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A CHART OF BACKWARD ERRORS FOR SINGLY AND DOUBLY STRUCTURED EIGENVALUE PROBLEMS*

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Abstract. We present a chart of structured backward errors for approximate eigenpairs of singly and doubly structured eigenvalue problems. We aim to give, wherever possible, formulae that are inexpensive to compute so that they can be used routinely in practice. We identify a number of problems for which the structured backward error is within a factor $\sqrt{2}$ of the unstructured backward error. This paper collects, unifies, and extends existing work on this subject.

Key words. eigenvalue, eigenvector, symmetric matrix, Hermitian matrix, skew-symmetric matrix, skew-Hermitian matrix, symplectic matrix, conjugate symplectic matrix, Hamiltonian matrix, backward error, condition number

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1. Introduction.

Bunse-Gerstner, Byers, and Mehrmann [8] present a chart of numerical methods for structured eigenvalue problems for which the matrices have more than one of the properties defined as follows:

$A \in \mathbb{C}^{m\times m}$ is	$A \in \mathbb{R}^{m\times m}$ is
Hermitian if $A^* = A$,	symmetric if $A^T = A$,
skew-Hermitian if $A^* = -A$,	skew-symmetric if $A^T = -A$,
unitary if $A^*A = I$,	orthogonal if $A^TA = I$,
conjugate symplectic if $m = 2n$ and $A^*JA = J$,	symplectic if $m = 2n$ and $A^TJA = J$,
Hamiltonian if $m = 2n$ and $(JA) = (JA)^*$,	J-symmetric if $m = 2n$ and $(JA) = (JA)^T$,
skew-Hamiltonian if $m = 2n$ and $(JA) = -(JA)^*$,	J-skew symmetric if $m = 2n$ and $(JA) = -(JA)^T$,

where $J = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$, I_n being the $n\times n$ identity matrix. Structured eigenvalue problems occur in numerous applications and we refer to [8] for a list of them and pointers to the relevant literature. In this paper we present a chart of computable backward errors for approximate eigenpairs and condition numbers for simple eigenvalues of matrices having one or two of these special structures.

The importance of condition numbers for characterizing the sensitivity of solutions to problems and backward errors for assessing the stability and quality of numerical algorithms is widely appreciated. A backward error of an approximate eigenpair (x, λ) of a matrix A is a measure of the smallest perturbation E such that $(A+E)x = \lambda x$. This backward error has two main uses. First, it can be used to determine if (x, λ) solves a nearby problem by comparing the backward error with the size of any

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uncertainties in the data matrix A. Second, a bound on the forward error can be obtained in terms of the backward error and an appropriate condition number.

A natural definition of the normwise backward error of an approximate eigenpair (x, λ) is

$$\eta(x,\lambda) = \min\left\{\alpha^{-1}\|E\| : \ (A+E)x = \lambda x \ \right\}, \tag{1.1}$$

where α is a positive parameter that allows freedom in how the perturbations are measured and $\|\cdot\|$ denotes any vector norm and the corresponding subordinate matrix norm. Deif [9] derived the explicit expression for the 2-norm (also valid for any subordinate norm and the Frobenius norm),

$$\eta(x,\lambda) = \alpha^{-1}\|(A-\lambda I)x\|/\|x\|,$$

showing that the normwise backward error is a scaled residual. Also of interest is the backward error of a set of approximate eigenpairs $(x_j, \lambda_j)_{j=1}^k$, which we collect into matrices $X_k = [x_1, x_2, \dots, x_k]$ and $\Lambda_k = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$. For a measure of the backward error we use the natural generalization of (1.1),

$$\eta(X_k,\Lambda_k) = \min\left\{\alpha^{-1}\|E\| : (A+E)X_k = X_k\Lambda_k\right\}, \tag{1.2}$$

for which an explicit expression is available for any unitarily invariant norm if $\mathrm{rank}(X_k) = k$ [26, Thm. 2.4.2],

$$\eta(X_k,\Lambda_k) = \alpha^{-1}\|R_k X_k^+\|, \tag{1.3}$$

where $R_k = X_k\Lambda_k - AX_k$ is the residual matrix and X_k^+ is the pseudoinverse of X_k.

The measure η is not entirely appropriate for our structured eigenvalue problems, as it does not respect any structure in A. Similar remarks can be made about condition numbers. Standard condition numbers are derived without requiring that perturbations preserve structure. As a consequence, standard condition numbers usually exceed the actual condition number for an eigenvalue problem subject to structured perturbation. In the last few years, efforts have been concentrated on deriving new structure-preserving algorithms for the solution of structured eigenvalue problems for both the dense case [1], [4], [12], [14], [23] and the large and sparse case [2], [3], [5], [21], to cite just a few articles. It is therefore of interest to develop backward errors and condition numbers that fully respect the inherent structure of these problems.

Let $A \in \mathcal{C}_{\mathbb{K}} \subset \mathbb{K}^{m\times m}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) be a singly or doubly structured matrix, where $\mathcal{C}_{\mathbb{K}}$ is the set of matrices having the structure of interest. We extend the definition of the normwise backward error for a set of eigenpairs (X_k, Λ_k) in (1.2) to the structured case by

$$\eta_{\mathbb{K}}(X_k,\Lambda_k) = \min\left\{\alpha^{-1}\|E\|_F : \ (A+E)X_k = X_k\Lambda_k, \ A+E \in \mathcal{C}_{\mathbb{K}}\right\}. \tag{1.4}$$

The contribution of this work is to unify and extend explicit expressions of backward errors for singly and doubly structured eigenproblems. These expressions allow structured backward errors to be computed more efficiently than if (1.4) were treated as a general nonlinear optimization problem.

In section 2 we recall some basic properties of the structured matrices under consideration and give some useful lemmas. We recall in the first part of section 3 that for linear structures a Kronecker product approach can be used to rewrite the minimization problem in (1.4) in terms of the minimal 2-norm solution to an

TABLE 2.1
Eigenvalue properties of the singly structured matrices $A \in \mathbb{C}^{m\times m}$ under consideration.

Class of matrices	Eigenvalues	Class of matrices	Eigenvalues
$A^* = A$	real eigenvalues	$A^T = A$	arbitrary
$A^* = -A$	purely imaginary	$A^T = -A$	0 and/or pairs μ, $-\mu$, ($\mu \neq 0$)
$A^*A = I$	$\lvert\mu\rvert = 1$	$A^TA = I$	± 1 and/or pairs $\mu, 1/\mu$, ($\mu^2 \neq 1$)
$A^*JA = J$	pairs $\mu, 1/\bar{\mu}$	$A^TJA = J$	pairs $\mu, 1/\mu$
$(JA) = (JA)^*$	pairs $\mu, -\bar{\mu}$	$(JA) = (JA)^T$	pairs $\mu, -\mu$
$(JA) = -(JA)^*$	pairs $\mu, \bar{\mu}$	$(JA) = -(JA)^T$	double eigenvalues

underdetermined system. The dimension of the underdetermined system may make the computation of backward error expensive. Fortunately, there are particular classes of linear structured problems for which we can characterize the set of solutions to the constraints in (1.4) and identify the solution of minimal Frobenius norm. This yields backward error formulae that are cheaper to compute and easier to analyze and understand than with the Kronecker product approach. As a result we show that, in some instances, forcing the backward error matrix to have a particular structure has little effect on its norm.

Backward errors for eigenproblems with nonlinear structure are harder to derive. Sun [25] characterizes the complete set of solutions to the constraints in (1.4) for the class of unitary matrices and derives a structured backward error for this class of problems. We use his approach and extend it to the classes of Hermitian unitary, symplectic unitary, and symmetric orthogonal matrices. Many problems remain open. Following the presentation in [8], we give in the second part of section 3 a chart of structured backward errors. For each class of matrices, we either recall an existing known explicit formula for the structured backward error, or derive a new explicit formula, or identify obtaining such a formula as an open problem. We aim to provide formulae that are cheap to compute so that they can be used in the course of a computation. We identify several cases in which the structured backward error is within a factor $\sqrt{2}$ of the unstructured backward error. For completeness, we recall in section 4 how to compute structured condition numbers of simple eigenvalues of matrices depending linearly on a set of parameters.

2. Basics.

2.1. Background material and definitions. We summarize in Table 2.1 the properties of the eigenvalues of the singly structured matrices considered in this paper. If the matrix is real, then its spectrum is symmetric with respect to the real axis. For doubly structured matrices the eigenvalue properties combine. For example, the eigenvalues of a real Hamiltonian matrix come in quadruples $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$ if $\mathrm{Re}(\lambda) \neq 0$, and the eigenvalues of a Hermitian Hamiltonian matrix come in pairs $(\lambda, -\lambda)$ with λ real.

For $A \in \mathbb{C}^{m\times k}$ with $m \geq k$, there exists a matrix $U \in \mathbb{C}^{m\times k}$ with orthonormal columns, and a unique Hermitian positive semidefinite matrix $H \in \mathbb{C}^{k\times k}$, such that $A = UH$. This is called the *polar decomposition* of A.

For a Hermitian matrix A, we define $\mathrm{sign}(A)$ by $\mathrm{sign}(A) = Q\,\mathrm{sign}(D)Q^*$, where $A = QDQ^*$ is the eigendecomposition of A with $Q^*Q = I$, $\mathrm{sign}(D) = \mathrm{diag}(\mathrm{sign}(d_i))$, and $\mathrm{sign}(0) = 1$.

We define the symplectic quasi-QR factorization of a $2n \times k$ matrix A by

$$A = QT, \quad T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix},$$

where Q is unitary conjugate symplectic, $T_1 \in \mathbb{C}^{n\times k}$ is upper trapezoidal, and $T_2 \in \mathbb{C}^{n\times k}$ is strictly upper trapezoidal. This factorization is discussed in [7, Cor. 4.5(ii)] and [27].

We make frequent use of the following lemmas.

LEMMA 2.1. *Let $A \in \mathbb{C}^{m\times m}, Y_1 \in \mathbb{C}^{m\times k}, m \geq k$, and $Y = [Y_1, Y_2]$ be unitary and let $B \in \mathbb{C}^{k\times k}$. Then*

$$\|Y_1 B - AY_1\|_F^2 = \|B - Y_1^* AY_1\|_F^2 + \|Y_2^* AY_1\|_F^2.$$

Proof. The proof is immediate using $Y_1Y_1^* + Y_2Y_2^* = I$ and

$$Y_1 B - AY_1 = Y \begin{bmatrix} B - Y_1^* AY_1 \\ -Y_2^* AY_1 \end{bmatrix}. \qquad \square$$

LEMMA 2.2 ([25, Lem. 2.4]). *Let $A \in \mathbb{C}^{m\times m}$ be unitary, $Y_1 \in \mathbb{C}^{m\times k}$ with $2k \leq m$, $Y = [Y_1, Y_2]$ be unitary, and let H_1 and H_2 be the Hermitian polar factors of $Y_1^* AY_1$ and $Y_2^* AY_2$, respectively. Then for any unitarily invariant norm,*

$$\|I - H_1\| = \|I - H_2\| \quad \text{and} \quad \|Y_1^* AY_2\| = \|Y_2^* AY_1\|.$$

Proof. By the CS decomposition [22] there are unitary matrices $U = \mathrm{diag}(U_1, U_2)$ and $V = \mathrm{diag}(V_1, V_2)$ with $U_1, V_1 \in \mathbb{C}^{k\times k}$ such that

$$U^* Y^* AY V = \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & I \end{bmatrix},$$

where C, S are $k \times k$ diagonal matrices with nonnegative diagonal elements and $C^2 + S^2 = I$. Then

$$Y_1^* AY_1 = U_1 C V_1^*, \qquad Y_2^* AY_2 = U_2 \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} V_2^*$$

so that $H_1 = V_1 C V_1^*$ and $H_2 = V_2 \,\mathrm{diag}(C, I) V_2^*$. Hence, $\|I - H_2\| = \|I - C\| = \|I - H_1\|$. The second equality follows from

$$Y_2^* AY_1 = U_2 \begin{bmatrix} S \\ 0 \end{bmatrix} V_1, \quad Y_1^* AY_2 = U_1 \begin{bmatrix} -S & 0 \end{bmatrix} V_2. \qquad \square$$

2.2. Structured matrix problems. Before deriving structured backward errors, we need some results on the following structured matrix problem: *Given a class of structured matrices $\mathcal{C}_{\mathbb{K}} \subset \mathbb{K}^{m\times m}$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, characterize*

1. *pairs of matrices $Y, B \in \mathbb{K}^{m\times k}$ for which there exists a matrix $A \in \mathcal{C}_{\mathbb{K}}$ such that $AY = B$;*
2. *the set $\mathcal{S}_{\mathcal{C}_{\mathbb{K}}} = \{A \in \mathcal{C}_{\mathbb{K}} : \ AY = B\}$.*

The lemmas in this section give a solution to this problem for several classes of structured matrices and give, whenever possible, the optimal solution A_{opt} defined by

$$\|A_{\mathrm{opt}}\|_F = \min\{\|A\|_F : \ A \in \mathcal{S}_{\mathcal{C}_{\mathbb{K}}}\}.$$

First, we need to set the notation. Define the full and reduced singular value decompositions of Y by

$$Y = U \begin{bmatrix} \Sigma_Y & 0 \\ 0 & 0 \end{bmatrix} V^* = U_1 \Sigma_Y V_1^*, \tag{2.1}$$

where $U = [U_1, U_2]$ and $V = [V_1, V_2]$ are unitary with $U_1 \in \mathbb{K}^{m\times r}$, $V_1 \in \mathbb{K}^{k\times r}$, and $\Sigma_Y = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$, $\sigma_i > 0$, $i = 1\colon r$, $r = \operatorname{rank}(Y)$. In what follows, Y^+ denotes the pseudoinverse of Y, $P_Y = YY^+ = U_1U_1^*$ is the orthogonal projector onto range(Y), and $P_Y^\perp = I - P_Y$.

The first result is from [24, Lem. 1.4] and concerns the class of Hermitian matrices when $\mathbb{K} = \mathbb{C}$ and the class of symmetric matrices when $\mathbb{K} = \mathbb{R}$. We give the proof for completeness.

LEMMA 2.3. *Let $Y, B \in \mathbb{K}^{m\times k}$, $m \geq k$, be given and let*

$$\mathcal{C}_\mathbb{K} = \{A \in \mathbb{K}^{m\times m} : A = A^*\}.$$

Then $\mathcal{S}_{\mathcal{C}_\mathbb{K}} \neq \emptyset$ if and only if $BP_{Y^*} = B$ and $P_Y BY^+ \in \mathcal{C}_\mathbb{K}$, and if $\mathcal{S}_{\mathcal{C}_\mathbb{K}} \neq \emptyset$, then

$$\begin{aligned} \mathcal{S}_{\mathcal{C}_\mathbb{K}} &= \{BY^+ + (BY^+)^* P_Y^\perp + P_Y^\perp H P_Y^\perp :\ H \in \mathcal{C}_\mathbb{K}\}, \\ A_{\mathrm{opt}} &= BY^+ + (BY^+)^* P_Y^\perp. \end{aligned}$$

Proof. Substituting (2.1) for Y in $AY = B$ and letting

$$U^*AU = \widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \quad U^*BV = \begin{bmatrix} \widetilde{B}_{11} & \widetilde{B}_{12} \\ \widetilde{B}_{21} & \widetilde{B}_{22} \end{bmatrix}, \tag{2.2}$$

with $\widetilde{A}_{11}, \widetilde{B}_{11} \in \mathbb{K}^{r\times r}$, we obtain

$$\begin{bmatrix} \widetilde{A}_{11}\Sigma_Y & 0 \\ \widetilde{A}_{21}\Sigma_Y & 0 \end{bmatrix} = \begin{bmatrix} \widetilde{B}_{11} & \widetilde{B}_{12} \\ \widetilde{B}_{21} & \widetilde{B}_{22} \end{bmatrix}. \tag{2.3}$$

Hence, solutions to $AY = B$ exist if and only if

$$U^*BV = \begin{bmatrix} \widetilde{B}_{11} & 0 \\ \widetilde{B}_{21} & 0 \end{bmatrix}, \quad (\widetilde{B}_{11}\Sigma_Y^{-1})^* = \widetilde{B}_{11}\Sigma_Y^{-1}.$$

The first condition is equivalent to

$$B = U \begin{bmatrix} \widetilde{B}_{11} & 0 \\ \widetilde{B}_{21} & 0 \end{bmatrix} V^* = [\, U_1 \quad U_2 \,] \begin{bmatrix} U_1^*BV_1 & 0 \\ U_2^*BV_1 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2 \end{bmatrix} = BV_1V_1^* = BY^+Y = BP_{Y^*}.$$

The second condition is equivalent to $(P_Y BY^+)^* = P_Y BY^+$.

We now prove that $\mathcal{S}_{\mathcal{C}_\mathbb{K}} = \widetilde{\mathcal{S}}_{\mathcal{C}_\mathbb{K}}$, where $\widetilde{\mathcal{S}}_{\mathcal{C}_\mathbb{K}} = \{BY^+ + (BY^+)^* P_Y^\perp + P_Y^\perp H P_Y^\perp :\ H \in \mathcal{C}_\mathbb{K}\}$. First, we assume that $A \in \mathcal{S}_{\mathcal{C}_\mathbb{K}} \neq \emptyset$. Then from (2.3) we have

$$\begin{aligned} A &= U \begin{bmatrix} \widetilde{B}_{11}\Sigma_Y^{-1} & \Sigma_Y^{-1}\widetilde{B}_{21}^* \\ \widetilde{B}_{21}\Sigma_Y^{-1} & \widetilde{A}_{22} \end{bmatrix} U^* \\ &= U_1U_1^*BV_1\Sigma_Y^{-1}U_1^* + U_2U_2^*BV_1\Sigma_Y^{-1}U_1^* + U_1\Sigma_Y^{-1}V_1^*B^*U_2U_2^* + U_2U_2^*AU_2U_2^* \\ &= BV_1\Sigma_Y^{-1}U_1^* + Y^{+*}B^*(I - U_1U_1^*) + (I - U_1U_1^*)A(I - U_1U_1^*) \\ &= BY^+ + (BY^+)^* P_Y^\perp + P_Y^\perp A P_Y^\perp \end{aligned}$$

so that $A \in \widetilde{\mathcal{S}}_{\mathcal{C}_{\mathbb{K}}}$ and $\mathcal{S}_{\mathcal{C}_{\mathbb{K}}} \subset \widetilde{\mathcal{S}}_{\mathcal{C}_{\mathbb{K}}}$. Now it is easy to verify that if $BP_{Y^*} = B$ and $P_Y BY^+$ is Hermitian, then any $A \in \widetilde{\mathcal{S}}_{\mathcal{C}_{\mathbb{K}}}$ satisfies $AY = B$ and $A^* = A$ so that $\widetilde{\mathcal{S}}_{\mathcal{C}_{\mathbb{K}}} \subset \mathcal{S}_{\mathcal{C}_{\mathbb{K}}}$, which completes the proof of the first part of the lemma.

For the second part, we have

$$\begin{aligned}
\|A\|_F^2 &= \|\widetilde{A}\|_F^2 \\
&= \left\| \begin{bmatrix} \widetilde{B}_{11}\Sigma_Y^{-1} \\ \widetilde{B}_{21}\Sigma_Y^{-1} \end{bmatrix} \right\|_F^2 + \| \begin{bmatrix} \widetilde{B}_{11}\Sigma_Y^{-1} & \Sigma_Y^{-1}\widetilde{B}_{21}^* \end{bmatrix} \|_F^2 - \|\widetilde{B}_{11}\Sigma_Y^{-1}\|_F^2 + \|\widetilde{A}_{22}\|_F^2 \\
&= 2\|BY^+\|_F^2 - \|U_1^* B V_1 \Sigma_Y^{-1}\|_F^2 + \|\widetilde{A}_{22}\|_F^2.
\end{aligned}$$

Hence $\|A\|_F$ is minimized by setting $\widetilde{A}_{22} = 0$, which implies $P_Y^{\perp} A P_Y^{\perp} = 0$. The expression for A_{opt} follows. ∎

The result of Lemma 2.3 can be extended to other classes of matrices.

LEMMA 2.4. *Let $Y, B \in \mathbb{K}^{m\times k}$, $m \ge k$, be given.*

1. *Let $\mathcal{C}_{\mathbb{K}} = \{A \in \mathbb{K}^{m\times m} : A = -A^*\}$. Then $\mathcal{S}_{\mathcal{C}_{\mathbb{K}}} \neq \emptyset$ if and only $BP_{Y^*} = B$ and $P_Y BY^+ = -(P_Y BY^+)^*$, and if $\mathcal{S}_{\mathcal{C}_{\mathbb{K}}} \neq \emptyset$, then*

$$\begin{aligned}
\mathcal{S}_{\mathcal{C}_{\mathbb{K}}} &= \{BY^+ - (BY^+)^* P_Y^{\perp} + P_Y^{\perp} H P_Y^{\perp} :\ H \in \mathcal{C}_{\mathbb{K}}\}, \\
A_{\text{opt}} &= BY^+ - (BY^+)^* P_Y^{\perp}.
\end{aligned}$$

2. *Let $\mathcal{C}_{\mathbb{C}} = \{A \in \mathbb{C}^{m\times m} : A = A^T\}$. Then $\mathcal{S}_{\mathcal{C}_{\mathbb{C}}} \neq \emptyset$ if and only $BP_{Y^*} = B$ and $P_{\overline{Y}} BY^+ = (P_{\overline{Y}} BY^+)^T$, and if $\mathcal{S}_{\mathcal{C}_{\mathbb{C}}} \neq \emptyset$, then*

$$\begin{aligned}
\mathcal{S}_{\mathcal{C}_{\mathbb{C}}} &= \{BY^+ + (BY^+)^T P_Y^{\perp} + P_{\overline{Y}}^{\perp} H P_Y^{\perp} :\ H \in \mathcal{C}_{\mathbb{C}}\}, \\
A_{\text{opt}} &= BY^+ + (BY^+)^T P_Y^{\perp}.
\end{aligned}$$

3. *Let $\mathcal{C}_{\mathbb{C}} = \{A \in \mathbb{C}^{m\times m} : A = -A^T\}$. Then $\mathcal{S}_{\mathcal{C}_{\mathbb{C}}} \neq \emptyset$ if and only $BP_{Y^*} = B$ and $P_{\overline{Y}} BY^+ = -(P_{\overline{Y}} BY^+)^T$, and if $\mathcal{S}_{\mathcal{C}_{\mathbb{C}}} \neq \emptyset$, then*

$$\begin{aligned}
\mathcal{S}_{\mathcal{C}_{\mathbb{C}}} &= \{BY^+ - (BY^+)^T P_Y^{\perp} + P_{\overline{Y}}^{\perp} H P_Y^{\perp} :\ H \in \mathcal{C}_{\mathbb{C}}\}, \\
A_{\text{opt}} &= BY^+ - (BY^+)^T P_Y^{\perp}.
\end{aligned}$$

Proof. All these results are proved in a similar way to Lemma 2.3. For the symmetric or skew-symmetric case, the matrices $\widetilde{A}$ and $\widetilde{B}$ in (2.2) are defined by $\widetilde{A} = U^T A U$ and $\widetilde{B} = U^T B V$. ∎

Note that Lemma 2.3 solves the Hamiltonian structured matrix problem since JA is Hermitian, and for similar reasons Lemma 2.4 solves the skew-Hamiltonian, J-symmetric, and J-skew-symmetric structured matrix problems.

In the next lemma, we extend a result of Kahan, Parlett, and Jiang [19]. Here, Y and X do not have to have orthonormal columns, we do not require X^*Y to be nonsingular, and Y and X may have different ranks.

LEMMA 2.5. *Let $Y, X, B, C \in \mathbb{K}^{m\times k}$, $m \ge k$, be given with* $\operatorname{rank}(Y) = r$ *and* $\operatorname{rank}(X) = s$, *and let $\mathcal{S}_{\mathbb{K}} = \{A \in \mathbb{K}^{m\times m} :\ AY = B,\ A^*X = C\}$. If $C^*Y = X^*B$, then*

$$\begin{aligned}
\mathcal{S}_{\mathbb{K}} &= \{BY^+ + (CX^+)^* P_Y^{\perp} + P_X^{\perp} H P_Y^{\perp},\ H \in \mathbb{K}^{m\times m}\} \\
&= \{(C^*X^+)^* + P_X^{\perp} BY^+ + P_X^{\perp} H P_Y^{\perp},\ H \in \mathbb{K}^{m\times m}\}, \\
A_{\text{opt}} &= BY^+ + (CX^+)^* P_Y^{\perp} = (C^*X^+)^* + P_X^{\perp} BY^+.
\end{aligned}$$

Proof. Let $\widetilde{\mathcal{S}}^1_{\mathbb{K}} = \{BY^+ + (CX^+)^* P_Y^\perp + P_X^\perp H P_Y^\perp,\ H \in \mathbb{K}^{m\times m}\}$ and $\widetilde{\mathcal{S}}^2_{\mathbb{K}} = \{(C^*X^+)^* + P_X^\perp BY^+ + P_X^\perp H P_Y^\perp,\ H \in \mathbb{K}^{m\times m}\}$. First, we assume that $A \in \mathcal{S}_{\mathbb{K}}$. Let

$$Y = U \begin{bmatrix} \Sigma_Y & 0 \\ 0 & 0 \end{bmatrix} V^* = U_1 \Sigma_Y V_1^*, \quad X = W \begin{bmatrix} \Sigma_X & 0 \\ 0 & 0 \end{bmatrix} Z^* = W_1 \Sigma_X Z_1^*$$

be the full and reduced singular value decompositions of Y and X, $U = [U_1, U_2]$, $W = [W_1, W_2]$ with $U_1 \in \mathbb{K}^{m\times r}$, $W_1 \in \mathbb{K}^{m\times s}$. Partition $V = [V_1, V_2]$ and $Z = [Z_1, Z_2]$ accordingly to U and W and let

$$\widetilde{A} = W^* A U = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}.$$

Then

$$\begin{aligned} \widetilde{A}_{11} &= W_1^* A U_1 = W_1^* B V_1 \Sigma_Y^{-1} = \Sigma_X^{-1} Z_1^* C^* U_1, & \widetilde{A}_{12} &= W_1^* A U_2 = \Sigma_X^{-1} Z_1^* C^* U_2, \\ \widetilde{A}_{21} &= W_2^* A U_1 = W_2^* B V_1 \Sigma_Y^{-1}, & \widetilde{A}_{22} &= W_2^* A U_2. \end{aligned}$$

Now,

$$A = W \widetilde{A} U^* = W_1 \widetilde{A}_{11} U_1^* + W_1 \widetilde{A}_{12} U_2^* + W_2 \widetilde{A}_{21} U_1^* + W_2 \widetilde{A}_{22} U_2^*.$$

Then replacing $\widetilde{A}_{11}$, $\widetilde{A}_{12}$, and $\widetilde{A}_{21}$ by the expressions above yields, for $\widetilde{A}_{11} = W_1^* B V_1 \Sigma_Y^{-1}$,

$$A = BY^+ + (CX^+)^* P_Y^\perp + P_X^\perp A P_Y^\perp,$$

and for $\widetilde{A}_{11} = \Sigma_X^{-1} Z_1^* C^* U_1$,

$$A = (C^*X^+)^* + P_X^\perp BY^+ + P_X^\perp A P_Y^\perp.$$

Hence $\mathcal{S}_{\mathbb{K}} \subset \widetilde{\mathcal{S}}^1_{\mathbb{K}}$ and $\mathcal{S}_{\mathbb{K}} \subset \widetilde{\mathcal{S}}^2_{\mathbb{K}}$. It is easy to verify that if $C^*Y = X^*B$, then any $A \in \widetilde{\mathcal{S}}^1_{\mathbb{K}}$ and any $A \in \widetilde{\mathcal{S}}^2_{\mathbb{K}}$ satisfy $AY = B$ and $A^*X = C$ so that $\widetilde{\mathcal{S}}^1_{\mathbb{K}} = \mathcal{S}_{\mathbb{K}} = \widetilde{\mathcal{S}}^2_{\mathbb{K}}$.

We have

$$\begin{aligned} \|A\|_F^2 &= \|\widetilde{A}\|_F^2 \\ &= \left\| \begin{bmatrix} \widetilde{A}_{11} \\ \widetilde{A}_{21} \end{bmatrix} \right\|_F^2 + \| \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \end{bmatrix} \|_F^2 - \|\widetilde{A}_{11}\|_F^2 + \|\widetilde{A}_{22}\|_F^2 \\ &= \|BV_1\Sigma_Y^{-1}\|_F^2 + \|\Sigma_X^{-1} Z_1^* C^*\|_F^2 - \|W_1^* B V_1 \Sigma_Y^{-1}\|_F^2 + \|\widetilde{A}_{22}\|_F^2. \end{aligned}$$

Hence $\|A\|_F$ is minimized by setting $\widetilde{A}_{22} = 0$, which implies $P_X^\perp A P_Y^\perp = 0$, and the expressions for A_{opt} follow. □

This last result is from [25, Lem. 2.2]. We give the proof for completeness.

LEMMA 2.6. *Let $Y, B \in \mathbb{K}^{m\times k}$, $m \geq k$, be given and let $\mathcal{C}_{\mathbb{K}} = \{A \in \mathbb{K}^{m\times m} : A^*A = I\}$. Then, $\mathcal{S}_{\mathcal{C}_{\mathbb{K}}} \neq \emptyset$ if and only if $Y^*Y = B^*B$, and if $\mathcal{S}_{\mathcal{C}_{\mathbb{K}}} \neq \emptyset$, then*

$$\mathcal{S}_{\mathcal{C}_{\mathbb{K}}} = \{BY^+ + QP_Y^\perp :\ Q \in \mathcal{C}_{\mathbb{K}},\ QP_Y = P_B Q\}.$$

Proof. If $\mathcal{S}_{\mathcal{C}_{\mathbb{K}}} \neq \emptyset$, then $Y^*Y = B^*B$. Now assume that $Y^*Y = B^*B$. Substituting Y by (2.1) into $Y^*Y = B^*B$ gives $BV_2 = 0$ and $BV_1 = Q_1\Sigma$, where $Q_1 \in \mathbb{K}^{m\times r}$ with $Q_1^*Q_1 = I$. Hence

$$B = Q \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*,$$

where $Q = [Q_1, Q_2]$ is unitary. Then $A = QU^* \in \mathcal{S}_{\mathcal{C}_\mathbb{K}}$ and therefore $\mathcal{S}_{\mathcal{C}_\mathbb{K}} \neq \emptyset$.

Let $\widetilde{\mathcal{S}}_{\mathcal{C}_\mathbb{K}} = \{BY^+ + Q(I - YY^+) : \ Q^*Q = I,\ QP_Y = P_BQ\}$. First, we assume that $A \in \mathcal{S}_{\mathcal{C}_\mathbb{K}}$. We can rewrite A as $A = BY^+ + A(I - YY^+)$. Note that since A is unitary, $Y^+ = (A^*B)^+ = B^+A$. Also, $AP_Y = BY^+ = P_BA$ so that $A \in \widetilde{\mathcal{S}}_{\mathcal{C}_\mathbb{K}}$ and $\mathcal{S}_{\mathcal{C}_\mathbb{K}} \subset \widetilde{\mathcal{S}}_{\mathcal{C}_\mathbb{K}}$.

Assume that $A \in \widetilde{\mathcal{S}}_{\mathcal{C}_\mathbb{K}}$. Hence $A = BY^+ + QP_Y^\perp$ for some unitary Q such that $QYY^+ = BB^+Q$. From $AY = B$, $Y^+ = B^+A$, and $YY^+ = (YY^+)^*$ it is easy to show that $Y^+Y = B^+B$. We have

$$\begin{aligned} A^*A &= ((BY^+)^* + (I - P_Y)Q^*)(BY^+ + Q(I - P_Y)) \\ &= (BY^+)^*BY^+ + (BY^+)^*Q(I - P_Y) + (I - P_Y)Q^*BY^+ + I - P_Y. \end{aligned}$$

First,

$$(BY^+)^*BY^+ = Y^{+*}B^*BY^+ = Y^{+*}Y^*YY^* = (YY^+)^*(YY^+) = P_Y,$$

and second,

$$\big((BY^+)^*Q(I - P_Y)\big)^* = (I - P_Y)Q^*BY^+ = Q^*(I - P_B)BY^+ = 0.$$

Hence $A^*A = P_Y + 0 + 0 + I - P_Y = I$. Also $AY = BY^+Y + Q(I - YY^+)Y = BB^+B = B$ so that $A \in \mathcal{S}_{\mathcal{C}_\mathbb{K}}$ and $\widetilde{\mathcal{S}}_{\mathcal{C}_\mathbb{K}} \subset \mathcal{S}_{\mathcal{C}_\mathbb{K}}$, which completes the proof. ◻

3. Structured normwise backward errors.

3.1. Kronecker product approach. Assume that A depends linearly on $t \le m^2$ free parameters and that every element of A is a multiple of a single parameter. We write this dependence as $A = A[p]$ with $p \in \mathbb{K}^t$. Higham and Higham [15], [16] extend the notion of componentwise backward error to allow dependence of the perturbations on a set of parameters, and they define structured componentwise backward errors. We use their approach to rewrite the constraint $A + E \in \mathcal{C}_\mathbb{K}$ in (1.4) as $A + E = A[p + \Delta p]$ or, equivalently, $E = E[\Delta p]$, where Δp is a t-vector of perturbed parameters. Note that if any sparsity of A is included in the structure, then E will have the same sparsity as A.

Applying the vec operator (which stacks the columns of a matrix into one long vector) to the constraints in (1.4) gives

$$(X_k^T \otimes I_m)\,\mathrm{vec}(E) = \mathrm{vec}(R_k), \quad \mathrm{vec}(E) = B\Delta p, \tag{3.1}$$

where $\otimes$ denotes the Kronecker product, $B \in \mathbb{K}^{m^2 \times t}$ is of full rank, and R_k is the residual matrix. We refer to Lancaster and Tismenetsky [20, Chap. 12] for properties of the vec operator and the Kronecker product. Let D be a diagonal matrix such that

$$\|E\|_F = \|D\Delta p\|_2,$$

and let $y = D\Delta p$, $M_k = (X_k^T \otimes I_m)BD^{-1} \in \mathbb{K}^{km \times t}$, and $s_k = \mathrm{vec}(R_k)$. Then we can rewrite (3.1) as the linear system $M_k y = s_k$ and therefore

$$\eta_\mathbb{K}(X_k, \Lambda_k) = \alpha^{-1} \min_{y \in \mathbb{K}^t} \big\{\|y\|_2 \ : \ M_k y = s_k\big\}.$$

This shows that the structured normwise backward error is given in terms of the minimal 2-norm solution to an overdetermined system if $t < km$ or an underdetermined system otherwise. There may be no solution to the system if M_k is rank deficient or if the system is overdetermined. If the system is underdetermined and consistent, then the minimal 2-norm solution is given in terms of the pseudoinverse by $y = M_k^+ s_k$. In this case

$$\eta_{\mathbb{K}}(X_k, \Lambda_k) = \alpha^{-1} \|M_k^+ s_k\|_2. \tag{3.2}$$

When the data A, X_k, Λ_k are all real, then Δp is automatically real. In certain circumstances it is appropriate to restrict Δp to be real even though the data A, X_k, Λ_k are complex. This happens when the constraints on A's structure involve conjugation of its coefficients or, in the case of real structured backward error, when A is real and λ or x is complex. In these cases, the backward error derivation must be modified by taking real and imaginary parts in the constraint $(A+E)x = \lambda x$ to obtain a real system of equations. For example, consider a 2×2 skew-Hermitian matrix E and a single eigenpair (x, λ) $(k = 1)$. Taking real and imaginary parts in the constraint $Ex = r$ yields

$$[F, G] \begin{bmatrix} \mathrm{Re}(x) & \mathrm{Im}(x) \\ -\mathrm{Im}(x) & \mathrm{Re}(x) \end{bmatrix} = [\mathrm{Re}(r), \mathrm{Im}(r)],$$

where $F = \mathrm{Re}(E)$ is skew-symmetric and $G = \mathrm{Im}(E)$ is symmetric. The 2×2 skew-Hermitian E can be parameterized by

$$E = \begin{bmatrix} 0 & -\Delta p_1 \\ \Delta p_1 & 0 \end{bmatrix} + i \begin{bmatrix} \Delta p_2 & \Delta p_3 \\ \Delta p_3 & \Delta p_4 \end{bmatrix}, \quad \Delta p_j \in \mathbb{R}, \quad j = 1\colon 4,$$

so that

$$\mathrm{vec}([F, G]) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta p_1 \\ \Delta p_2 \\ \Delta p_3 \\ \Delta p_4 \end{bmatrix} := B \Delta p.$$

In this case, M_1 in (3.2) is given by

$$M_1 = \left(\begin{bmatrix} \mathrm{Re}(x) & \mathrm{Im}(x) \\ -\mathrm{Im}(x) & \mathrm{Re}(x) \end{bmatrix}^T \otimes I_2 \right) B D^{-1} \in \mathbb{R}^{4 \times 4},$$

with $D = \mathrm{diag}([\|b_1\|_1, \ldots, \|b_4\|_1]) = \mathrm{diag}([2, 1, 2, 1])$ and with b_j being the jth column of B.

Generally, the size of M_k makes the computation of $\eta_{\mathbb{K}}(X_k, \Lambda_k)$ expensive. Thus (3.2) is not a formula we would evaluate routinely in the course of solving a problem. Nevertheless, it is useful as a tool when testing the stability of newly developed structure-preserving algorithms, as shown in [27], or to illustrate instability of well-known algorithms.

As we shall see in the next section, for certain classes of structured matrices it is possible to express the structured backward error in a form that is much less expensive to evaluate than (3.2). We also consider some nonlinear structures that are not covered by this Kronecker product approach.

3.2. A chart of structured backward errors. This section provides a chart of structured backward errors for a set of approximate eigenpairs $(x_j, \lambda_j)_{j=1}^k$ for the singly and doubly structured matrices under consideration. We aim to give, whenever possible, formulae that are cheap to compute so that they can be used routinely in practice. We give an expression for E_{opt}, the solution of minimal Frobenius norm to the constraints in (1.4). We assume that for each class of problems the set of eigenvalues $\{\lambda_j\}_{j=1}^k$ satisfies the relevant eigenvalue properties listed in Table 2.1, since otherwise $\eta_{\mathbb{K}}(X_k, \Lambda_k) = \infty$. For the structured backward error to exist, we may also need to impose some restrictions on X_k.

The first chart, in Table 3.1, covers the complex case, and the second chart, in Table 3.3, covers the real case. They both list structured backward errors that may be applied to the corresponding structured eigenvalue problems. Question marks indicate cases for which explicit expressions for the structured backward errors are not yet known. An **X** or X indicates that an explicit expression for $\eta_{\mathbb{K}}(X_k, \Lambda_k)$ exists. The symbol **X** emphasizes that the structured backward error is at most a factor $\sqrt{2}$ larger than the corresponding unstructured backward error (never smaller). Finally, entries marked with $\otimes$ indicate that an explicit expression for $\eta_{\mathbb{K}}(X_k, \Lambda_k)$ is obtained via the Kronecker product approach described in section 3.1 (which we recall is applicable to linear structure only). Table 3.2 provides the block structure of the corresponding doubly structured matrices together with the matrix properties of the blocks and is useful in forming the matrix B in (3.1).

In the following, "W-trick"[1] refers to the unitary similarity transformation

$$W^*AW = \frac{1}{2}\begin{bmatrix} A_{11} + A_{22} + i(A_{12} - A_{21}) & A_{11} - A_{22} - i(A_{12} + A_{21}) \\ A_{11} - A_{22} + i(A_{12} + A_{21}) & A_{11} + A_{22} - i(A_{12} - A_{21}) \end{bmatrix},$$

where $W = 2^{-\frac{1}{2}}\left[\begin{smallmatrix} I & I \\ iI & -iI \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right] \in \mathbb{K}^{2n\times 2n}$. We define

$$Y_k = \begin{bmatrix} Y_{k,1} \\ Y_{k,2} \end{bmatrix} := W^*X_k \quad \text{and} \quad S_k = \begin{bmatrix} S_{k,1} \\ \mathcal{S}_{k,2} \end{bmatrix} := W^*R_k.$$

The superscript (i,j) in $\eta_{\mathbb{K}}^{(i,j)}$ refers to the class of matrices in position (i,j) of the complex chart if $\mathbb{K} = \mathbb{C}$ and of the real chart if $\mathbb{K} = \mathbb{R}$. Recall that $R_k = X_k\Lambda_k - AX_k$.

3.2.1. Complex chart ($\mathbb{K} = \mathbb{C}$).

Position (1,1): $\mathcal{C}_{\mathbb{C}}^{(1,1)} = \{A \in \mathbb{C}^{m\times m} : A^* = A\}$ is the class of Hermitian matrices. First, we assume that X_k has orthonormal columns,[2] since otherwise $\eta_{\mathbb{C}}^{(1,1)}(X_k, \Lambda_k) = \infty$. We have $X_k^+ = X_k^*$ so that $R_k P_{X_k^*} = R_k$ and $P_{X_k} R_k X_k^* = X_k\Lambda_k X_k^* - X_k X_k^* A X_k X_k^*$ is Hermitian. Hence, from Lemma 2.3 the optimal solution to the constraints in (1.4) is given by

$$E_{\text{opt}} = R_k X_k^* + (X_k R_k^*) P_{X_k}^{\perp}$$

so that

$$\eta_{\mathbb{C}}^{(1,1)}(X_k, \Lambda_k) = \alpha^{-1}\sqrt{\operatorname{trace}(E_{\text{opt}}^* E_{\text{opt}})} = \alpha^{-1}\sqrt{2\|R_k\|_F^2 - \|X_k^* R_k\|_F^2},$$

[1]The term "X-trick" is used in [8]. We use W-trick to avoid confusion with our notation.

[2]In practice, if X_k has columns that are close to being orthonormal, then one can replace them by the unitary factor of either its QR factorization or its polar decomposition.

TABLE 3.1
Summary of the structured backward errors.

		1	2	3	4	5	6	7	8	9	10	11	12
		$A^* = A$	$A^* = -A$	$A^*A = I$	$A^*JA = J$	$JA = (JA)^*$	$JA = -(JA)^*$	$A^T = A$	$A^T = -A$	$A^TA = I$	$A^TJA = J$	$JA = (JA)^T$	$JA = -(JA)^T$
1	$A^* = A$	**X**	∅	**X**	?	X	**X**	**X**	**X**	?	?	⊗	⊗
2	$A^* = -A$		**X**	**X**	?	**X**	X	**X**	**X**	?	?	⊗	⊗
3	$A^*A = I$			**X**	**X**	?	?	?	?	?	?	?	?
4	$A^*JA = J$				?	?	?	?	?	?	?	?	?
5	$JA = (JA)^*$					X	∅	⊗	⊗	?	?	X	X
6	$JA = -JA^*$						X	⊗	⊗	?	?	X	X
7	$A^T = A$							X	∅	?	?	X	**X**
8	$A^T = -A$								X	?	?	X	X
9	$A^TA = I$									?	?	?	?
10	$A^TJA = J$										?	?	?
11	$JA = (JA)^T$											X	∅
12	$JA = -(JA)^T$												X

X: explicit expression for $\eta_{\mathbb{C}}(X_k, \Lambda_k)$ is available and within a factor $\sqrt{2}$ of the unstructured backward error.
X: explicit expression for $\eta_{\mathbb{C}}(X_k, \Lambda_k)$ is available.
?: no explicit expression known for $\eta_{\mathbb{C}}(X_k, \Lambda_k)$.
∅: no nontrivial matrices with the prescribed pair of structures.
⊗: expression available from Kronecker product approach. See Table 3.2 for the block structure of the matrices.

TABLE 3.2
Block structure and block property of some doubly structured matrices.

	$(JA) = (JA)^*$	$(JA) = -(JA)^*$
$A^T = A$	$\begin{bmatrix} A_1 & A_2 \\ \bar{A}_2 & -\bar{A}_1 \end{bmatrix}$, $A_1 = A_1^T$, $\bar{A}_2 = A_2^T$	$\begin{bmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{bmatrix}$, $A_1 = A_1^T$, $A_2^T = -\bar{A}_2$
$A^T = -A$	$\begin{bmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{bmatrix}$, $A_1 = -A_1^T$, $A_2 = A_2^*$	$\begin{bmatrix} A_1 & A_2 \\ \bar{A}_2 & -\bar{A}_1 \end{bmatrix}$, $A_1 = -A_1^T$, $A_2 = -A_2^*$

	$(JA) = (JA)^T$	$(JA) = -(JA)^T$
$A^* = A$	$\begin{bmatrix} A_1 & A_2 \\ A_2^* & -A_1^T \end{bmatrix}$, $A_1 = A_1^*$, $A_2 = A_2^T$	$\begin{bmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{bmatrix}$, $A_1 = A_1^*$, $A_2 = -A_2^T$
$A^* = -A$	$\begin{bmatrix} A_1 & A_2 \\ -A_2^* & -A_1^T \end{bmatrix}$, $A_1 = -A_1^*$, $A_2 = A_2^T$	$\begin{bmatrix} A_1 & A_2 \\ -A_2^* & A_1^T \end{bmatrix}$, $A_1 = -A_1^*$, $A_2 = -A_2^T$

where the second equality follows after some algebra. The expression for $\eta_{\mathbb{C}}^{(1,1)}(X_k, \Lambda_k)$ was obtained in [26, Thm. 2.5.9]. If $\eta(X_k, \Lambda_k)$ denotes the unstructured backward error in (1.3), then

$$\eta(X_k, \Lambda_k) \leq \eta_{\mathbb{C}}^{(1,1)}(X_k, \Lambda_k) \leq \sqrt{2}\, \eta(X_k, \Lambda_k).$$

The first inequality is due to the fact that the class of admissible perturbations is larger for the unstructured case than for the structured case. These inequalities show, as for the structured backward error for Hermitian linear systems [6], [18, Prob. 7.12], that forcing the backward error matrix to be Hermitian has little effect on its norm. Note that for a single eigenpair (x, λ) with x of unit 2-norm and $r = (\lambda I - A)x$ being the residual, E_{opt} is given by

$$E_{\mathrm{opt}} = rx^* + xr^* - (r^*x)xx^*,$$

which is a well-known result in the fields of nonlinear equations and optimization [10], [11, p. 171] and numerical linear algebra [6], [19]. In this case,

$$\eta_{\mathbb{C}}^{(1,1)}(x, \lambda) = \alpha^{-1}\sqrt{2\|r\|_2^2 - (\lambda - x^*Ax)^2}.$$

Position (1,3): $\mathcal{C}_{\mathbb{C}}^{(1,3)} = \{A \in \mathbb{C}^{m\times m} : \ A^* = A,\ A^*A = I\}$ is the class of Hermitian unitary matrices. We assume that the columns of X_k are orthonormal and that $\Lambda_k = \mathrm{diag}(\pm 1)$. The derivation of $\eta_{\mathbb{C}}$ is along the same lines as that for the class of unitary matrices (see position (3,3)) but with an extra constraint in the minimization problem. Therefore, we give just an outline and refer to position (3,3) for a detailed derivation. Let $X = [X_k, \widetilde{X}]$ be unitary. From (3.5) below we have that

$$\begin{aligned} \alpha^2 \eta_{\mathbb{C}}^{(1,3)}(X_k, \Lambda_k)^2 &= \|R\|_F^2 + \min_{\substack{\widetilde{Z}^*\widetilde{Z}=I \\ \widetilde{Z}^*=\widetilde{Z}}} \|\widetilde{X}\widetilde{Z} - A\widetilde{X}\|_F^2 \\ &= \|R\|_F^2 + \|X_k^* A\widetilde{X}\|_F^2 + \min_{\substack{\widetilde{Z}^*\widetilde{Z}=I \\ \widetilde{Z}^*=\widetilde{Z}}} \|\widetilde{Z} - \widetilde{X}^* A \widetilde{X}\|_F^2, \end{aligned} \tag{3.3}$$

where the second equality is obtained using Lemma 2.1. The minimization problem in (3.3) is a nearness problem whose solution is given in terms of $\mathrm{sign}(\widetilde{X}^* A\widetilde{X})$ by [17]

$$\min_{\substack{\widetilde{Z}^*\widetilde{Z}=I \\ \widetilde{Z}^*=\widetilde{Z}}} \|\widetilde{Z} - \widetilde{X}^* A\widetilde{X}\|_F = \|\,\mathrm{sign}(\widetilde{X}^* A\widetilde{X}) - \widetilde{X}^* A\widetilde{X}\|_F.$$

Let $U_k H_k$ and $\widetilde{U}\widetilde{H}$ be the polar decompositions of $X_k^* A X_k$ and $\widetilde{X}^* A\widetilde{X}$, respectively. If $\widetilde{X}^* A\widetilde{X} = QDQ^*$ is the eigendecomposition of $\widetilde{X}^* A\widetilde{X}$ with Q unitary and D real diagonal, then $\widetilde{X}^* A\widetilde{X} = Q\,\mathrm{sign}(D)Q^* Q|D|Q^* = \mathrm{sign}(\widetilde{X}^* A\widetilde{X})Q|D|Q^*$ with $\mathrm{sign}(\widetilde{X}^* A\widetilde{X})$ unitary and $Q|D|Q^*$ Hermitian positive definite. Hence, we have $\widetilde{U} = \mathrm{sign}(\widetilde{X}^* A\widetilde{X})$ and $\widetilde{H} = Q|D|Q^*$ so that

$$\|\,\mathrm{sign}(\widetilde{X}^* A\widetilde{X}) - \widetilde{X}^* A\widetilde{X}\|_F = \|\widetilde{U} - \widetilde{U}\widetilde{H}\|_F = \|I - \widetilde{H}\|_F.$$

By Lemmas 2.2 and 2.1 we have

$$\|I - \widetilde{H}\|_F^2 = \|I - H_k\|_F^2 = \|U_k - X_k^* A X_k\|_F^2 = \|X_k U_k - A X_k\|_F^2 - \|X_k^* A\widetilde{X}\|_F^2.$$

Then replacing the minimization problem in (3.3) by the above expression yields

$$\eta_{\mathbb{C}}^{(1,3)}(X_k, \Lambda_k) = \alpha^{-1}\sqrt{\|R_k\|_F^2 + \|X_k U_k - A X_k\|_F^2}.$$

Note that $\eta_{\mathbb{C}}^{(1,3)}(X_k, \Lambda_k) = \eta_{\mathbb{C}}^{(3,3)}(X_k, \Lambda_k)$, where $(3,3)$ refers to the class of unitary matrices. Let $\eta(X_k, \Lambda_k)$ be the unstructured backward error. Then, as in position (3,3), we have

$$\eta(X_k, \Lambda_k) \le \eta_{\mathbb{C}}^{(1,3)}(X_k, \Lambda_k) \le \sqrt{2}\,\eta(X_k, \Lambda_k),$$

showing that forcing $A + E$ to be Hermitian and unitary has little effect on its norm.

Position (1,5): $\mathcal{C}_{\mathbb{C}}^{(1,5)} = \{A \in \mathbb{C}^{2n\times 2n} :\ A^* = A,\ JA = (JA)^*\}$ is the class of Hermitian Hamiltonian matrices. Note that $A \in \mathcal{C}_{\mathbb{C}}$ has the form $\left[\begin{smallmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{smallmatrix}\right]$ with $A_1 = A_1^*$ and $A_2 = A_2^*$ and that the W-trick gives

$$W^* A W = \begin{bmatrix} 0 & \widetilde{A} \\ \widetilde{A}^* & 0 \end{bmatrix}, \quad \widetilde{A} = A_1 - iA_2.$$

Hence, using the W-trick, the constraints in (1.4) can be rewritten as

$$\widetilde{E} Y_{k,2} = S_{k,1}, \qquad \widetilde{E}^* Y_{k,1} = S_{k,2}, \qquad \widetilde{E} = E_1 - iE_2 \in \mathbb{C}^{n\times n},$$

because E is transformed in the same way as A. If $S_{k,2}^* Y_{k,2} = Y_{k,1}^* S_{k,1}$, then

$$\eta_{\mathbb{C}}^{(1,5)}(X_k, \Lambda_k) = \frac{\sqrt{2}}{\alpha}\|E_{\mathrm{opt}}\|_F, \quad \widetilde{E}_{\mathrm{opt}} = S_{k,1} Y_{k,2}^+ + (S_{k,2} Y_{k,1}^+)^* P_{Y_{k,2}}^\perp,$$

using Lemma 2.5.

Note that if X_k and Λ_k are such that $X_k^* X_k = I$, $X_k^* J X_k = J$ and $J\Lambda_k = (J\Lambda_k)^*$ is Hamiltonian, then we can show that the assumption $S_{k,2}^* Y_{k,2} = Y_{k,1}^* S_{k,1}$ is satisfied.

Position (1,6): $\mathcal{C}_{\mathbb{C}}^{(1,6)} = \{A \in \mathbb{C}^{2n\times 2n} :\ A^* = A,\ JA = -(JA)^*\}$ is the class of Hermitian skew-Hamiltonian matrices. Note that $A \in \mathcal{C}_{\mathbb{C}}$ has the form $\left[\begin{smallmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{smallmatrix}\right]$

with $A_1 = A_1^*$ and $A_2 = -A_2^*$ and that the W-trick diagonalizes A, $\widetilde{A} = W^*AW = \mathrm{diag}(\widetilde{A}_1, \widetilde{A}_2)$, where $\widetilde{A}_1 = A_1 + iA_2$ and $\widetilde{A}_2 = A_1 - iA_2$ are Hermitian. Hence, the $2n \times 2n$ skew-Hermitian problem can be reduced to two $n \times n$ Hermitian eigenproblems that can be solved independently. We refer to position (1,1) for the corresponding backward error.

Positions (1,7), (1,8): $A \in \mathbb{C}^{m\times m}$, $A^* = A$, and $A^T = A$ (or $A^T = -A$) imply that A is real symmetric (or iA is real skew-symmetric). Hence

$$\eta_{\mathbb{C}}^{(1,7)}(X_k, \Lambda_k) = \eta_{\mathbb{R}}^{(1,1)}(X_k, \Lambda_k), \qquad \eta_{\mathbb{C}}^{(1,8)}(X_k, \Lambda_k) = \eta_{\mathbb{R}}^{(2,2)}(X_k, i\Lambda_k).$$

Positions $(2, k)$, $k = 2\colon 12$: Each of these classes consists of matrices which are the scalar i times matrices in the corresponding classes in row 1. Hence

$$\begin{aligned} \eta_{\mathbb{C}}^{(2,2)}(X_k, \Lambda_k) &= \eta_{\mathbb{C}}^{(1,1)}(X_k, i\Lambda_k), & \eta_{\mathbb{C}}^{(2,3)}(X_k, \Lambda_k) &= \eta_{\mathbb{C}}^{(1,3)}(X_k, i\Lambda_k), \\ \eta_{\mathbb{C}}^{(2,5)}(X_k, \Lambda_k) &= \eta_{\mathbb{C}}^{(1,6)}(X_k, i\Lambda_k), & \eta_{\mathbb{C}}^{(2,6)}(X_k, \Lambda_k) &= \eta_{\mathbb{C}}^{(1,5)}(X_k, i\Lambda_k), \\ \eta_{\mathbb{C}}^{(2,7)}(X_k, \Lambda_k) &= \eta_{\mathbb{C}}^{(1,7)}(X_k, i\Lambda_k), & \eta_{\mathbb{C}}^{(2,8)}(X_k, \Lambda_k) &= \eta_{\mathbb{C}}^{(1,8)}(X_k, i\Lambda_k). \end{aligned}$$

Position (3,3): $\mathcal{C}_{\mathbb{C}}^{(3,3)} = \{A \in \mathbb{C}^{m\times m} : \ A^*A = I\}$ is the class of unitary matrices. We use Sun's approach [25] to derive $\eta_{\mathbb{C}}(X_k, \Lambda_k)$. First, we assume that the columns of X_k are orthonormal. As $X_k^*X_k = (X_k\Lambda_k)^*X_k\Lambda_k = I_k$, then from Lemma 2.6, solutions of $(A+E)X_k = X_k\Lambda_k$ with $A+E$ unitary exist and have the form

$$A + E = X_k\Lambda_k X_k^* + Q(I - X_kX_k^*) \tag{3.4}$$

with $Q \in \mathcal{C}_{\mathbb{C}}^{(3,3)}$ such that $QX_kX_k^* = X_kX_k^*Q$. Substituting $X_kX_k^* = X\,\mathrm{diag}(I_k, 0)X^*$, where $X = [X_k, \widetilde{X}]$ is unitary, into $QX_kX_k^* = X_kX_k^*Q$, yields

$$X^*QX\,\mathrm{diag}(I_k, 0) = \mathrm{diag}(I_k, 0)X^*QX$$

which implies $\widetilde{X}^*QX_k = 0$ and $X_k^*Q\widetilde{X} = 0$ or, equivalently,

$$Q = X\begin{bmatrix} Z_k & 0 \\ 0 & \widetilde{Z}\end{bmatrix}X^*, \quad Z = \mathrm{diag}(Z_k, \widetilde{Z}) \in \mathcal{C}_{\mathbb{C}}^{(3,3)}.$$

Hence, from (3.4)

$$E = X_k\Lambda_kX_k^* + \widetilde{X}\widetilde{Z}\widetilde{X}^* - A = [(X_k\Lambda_k - AX_k), (\widetilde{X}\widetilde{Z} - A\widetilde{X})]X^*$$

so that

$$\alpha^2\eta_{\mathbb{C}}^{(3,3)}(X_k, \Lambda_k)^2 = \|R\|_F^2 + \min_{\widetilde{Z}^*\widetilde{Z}=I}\|\widetilde{X}\widetilde{Z} - A\widetilde{X}\|_F^2. \tag{3.5}$$

Let U_kH_k and $\widetilde{U}\widetilde{H}$ be the polar decompositions of $X_k^*AX_k$ and $\widetilde{X}^*A\widetilde{X}$, respectively. The minimization problem in (3.5) is a well-known Procrustes problem [13, p. 149] whose solution is given by

$$\min_{\widetilde{Z}^*\widetilde{Z}=I}\|\widetilde{X}\widetilde{Z} - A\widetilde{X}\|_F^2 = \|\widetilde{X}\widetilde{U} - A\widetilde{X}\|_F^2.$$

By applying Lemma 2.1, then Lemma 2.2, and finally Lemma 2.1 again, we have

$$\begin{aligned} \|\widetilde{X}\widetilde{U} - A\widetilde{X}\|_F^2 &= \|X_k^*A\widetilde{X}\|_F^2 + \|\widetilde{U} - \widetilde{X}^*A\widetilde{X}\|_F^2 \\ &= \|\widetilde{X}^*AX_k\|_F^2 + \|U_k - X_k^*AX_k\|_F^2 \\ &= \|X_kU_k - AX_k\|_F^2. \end{aligned}$$

Hence

$$\eta_{\mathbb{C}}^{(3,3)}(X_k,\Lambda_k)=\alpha^{-1}\sqrt{\|R_k\|_F^2+\|X_kU_k-AX_k\|_F^2}\le \alpha^{-1}\sqrt{2}\,\|R_k\|_F=\sqrt{2}\,\eta(X_k,\Lambda_k),$$

where the inequality follows from

$$\|X_kU_k-AX_k\|_F=\min_{Z_k^*Z_k=I}\|X_kZ_k-AX_k\|_F\le\|X_k\Lambda_k-AX_k\|_F=\|R_k\|_F.$$

This is another example in which forcing the backward error matrix to be unitary has little effect on its norm.

Position (3,4): $\mathcal{C}_{\mathbb{C}}^{(3,4)}=\{A\in\mathbb{C}^{2n\times 2n}:\ A^*A=I,\ A^*JA=J\}$ is the class of symplectic unitary matrices. Matrices in this class have the form $A=\left[\begin{smallmatrix}A_1 & -A_2\\ A_2 & A_1\end{smallmatrix}\right]$ and are diagonalized by the W-trick, $W^*AW=\operatorname{diag}(\widetilde{A}_1,\widetilde{A}_2)$ with $\widetilde{A}_1=A_1-iA_2$, $\widetilde{A}_2=A_1+iA_2$ unitary. Hence, the $2n\times 2n$ original eigenvalue problem can be reduced to two $n\times n$ unitary eigenproblems that can be solved independently. Position (3,3) provides an explicit expression of the corresponding structured backward error.

Position (5,5): $\mathcal{C}_{\mathbb{C}}^{(5,5)}=\{A\in\mathbb{C}^{2n\times 2n}:\ JA=(JA)^*\}$ is the class of Hamiltonian matrices. The constraints in (1.4) can be rewritten as $JEX_k=JR_k$ with JE Hermitian. If

$$JR_kP_{X_k^*}=JR_k\quad\text{and}\quad P_{X_k}(JR_k)X_k^+\ \text{is Hermitian,}\tag{3.6}$$

then

$$\eta_{\mathbb{C}}^{(5,5)}(X_k,\Lambda_k)=\alpha^{-1}\|E_{\rm opt}\|_F\quad\text{with}\quad E_{\rm opt}=R_kX_k^++(JR_kX_k^+J)^*P_{X_k}^{\perp}$$

using Lemma 2.3.

For a single approximate eigenpair (x,λ), the assumptions in (3.6) are always satisfied and, for x of unit 2-norm,

$$\eta_{\mathbb{C}}^{(5,5)}(x,\lambda)=\alpha^{-1}\sqrt{2\|r\|_2^2-\|x^*Jr\|_2^2}\le\sqrt{2}\,\eta(x,\lambda).$$

Hence, for a single eigenpair, forcing the backward error matrix to be Hamiltonian has little effect on its norm.

For a set of k approximate eigenpairs (X_k,Λ_k), if Λ_k is Hamiltonian, which implies that $k=2r$ is even and $\Lambda_k=\operatorname{diag}(\widetilde{\Lambda}_r,\widetilde{\Lambda}_r^*)$, and if $X_k^*JX_k=J$ with X_k of full rank, then we can show that the assumptions in (3.6) are satisfied and therefore $\eta_{\mathbb{C}}^{(5,5)}(X_k,\Lambda_k)$ is guaranteed to be finite.

Position (5,11): $\mathcal{C}_{\mathbb{C}}^{(5,11)}=\{A\in\mathbb{C}^{2n\times 2n}:\ JA=(JA)^*,\ JA=(JA)^T\}$. Matrices in this class are real and therefore

$$\eta_{\mathbb{C}}^{(5,11)}(X_k,\Lambda_k)=\eta_{\mathbb{R}}^{(5,5)}(X_k,\Lambda_k),$$

where $\eta_{\mathbb{R}}^{(5,5)}$ refer to position (5,5) of the real chart (see Table 3.3).

Position (5,12): $\mathcal{C}_{\mathbb{C}}^{(5,12)}=\{A\in\mathbb{C}^{2n\times 2n}:\ JA=(JA)^*,\ JA=-(JA)^T\}$. $A\in\mathcal{C}_{\mathbb{C}}$ implies that (iA) is real and satisfies $(J(iA))=(J(iA))^T$. Hence

$$\eta_{\mathbb{C}}^{(5,12)}(X_k,\Lambda_k)=\eta_{\mathbb{R}}^{(5,5)}(X_k,i\Lambda_k).$$

Positions $(6,j)$, $j=6\colon 12$: Each of these classes consists of matrices which are the scalar i times matrices in the corresponding classes in row 5.

Position (7,7): $\mathcal{C}_{\mathbb{C}}^{(7,7)} = \{A \in \mathbb{C}^{m\times m} : A^T = A\}$ is the class of complex symmetric matrices. From Lemma 2.4 if $R_k X_k^+ X_k = R_k$ and $\bar{X}_k \bar{X}_k{}^+ R_k X_k^+ = (\bar{X}_k \bar{X}_k{}^+ R_k X_k^+)^T$, then

$$\eta_{\mathbb{C}}^{(7,7)}(X_k, \Lambda_k) = \alpha^{-1}\|E_{\rm opt}\|_F \quad \text{with} \quad E_{\rm opt} = R_k X_k^+ + (R_k X_k^+)^T P_{\bar{X}_k}^{\perp}.$$

Position (7,11): $\mathcal{C}_{\mathbb{C}}^{(7,11)} = \{A \in \mathbb{C}^{m\times m} : A^T = A,\ JA = (JA)^T\}$. Matrices in this class have the form $\left[\begin{smallmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{smallmatrix}\right]$ with A_1, A_2 complex symmetric. The W-trick gives

$$W^*AW = \begin{bmatrix} 0 & \widetilde{A}_1 \\ \widetilde{A}_2 & 0 \end{bmatrix}, \quad \widetilde{A}_1 = \widetilde{A}_1^T,\ \widetilde{A}_2 = \widetilde{A}_2^T.$$

Hence using the W-trick, the constraints in (1.4) can be rewritten as

$$\widetilde{E}_1 Y_{k,2} = S_{k,1}, \quad \widetilde{E}_2 Y_{k,1} = S_{k,2}, \qquad \widetilde{E}_1 = \widetilde{E}_1^T,\ \widetilde{E}_2 = \widetilde{E}_2^T \in \mathbb{C}^{n\times n}. \tag{3.7}$$

If $S_{k,1} P_{Y_{k,2}^*} = S_{k,1}$ and $S_{k,2} P_{Y_{k,1}^*} = S_{k,2}$, and if $P_{\overline{Y}_{k,2}} S_{k,1} Y_{k,2}^+$ and $P_{\overline{Y}_{k,1}} S_{k,2} Y_{k,1}^+$ are complex symmetric, then

$$\eta_{\mathbb{C}}^{(7,11)}(X_k, \Lambda_k) = \alpha^{-1}\sqrt{\|\widetilde{E}_{1_{\rm opt}}\|_F^2 + \|\widetilde{E}_{2_{\rm opt}}\|_F^2},$$

where, using Lemma 2.4,

$$\widetilde{E}_{1_{\rm opt}} = S_{k,1} Y_{k,2}^+ + (S_{k,1} Y_{k,2}^+)^T P_{\bar{Y}_{k,2}}^{\perp}, \quad \widetilde{E}_{1_{\rm opt}} = S_{k,2} Y_{k,1}^+ + (S_{k,2} Y_{k,1}^+)^T P_{\bar{Y}_{k,1}}^{\perp}.$$

Position (7,12): $\mathcal{C}_{\mathbb{C}}^{(7,12)} = \{A \in \mathbb{C}^{m\times m} : A^T = A,\ JA = -(JA)^T\}$. Matrices in this class have the form $\left[\begin{smallmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{smallmatrix}\right]$ with A_1 complex symmetric and A_2 complex skew-symmetric, and they are diagonalized by the W-trick, $W^*AW = \operatorname{diag}(\widetilde{A}_1, \widetilde{A}_1^T)$. Then the $2n \times 2n$ original eigenvalue problem is reduced to one $n \times n$ eigenproblem with $\widetilde{A}_1$ of no particular structure. Hence, one can use the formula for the unstructured backward error in (1.3).

Position (8,8): $\mathcal{C}_{\mathbb{C}}^{(8,8)} = \{A \in \mathbb{C}^{m\times m} : A^T = -A\}$ is the class of complex skew-symmetric matrices. From Lemma 2.4, if $R_k X_k^+ X_k = R_k$ and $\overline{X}_k \overline{X}_k^+ R_k X_k^+ = -(\overline{X}_k \overline{X}_k^+ R_k X_k^+)^T$, then the optimal solution to the constraints in (1.4) is given by

$$E_{\rm opt} = R_k X_k^+ - (R_k X_k^+)^T P_{\bar{X}_k}^{\perp}$$

and then

$$\eta_{\mathbb{C}}^{(8,8)}(X_k, \Lambda_k) = \alpha^{-1}\|E_{\rm opt}\|_F.$$

Position (8,11): $\mathcal{C}_{\mathbb{C}}^{(8,11)} = \{A \in \mathbb{C}^{m\times m} : A^T = -A,\ JA = (JA)^T\}$. Matrices in this class have the form $\left[\begin{smallmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{smallmatrix}\right]$ with $A_1^T = -A_1$ and $A_2^T = -A_2$. They are diagonalized by the W-trick, $W^*AW = \operatorname{diag}(\widetilde{A}_1, \widetilde{A}_2)$ with $\widetilde{A}_1$, $\widetilde{A}_2$ complex skew-symmetric. Hence, the $2n \times 2n$ original eigenvalue problem can be reduced to two $n \times n$ complex skew-symmetric eigenvalue problems that can be solved independently. We refer to position (8,8) for an explicit expression of the corresponding structured backward error.

Table 3.3
Summary of the structured backward errors for real matrices.

	1 $A^T = A$	2 $A^T = -A$	3 $A^TA = I$	4 $A^TJA = J$	5 $JA = (JA)^T$	6 $JA = -(JA)^T$
1 $A^T = A$	**X**	$\emptyset$	**X**	?	S,$\otimes$	S,$\otimes$
2 $A^T = -A$		**X**	?	?	S,$\otimes$	$\otimes$
3 $A^TA = I$			**X**	?	?	?
4 $A^TJA = J$				?	?	?
5 $JA = (JA)^T$					X	$\emptyset$
6 $JA = -(JA)^T$						X

X: explicit expression for $\eta_{\mathbb{R}}(X_k, \Lambda_k)$ is available and within a factor $\sqrt{2}$ of the unstructured backward error.
X: explicit expression for $\eta_{\mathbb{R}}(X_k, \Lambda_k)$ is available.
S: explicit backward error available for a single eigenpair (x, λ).
?: no explicit backward error known.
$\emptyset$: no nontrivial matrices with the prescribed pair of structures.
$\otimes$: expression available from Kronecker product approach.
See Table 3.2 for the block structure of the matrices.

Position (8,12): $\mathcal{C}_{\mathbb{C}}^{(8,12)} = \{A \in \mathbb{C}^{2n\times 2n} : \ A^T = -A, \ JA = -(JA)^T\}$. Matrices in this class have the form $\left[\begin{smallmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{smallmatrix}\right]$ with A_1, A_2 complex skew-symmetric. Using the W-trick, the constraints in (1.4) become

$$\widetilde{E}_1 Y_{k,2} = S_{k,1}, \qquad \widetilde{E}_2 Y_{k,1} = S_{k,2}, \qquad \widetilde{E}_1^T = -\widetilde{E}_1, \ \widetilde{E}_2^T = -\widetilde{E}_2 \in \mathbb{C}^{n\times n}$$

with $\widetilde{E}_1 = E_1 - iE_2$ and $\widetilde{E}_2 = E_1 + iE_2$. If the assumptions in Lemma 2.4 are satisfied, then

$$\eta_{\mathbb{C}}^{(8,12)}(X_k, \Lambda_k) = \sqrt{\|S_{k,1}Y_{k,2}^+ - (S_{k,1}Y_{k,2}^+)^T P_{Y_{k,2}}^{\perp}\|_F^2 + \|S_{k,2}Y_{k,1}^+ - (S_{k,2}Y_{k,1}^+)^T P_{Y_{k,1}^{\perp}}\|_F^2}.$$

Position (11,11): $\mathcal{C}_{\mathbb{C}}^{(11,11)} = \{A \in \mathbb{C}^{2n\times 2n} : \ JA = (JA)^T\}$ is the class of J-symmetric Hamiltonian matrices. If $R_k P_{X_k^*} = R_k$ and $P_{\bar{X}_k} JR_k X_k^+ = (P_{\bar{X}_k} JR_k X_k^+)^T$, then from Lemma 2.4

$$\eta_{\mathbb{C}}^{(11,11)}(X_k, \Lambda_k) = \alpha^{-1}\|JR_kX_k^+ + (JR_kX_k^+)^T P_{X_k^{\perp}}\|_F.$$

Position (12,12): $\mathcal{C}_{\mathbb{C}}^{(12,12)} = \{A \in \mathbb{C}^{2n\times 2n} : \ JA = -(JA)^T\}$ is the class of J-symmetric Hamiltonian matrices. If $R_k P_{X_k^*} = R_k$ and $P_{\bar{X}_k} JR_k X_k^+ = -(P_{\bar{X}_k} JR_k X_k^+)^T$, then from Lemma 2.4

$$\eta_{\mathbb{C}}^{(12,12)}(X_k, \Lambda_k) = \alpha^{-1}\|JR_kX_k^+ - (JR_kX_k^+)^T P_{X_k^{\perp}}\|_F.$$

3.2.2. Real chart ($\mathbb{K} = \mathbb{R}$). When the matrix of the structured eigenvalue problem is real, it is natural to consider perturbation matrices E that are real, too. This problem is addressed in this section and the results are summarized in Table 3.3. The W-trick cannot be used since the transformation with W would send our real problem to the complex space. We have to use the Kronecker product approach instead.

Position (1,1): $\mathcal{C}_{\mathbb{R}}^{(1,1)} = \{A \in \mathbb{R}^{m\times m} : A^T = A\}$ is the class of real symmetric matrices. For Λ_k and X_k real and such that $X_k^T X_k = I$, we have $X_k^+ = X_k^T$ so that $R_k P_{X_k^T} = R_k$ and $P_{X_k} R_k X_k^T = X_k \Lambda_k X_k^T - X_k X_k^T A X_k X_k^T$ is symmetric. Hence, from Lemma 2.3 applied with $\mathbb{K} = \mathbb{R}$ the optimal solution to $EX_k = R_k$ with $E^T = E$ is given by

$$E_{\text{opt}} = R_k X_k^T + (X_k R_k^T) P_{X_k}^{\perp}$$

so that

$$\eta_{\mathbb{R}}^{(1,1)}(X_k, \Lambda_k) = \alpha^{-1}\sqrt{\operatorname{trace}(E_{\text{opt}}^T E_{\text{opt}})} = \alpha^{-1}\sqrt{2\|R_k\|_F^2 - \|X_k^T R_k\|_F^2}$$

and, as in the complex case,

$$\eta(X_k, \Lambda_k) \le \eta_{\mathbb{R}}^{(1,1)}(X_k, \Lambda_k) \le \sqrt{2}\,\eta(X_k, \Lambda_k).$$

Position (1,3): $\mathcal{C}_{\mathbb{R}}^{(1,3)} = \{A \in \mathbb{R}^{m\times m} : \ A^T = A, A^T A = I\}$ is the class of symmetric unitary matrices. As all the eigenvalues are ± 1, we can take X_k real and apply Lemma 2.6 with $\mathbb{K} = \mathbb{R}$. The derivation for the backward error for position (1,2) of the complex chart remains valid in real arithmetic and, therefore,

$$\eta_{\mathbb{R}}^{(1,3)}(X_k, \Lambda_k) = \alpha^{-1}\sqrt{\|R\|_F^2 + \|X_k U_k - A X_k\|_F^2},$$

where U_k is the orthogonal factor of the polar factorization of $X_k^T A X_k$.

Positions (1,5): $\mathcal{C}_{\mathbb{R}}^{(1,5)} = \{A \in \mathbb{R}^{2n\times 2n} : \ A^T = A, \ JA = (JA)^T\}$ is the class of symmetric Hamiltonian matrices. The backward error for this problem is considered in [27], where it is shown that for a single eigenpair (x, λ) with x of unit 2-norm,

$$\eta_{\mathbb{R}}^{(1,5)}(x, \lambda) = \alpha^{-1}\sqrt{2\|r\|_2^2 + 2(e_2^T Q^T r)^2},$$

with $e_2 = [0, 1, 0, \ldots, 0]^T$, $r = (\lambda I - A)x$, and Q the orthogonal factor in the symplectic quasi-QR factorization of $[x \ r]$. For a set of eigenpairs, an explicit expression for $\eta_{\mathbb{R}}^{(1,5)}(X_k, \Lambda_k)$ is obtained through the Kronecker product approach.

Positions (1,6): $\mathcal{C}_{\mathbb{R}}^{(1,6)} = \{A \in \mathbb{R}^{2n\times 2n} : \ A^T = A, \ JA = -(JA)^T\}$ is the class of symmetric skew-Hamiltonian matrices. The structured backward error for this class of problems is also considered in [27], where it is shown that for a single eigenpair (x, λ) with x of unit 2-norm,

$$\eta_{\mathbb{R}}^{(1,5)}(x, \lambda) = \alpha^{-1}\sqrt{2\|r\|_2^2 + 2(e_2^T \widetilde{Q}^T r)^2},$$

with $e_2 = [0, 1, 0, \ldots, 0]^T$, $r = (\lambda I - A)x$, and $\widetilde{Q}$ the orthogonal factor in the symplectic quasi-QR factorization of $[Jx \ r]$. For a set of eigenpairs, we need to use the Kronecker product approach.

Position (2,2): $\mathcal{C}_{\mathbb{R}}^{(2,2)} = \{A \in \mathbb{R}^{m\times m} : \ A^T = -A\}$ is the class of real skew-symmetric matrices. We assume that the spectrum of Λ_k is symmetric with respect to the real axis and that X_k has orthonormal columns. There exists a $k\times k$ unitary matrix N such that $Y_k = X_k N \in \mathbb{R}^{m\times k}$ and $\Omega_k = N^* \Lambda_k N \in \mathbb{R}^{k\times k}$ is block diagonal with 1×1 blocks equal to 0 and 2×2 blocks of the form $\left[\begin{smallmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{smallmatrix}\right]$. We have $\eta_{\mathbb{R}}^{(2,2)}(X_k, \Lambda_k) =$

$\eta_{\mathbb{R}}^{(2,2)}(Y_k, \Omega_k)$. Let $\widetilde{R}_k = Y_k\Omega_k - AY_k$. Since $\widetilde{R}_k P_{Y_k^T} = \widetilde{R}_k$ and $P_{Y_k}\widetilde{R}_k Y_k^T$ is skew-symmetric, Lemma 2.4 applies with $\mathbb{K} = \mathbb{R}$. The optimal solution to $EY_k = \widetilde{R}_k$ with $E^T = -E$ is given by

$$E_{\text{opt}} = \widetilde{R}_k Y_k^T + (Y_k \widetilde{R}_k^T) P_{Y_k}^{\perp}$$

so that

$$\eta_{\mathbb{R}}^{(2,2)}(X_k, \Lambda_k) = \alpha^{-1}\sqrt{2\|\widetilde{R}_k\|_F^2 - \|Y_k^T \widetilde{R}_k\|_F^2} = \alpha^{-1}\sqrt{2\|R_k\|_F^2 - \|X_k^T R_k\|_F^2}.$$

Hence $\eta_{\mathbb{R}}^{(2,2)}(X_k, \Lambda_k) \le \sqrt{2}\,\eta(X_k, \Lambda_k)$, showing that forcing the backward error matrix to be real skew-symmetric has little effect on its norm.

Positions (3,3): $\mathcal{C}_{\mathbb{R}}^{(3,3)} = \{A \in \mathbb{R}^{m\times m} :\ A^T A = I\}$ is the class of orthogonal matrices. We assume that the spectrum of Λ_k is symmetric with respect to the real axis and that X_k has orthonormal columns. There exists a unitary $k \times k$ matrix N such that $Y_k = X_k N \in \mathbb{R}^{m\times k}$ and $\Omega_k = N^* \Lambda_k N \in \mathbb{R}^{k\times k}$ is block diagonal with 1×1 blocks equal to ± 1 and 2×2 blocks of the form $\left[\begin{smallmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{smallmatrix}\right]$, $\sin\theta \ne 0$. We have $\eta_{\mathbb{R}}^{(3,3)}(X_k, \Lambda_k) = \eta_{\mathbb{R}}^{(3,3)}(Y_k, \Omega_k)$. With (X_k, Λ_k) replaced by (Y_k, Ω_k), the technique described in position $(3,3)$ constructs a real solution E_{opt} of minimal Frobenius norm to the constraints in (1.4). Finally, we end up with $\eta_{\mathbb{R}}^{(3,3)}(X_k, \Lambda_k) = \eta_{\mathbb{C}}^{(3,3)}(X_k, \Lambda_k)$.

Positions (2,5): $\mathcal{C}_{\mathbb{R}}^{(2,5)} = \{A \in \mathbb{R}^{2n\times 2n} :\ A^T = -A,\ JA = (JA)^T\}$ is the class of skew-symmetric Hamiltonian matrices. We assume that λ is purely imaginary and $x = \left[\begin{smallmatrix}x_1 \\ x_2\end{smallmatrix}\right]$ with $x_2 = \pm i x_1$ has unit 2-norm. It is shown in [27] that

$$\eta_{\mathbb{R}}^{(2,5)}(x, \lambda) = \alpha^{-1}\sqrt{2\|s\|_2^2 + 2(e_2^T Q^T s)^2},$$

where $e_2 = [0, 1, 0, \ldots, 0]^T$ and Q is the orthogonal factor in the symplectic quasi-QR factorization of $[w\ s] = [\,I \quad -(A + \sigma i\lambda I)\,]\left[\begin{smallmatrix}\mathrm{Re}(x_1) \\ -\sigma\,\mathrm{Im}(x_1)\end{smallmatrix}\right]$ with $\sigma = 1$ if $x_2 = i x_1$ or $\sigma = -1$ otherwise. The computation of $\eta_{\mathbb{R}}^{(2,5)}(x, \lambda)$ can be done in $O(n^2)$ operations. For a set of eigenpairs, we refer to the Kronecker product approach.

Position (5,5): $\mathcal{C}_{\mathbb{R}}^{(5,5)} = \{A \in \mathbb{R}^{2n\times 2n} :\ JA = (JA)^T\}$ is the class of Hamiltonian matrices. We assume that $k \le n$. The constraints in (1.4) can be rewritten as $JE\widetilde{X}_{2k} = J\widetilde{R}_{2k}$, $JE = (JE)^T$, where $\widetilde{X}_{2k} = [\mathrm{Re}(X_k)\ \ \mathrm{Im}(X_k)]$, $\widetilde{R}_{2k} = [\mathrm{Re}(R_k)\ \ \mathrm{Im}(R_k)]$. If

$$\widetilde{R}_{2k} P_{\widetilde{X}_{2k}}^T = \widetilde{R}_{2k} \quad \text{and} \quad P_{\widetilde{X}_{2k}} J\widetilde{R}_{2k}\widetilde{X}_{2k}^T = (P_{\widetilde{X}_{2k}} J\widetilde{R}_{2k}\widetilde{X}_{2k}^T)^T,$$

then $\eta_{\mathbb{R}}^{(5,5)}(X_k, \Lambda_k) = \alpha^{-1}\|E_{\text{opt}}\|_F$ where, using Lemma 2.3,

$$E_{\text{opt}} = \widetilde{R}_{2k}\widetilde{X}_{2k}^{+} + J(\widetilde{R}_{2k}\widetilde{X}_{2k}^{+})^T J P_{\widetilde{X}_{2k}}^{\perp} \in \mathbb{R}^{2n\times 2n}.$$

Position (6,6): $\mathcal{C}_{\mathbb{R}}^{(6,6)} = \{A \in \mathbb{R}^{2n\times 2n} :\ JA = -(JA)^T\}$ is the class of skew-Hamiltonian matrices. We assume that $k \le n$. The constraints in (1.4) can be rewritten as $JE\widetilde{X}_{2k} = J\widetilde{R}_{2k}$, $JE = (JE)^T$, where

$$\widetilde{X}_{2k} = [\mathrm{Re}(X_k),\ \mathrm{Im}(X_k)], \qquad \widetilde{R}_{2k} = [\mathrm{Re}(R_k),\ \mathrm{Im}(R_k)].$$

If $\widetilde{R}_{2k} P_{\widetilde{X}_{2k}}^T = \widetilde{R}_{2k}$, and if $P_{\widetilde{X}_{2k}} J\widetilde{R}_{2k}\widetilde{X}_{2k}^T$ is skew-symmetric, then using Lemma 2.4 we obtain

$$E_{\text{opt}} = \widetilde{R}_{2k}\widetilde{X}_{2k}^{+} - J(\widetilde{R}_{2k}\widetilde{X}_{2k}^{+})^T J P_{\widetilde{X}_{2k}}^{\perp} \quad \text{and} \quad \eta_{\mathbb{R}}^{(6,6)}(X_k, \Lambda_k) = \alpha^{-1}\|E_{\text{opt}}\|_F.$$

4. Structured normwise condition numbers. The condition number characterizes the sensitivity of solutions to problems. If λ is a simple, nonzero eigenvalue of a singly or doubly structured matrix $A \in \mathcal{C}_{\mathbb{K}}$, with corresponding right eigenvector x and left eigenvector y, then a structured normwise condition number of λ can be defined as follows:

$$\kappa_{\mathbb{K}}(\lambda) := \lim_{\epsilon \to 0} \sup \left\{ \frac{|\Delta\lambda|}{\epsilon|\lambda|} : (A+E)(x+\Delta x) = (\lambda+\Delta\lambda)(x+\Delta x), \right. \\ \left. A+E \in \mathcal{C}_{\mathbb{K}},\ \|E\|_F \le \epsilon\alpha \right\}, \tag{4.1}$$

where α is a positive parameter. The forward error, condition number, and backward error are related by the inequality (correct to first order in the backward error)

$$\textit{forward error} \ \le \ \textit{condition number} \ \times \ \textit{backward error}.$$

In this section, we consider only linear structure in A. Expanding the first constraint in (4.1) and premultiplying by y^* lead to

$$\Delta\lambda = \frac{y^* E x}{y^* x} + O(\epsilon^2).$$

To evaluate $\kappa_{\mathbb{K}}(\lambda)$ we need to obtain a sharp bound for the first term in this expansion. If the structure is linear, then with the same notation as in section 3.1 we have

$$Ex = \mathrm{vec}(Ex) = (x^T \otimes I_m)\,\mathrm{vec}(E) = (x^T \otimes I_m) B \Delta p = M D \Delta p,$$

where $\mathrm{vec}(E) = B\Delta p$, $M = (x^T \otimes I_m) B D^{-1}$, and D is such that $\|E\|_F = \|D\Delta p\|_2$. Hence,

$$|y^* E x| = \|y^* M D \Delta p\|_2 \le \|y^* M\|_2 \|E\|_F = \|y^* M\|_2 \|D\Delta p\|_2.$$

Equality is obtainable for a suitable Δp because equality is always possible in the Cauchy–Schwarz inequality. Therefore

$$\kappa_{\mathbb{K}}(\lambda) = \alpha \frac{\|y^* M\|_2}{|\lambda||y^* x|}. \tag{4.2}$$

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