.0$, with optimal perturbations of a particularly simple form in this case: $\Delta A = 0.125ee^{\mathrm{T}}$, $\Delta B = -0.25ee^{\mathrm{T}}$, $\Delta C =$

Table 1
Eigenvalues of original (hyperbolic) and perturbed (non-hyperbolic) quadratic eigenvalue problem from a damped mass–spring system

Original	Perturbed
−1.7403e1	−1.7403e1
−1.1745e1	−1.1745e1
−6.0824e0	−6.0824e0
−3.7321e0	−1.0000e0
−2.6795e−1	−1.0000e0
−2.6072e−1	−2.6072e−1
−2.5544e−1	−2.5544e−1
−2.5364e−1	−2.5364e−1

$0.125ee^{\mathrm{T}}$, where e is the vector of 1's. All of $A + \Delta A$, $B + \Delta B$ and $C + \Delta C$ are positive definite. By comparison, $\phi(A, B, C) = 3.4641$. The eigenvalues of the original and perturbed quadratic eigenvalue problems are shown in Table 1. In this example, loss of hyperbolicity coincides with loss of the gap between the primary and secondary eigenvalues.

3.3.2. Moving wiresaw

We consider a model of the motion of a wiresaw used to cut through silicon ingot [25]. The underlying partial differential equation has the form

$$u_{tt} - 2vu_{xt} - (1 - v^2)u_{xx} = 0, \tag{3.13}$$

where $u(x, t)$ is the unknown function and v is a parameter. Approximating

$$u(x, t) = \sum_{k=1}^{n} q_k(t) \sin(k\pi x) \tag{3.14}$$

and applying the Galerkin method gives the second-order differential equation

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0, \tag{3.15}$$

where $q(t) = [q_1(t), \ldots, q_n(t)]^{\mathrm{T}}$, $M = I_n/2$, $K = \mathrm{diag}_{1\leqslant j\leqslant n}\left(j^2\pi^2(1 - v^2)/2\right)$, and

$$C = -C^{\mathrm{T}} = (c_{ij}) \quad \text{with} \quad c_{ij} = \begin{cases} \frac{4ij}{j^2 - i^2}v & \text{if } i + j \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

This is a gyroscopic system with corresponding λ-matrix $G(\lambda) = \lambda^2 M + \lambda C + K$. Let

$$Q(\lambda) = -G(-\mathrm{i}\lambda) = \lambda^2 M + \lambda(\mathrm{i}C) - K.$$

The coefficient matrices are Hermitian and for $v < 1$ it is easily checked that K is positive definite and hence Q is hyperbolic. However, Q is not overdamped.

We took $n = 8$ and $v = 0.01$. We found that $d(M, \mathrm{i}C, -K) = 1.0$, with optimal perturbations $\Delta M = -ee^{\mathrm{T}}/16$, $\Delta C = \Delta K = 0$, and $\phi(M, \mathrm{i}C, -K) = 4.4427$.

The eigenvalues of $Q(\lambda)$ and the perturbed QEP come in pairs $(-\lambda, \lambda)$. In contrast with the previous example, the loss of hyperbolicity coincides with the loss of the non-singularity of M and the appearance of infinite eigenvalues.

3.3.3. Wave equation

We consider the free vibration of a string with clamped ends in a spatially inhomogeneous environment. The equation characterizing the wave motion can be described by [10]

$$\begin{cases} u_{tt} + \epsilon a(x) u_t = \Delta u, & x \in [0, \pi], \quad \epsilon > 0, \\ u(t, 0) = u(t, \pi) = 0. \end{cases}$$

Approximating u as in (3.14) and applying the Galerkin method again leads to a second-order differential equation (3.15), here with $M = (\pi/2) I_n$, $K = (\pi/2)\mathrm{diag}(j^2)$, and

$$C = (c_{kj}), \quad c_{kj} = \int_0^\pi \epsilon\, a(x) \sin(kx) \sin(jx)\, \mathrm{d}x.$$

In our experiments we take $n = 9$, $a(x) = x^2(\pi - x)^2 - \delta$, $\delta = 201$ and $\epsilon = 0.1$.

The sufficient conditions (3.7) and (3.8) for ellipticity are not satisfied but the condition 2 of Theorem 3.5 is satisfied, so the QEP is elliptic. We found that $d(M, C, K) = 0.0588$ and $\phi(M, C, K) = 28.1$. The spectra of the original and the perturbed QEP are shown in Fig. 4. The loss of ellipticity corresponds to the two eigenvalues closest to the real axis merging to become the double eigenvalue 1.

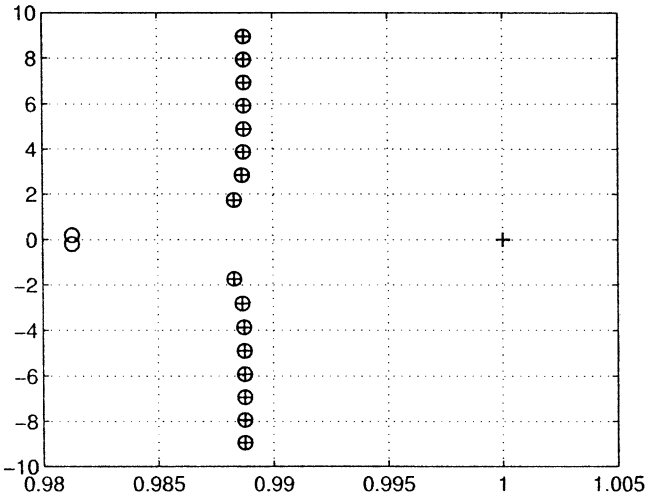

Fig. 4 Spectrum of $Q(\lambda)$ for the wave example. The eigenvalues of the original elliptic QEP are marked by "○" and those of the perturbed QEP by "+".

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